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On maximum degree (signless) Laplacian matrix of a graph

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Abstract

Let G be a simple graph on n vertices and $v_1, v_2, \ldots, v_n$ be the vertices of G. We denote the degree of a vertex v_i in G by $d_G(v_i) = d_i$. The maximum degree matrix of G, denoted by $M(G)$, is the real symmetric matrix with its ijth entry equal to $\max\{d_i, d_j\}$ if the vertices v_i and v_j are adjacent in G, 0 otherwise. In analogous to the definitions of Laplacian matrix and signless Laplacian matrix of a graph, we consider Laplacian and signless Laplacian for the maximum degree matrix, called the maximum degree Laplacian matrix and the maximum degree signless Laplacian matrix, respectively. Also, we introduce maximum degree Laplacian energy and maximum degree signless Laplacian energy of a graph. Then we determine the maximum degree (signless) Laplacian energy of some graphs in terms of ordinary energy, and (signless) Laplacian energy. We compute the maximum degree (signless) Laplacian spectra of some graph compositions. A lower and upper bound for the largest eigenvalue of the maximum degree (signless) Laplacian matrix is established and also we determine an upper bound for the second smallest eigenvalue of maximum degree Laplacian matrix in terms of vertex connectivity. We also determine bounds for the maximum degree (signless) Laplacian energy in terms of first Zagreb index.

Keywords: *Maximum degree matrix, maximum degree Laplacian matrix, maximum degree signless Laplacian matrix.*

AMS subject classification : *05C50.*

1. Introduction

Let G be a simple graph on n vertices and m edges. Let $V(G) = \{v_1, v_2, \dots, v_n\}$ be the vertex set of G. We use the notation $v_i \sim v_j$ ($i \sim j$) to denote that the vertices v_i and v_j are adjacent in G. Let $T(v_i) = \sum_{v_i \sim v_k} \max\{d_G(v_i), d_G(v_k)\}$ and $T(G) = \sum_{i=1}^{n} T(v_i)$. Let $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_{n-1}(G) \geq \lambda_n(G)$, $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_{n-1}(G) \geq \mu_n(G) = 0$ and $q_1(G) \geq q_2(G) \geq \cdots \geq q_{n-1}(G) \geq q_n(G)$ respectively, denote the eigenvalues of the adjacency matrix $A(G)$, Laplacian matrix $L(G) = D(G) - A(G)$ and signless Laplacian matrix $Q(G) = D(G) + A(G)$, where $D(G) = diag(d_1, d_2, \dots, d_n)$. Some studies and results on the spectra of these matrices can be found in [21, 5, 4] and therein cited references.

The energy of a graph G, denoted by $\mathcal{E}(G)$, was defined by I. Gutman [10] in the year 1978, as $\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i|$. The motivation for the definition of graph energy comes from Hückel theory, see [19] for details. Later in the year 2006, B. Zho and I. Gutman [14] introduced the concept of Laplacian energy $\mathcal{LE}(G)$ of a graph. It is defined as $\mathcal{LE}(G) = \sum_{i=1}^{n} \mid \mu_i - \frac{2m}{n} \mid$. In analogous to the definition of Laplacian energy of a graph, N. Abreu et al. [1] introduced the signless Laplacian energy $\mathcal{QE}(G)$ of a graph G, as $\mathcal{QE}(G) = \sum_{i=1}^{n} \mid q_i - \frac{2m}{n} \mid$. The concept of energy of a graph has slowly attracted many mathematicians and in recent years many papers on this topic are published. In fact there are more than 50 graph energies defined in literature, see [13, 11]. Studies on graph energy and (signless) Laplacian energy can be found in [19, 7, 8] and therein cited references.

Recently, several degree based graph matrices are introduced and there energies were studied. Some examples of degree based graph energies are Randić energy, harmonic energy, extended adjacency energy, Zagreb energy, arithmetic-geometric energy, etc, see [6, 18, 15] for more details. The maximum degree matrix of a graph was defined by C. Adiga and M. Smitha [3], recently. It is denoted by $M(G)$ and its ij-th entry is equal to $\max\{d_i, d_j\}$ if v_i and v_j are adjacent in G, 0 otherwise. Motivated by the definitions of Laplacian matrix and signless Laplacian matrix, we introduce maximum degree Laplacian matrix $L_M(G)$ and maximum degree signless Laplacian matrix $Q_M(G)$. These matrices are defined as $L_M(G) = D(G) - M(G)$ and $Q_M(G) = D(G) + M(G)$, where $D(G) = diag(T(v_1), T(v_2), \dots, T(v_n))$. We denote the eigenvalues of $L_M(G)$ and $Q_M(G)$ respectively, as $\partial_1^L(G) \geq \partial_2^L(G) \geq \cdots \geq \partial_{n-1}^L(G) \geq \partial_n^L(G) = 0$ and $\partial_1^Q(G) \geq \partial_2^Q(G) \geq \cdots \geq \partial_{n-1}^Q(G) \geq \partial_n^Q(G)$. The maximum degree

Laplacian energy $\mathcal{LE}_M(G)$ and maximum degree signless Laplacian energy $\mathcal{QE}_M(G)$ of a graph G are defined as

$$\mathcal{LE}_M(G) = \sum_{i=1}^{n} |\partial_i^L(G) - \frac{T(G)}{n}| \text{ and } \mathcal{QE}_M(G) = \sum_{i=1}^{n} |\partial_i^Q(G) - \frac{T(G)}{n}|.$$

One of the well studied degree based topological index is the first Zagreb index. For a graph G the first Zagreb index, denoted by $M_1(G)$, is defined as $M_1(G) = \sum_{i=1}^{n} d_G^2(v_i)$. Details about Zagreb indices can be found in [12].

In Section 2 of the paper, we determine the maximum degree (signless) Laplacian energy of some graphs in terms of ordinary energy and (signless) Laplacian energy. In Section 3, we compute the maximum degree (signless) Laplacian spectra of some graph compositions. In Section 4, a lower and upper bound for the largest eigenvalue of the (signless) Laplacian matrix is established and also we determine an upper bound for the second smallest eigenvalue of maximum degree Laplacian matrix in terms of vertex connectivity. We also determine bounds for the maximum degree (signless) Laplacian energy in terms of first Zagreb index.

2. Maximum degree (signless) Laplacian energy of some graphs

In this section, we determine the maximum degree (signless) Laplacian energy of some graphs in terms of ordinary energy and (signless) Laplacian energy. Also some basic results are presented.

The following definitions and lemmas will be used in this section.

Definition 2.1. [22] *The identity duplication of a graph G, denoted by $ID(G)$, is the graph obtained by taking two copies of the vertex set $V(G)$ and then joining a vertex in the first copy of $V(G)$ to a vertex in the second copy of $V(G)$ whenever they are adjacent in G.*

Definition 2.2. [17] *The double graph DG is the graph obtained by taking two copies of G and then joining a vertex in the first copy of G to a vertex in the second copy of G whenever they are adjacent in G.*

Lemma 2.3. [4] *Let $A = \begin{pmatrix} A_0 & A_1 \\ A_1 & A_0 \end{pmatrix}$ be a symmetric 2×2 block matrix. Then the spectrum of A is the union of spectrum of $A_0 + A_1$ and $A_0 - A_1$.*

Remark 2.4. *The matrices $L_M(G)$ and $Q_M(G)$ are positive semidefinite, because for any vector $X = (x_1, x_2, \ldots, x_n)$, we have*
$X^T L_M(G)X = \sum_{i\sim j} \max\{d_i, d_j\}(x_i - x_j)^2 \geq 0$ *and*
$X^T Q_M(G)X = \sum_{i\sim j} \max\{d_i, d_j\}(x_i + x_j)^2 \geq 0.$

For a vertex v of G, the neighborhood set of v is denoted by $N_G(v)$ and is defined as $N_G(v) = \{u \in V(G) : u \sim v\}$. The neighborhood degree sum of v is the sum of all degrees of vertices in $N_G(v)$ and is denoted by $N_d(v)$. We denote the maximum degree and minimum degree of a vertex in G by $\Delta(G) = \Delta$ and $\delta(G) = \delta$, respectively.

Proposition 2.5. *Let G be a graph. Suppose u and v are any two vertices of G such that $d_G(u) = d_G(v) = \Delta$ and $N_G(u)\backslash\{v\} = N_G(v)\backslash\{u\}$. Then Δ^2 is a maximum degree (signless) Laplacian eigenvalue of G if u and v are not adjacent in G, and if u and v are adjacent in G, then $\Delta^2 + \Delta$ is a maximum degree Laplacian eigenvalue of G and $\Delta^2 - \Delta$ is a maximum degree signless Laplacian eigenvalue of G.*

Proof. Suppose the vertices u and v are not adjacent in G. Since $d_G(u) = d_G(v) = \Delta$ and $N_G(u)\backslash\{v\} = N_G(v)\backslash\{u\}$, the matrices $L_M(G) - \Delta^2 I_n$ and $Q_M(G) - \Delta^2 I_n$ have two identical rows, namely the rows corresponding to the vertices u and v. Thus their determinants are zero, proving that Δ^2 is a maximum degree (signless) Laplacian eigenvalue of G. Similarly, if u and v are adjacent in G, then $\Delta^2 + \Delta$ is a maximum degree Laplacian eigenvalue of G and $\Delta^2 - \Delta$ is a maximum degree signless Laplacian eigenvalue of G. □

The proof of the following proposition is similar to Proposition 2.5.

Proposition 2.6. *Let G be a graph on n vertices. Suppose u and v are any two vertices of G such that $d_G(u) = d_G(v) = \delta$ and $N_G(u)\backslash\{v\} = N_G(v)\backslash\{u\}$. Then $N_d(u)$ is a maximum degree (signless) Laplacian eigenvalue of G if u and v are not adjacent in G, and if u and v are adjacent in G, then $N_d(u) + \delta$ is a maximum degree Laplacian eigenvalue of G and $N_d(u) - \delta$ is a maximum degree signless Laplacian eigenvalue of G.*

The following corollary follows from Proposition 2.6.

Corollary 2.7. *Let G be a graph. Let G^* be the graph obtained by attaching $p \geq 2$ pendant vertices to a vertex u of G. Then $d_G(u)$ is a maximum degree (signless) Laplacian eigenvalue of G^* with multiplicity at least $p-1$.*

Theorem 2.8. *Let G be an r regular graph of order n. Then $\mathcal{LE}_M(G) = \mathcal{QE}_M(G) = r\mathcal{E}(G)$.*

Proof. Since G is r regular, $L_M(G) = r(rI_n - A(G))$, $Q_M(G) = r(rI_n + A(G))$ and $T(G) = \sum_{i=1}^{n}\sum_{i\sim j}\max\{d_i, d_j\} = nr^2$. Thus for $1 \leq i \leq n$, the eigenvalues of $L_M(G)$ and $Q_M(G)$ are respectively, $\partial_i^L(G) = r(r - \lambda_{n-i+1}(G))$ and $\partial_i^Q(G) = r(r + \lambda_i(G))$. Hence, $\mathcal{LE}_M(G) = \sum\limits_{i=1}^{n} |\partial_i^L(G) - \frac{T(G)}{n}| = \sum\limits_{i=1}^{n} |r\lambda_i(G)| = r\mathcal{E}(G)$ and $\mathcal{QE}_M(G) = \sum\limits_{i=1}^{n} |\partial_i^Q(G) - \frac{T(G)}{n}| = \sum\limits_{i=1}^{n} |r\lambda_i(G)| = r\mathcal{E}(G)$. This completes the proof. $\square$

Let K_n, C_n, $Q_{2,n}$ and $H_{n,n}$ respectively, denote the complete graph on n vertices, the cycle graph on n vertices, the cocktail party graph on $2n$ vertices and crown graph on $2n$ vertices.

In the following corollary, we give the maximum degree (signless) Laplacian energy of some standard graphs.

Corollary 2.9. *We have*

$$\mathcal{LE}_M(G) = \mathcal{QE}_M(G) = \begin{cases} 2(n-1)^2 & \text{if } G = K_n, \\ 2\sum\limits_{j=1}^{n} |\cos\left(\frac{2\pi j}{n}\right)| & \text{if } G = C_n, \\ 8(n-1)^2 & \text{if } G = Q_{2,n}, \\ 4(n-1)^2 & \text{if } G = H_{n,n}. \end{cases}$$

Proof. We have $\mathcal{E}(K_n) = 2(n-1)$, $\mathcal{E}(C_n) = \sum\limits_{j=1}^{n} | \; cos\left(\frac{2\pi j}{n}\right) \; |$, $\mathcal{E}(K_{Q_{2,n}}) = 4(n-1)$ and $\mathcal{E}(H_{n,n}) = 4(n-1)$, see [4]. Therefore the corollary follows immediately from the above theorem. $\square$

Theorem 2.10. *Let G be a bi-regular graph such that no two vertices of degree δ are adjacent in G. Then $\mathcal{LE}_M(G) = \Delta\mathcal{LE}(G)$ and $\mathcal{QE}_M(G) = \Delta\mathcal{QE}(G)$.*

Proof. Since G is a bi-regular graph such that no two vertices of degree δ are adjacent, $L_M(G) = \Delta L(G)$, $Q_M(G) = \Delta Q(G)$ and $T(G) = 2m\Delta$. Thus the spectrum of $L_M(G)$ is $\{\Delta\mu_1, \Delta\mu_2, \ldots, \Delta\mu_n\}$ and the spectrum of

$Q_M(G)$ is $\{\Delta q_1, \Delta q_2, \ldots, \Delta q_n\}$. Therefore, $\mathcal{LE}_M(G) = \sum_{i=1}^{n} |\Delta\mu_i - \frac{T(G)}{n}| = \Delta \sum_{i=1}^{n} |\mu_i - \frac{2m}{n}| = \Delta\mathcal{LE}(G)$. Similarly, we have $\mathcal{QE}_M(G) = \Delta\mathcal{QE}(G)$. □

The following corollary is immediate from the above theorem.

Corollary 2.11. *Let P_n be the path graph on n vertices. Then $\mathcal{LE}_M(P_n) = \mathcal{QE}_M(P_n) = 2\mathcal{LE}(P_n) = 2\mathcal{QE}(P_n)$.*

Theorem 2.12. *Let G be a graph on n vertices and $ID(G)$ be the identity duplication graph of G. Then $\mathcal{LE}_M(ID(G)) = \mathcal{QE}_M(ID(G)) = \mathcal{LE}_M(G) + \mathcal{QE}_M(G)$.*

Proof. The graph $ID(G)$ is of order $2n$ with vertex set V(ID(G))=$V(G) \cup V(G)$ (disjoint union) and for any vertex $v \in V(ID(G))$, we have $d_{ID(G)}(v) = d_G(v)$. Therefore with suitable labeling of the vertices of $ID(G)$, the matrices $L_M(ID(G))$ and $Q_M(ID(G))$ can be written as

$$L_M(ID) = \begin{pmatrix} D(G) & -M(G) \\ -M(G) & D(G) \end{pmatrix} \text{ and } Q_M(ID) = \begin{pmatrix} D(G) & M(G) \\ M(G) & D(G) \end{pmatrix}.$$

Thus from Lemma 2.3, the spectrum of $L_M(ID(G))$ consists of the union of spectrum of $L_M(G)$ and the spectrum of $Q_M(G)$. Since $\frac{T(ID(G))}{2n} = \frac{T(G)}{n}$, we get $\mathcal{LE}_M(ID(G)) = \mathcal{LE}_M(G) + \mathcal{QE}_M(G)$. Similarly, we get $\mathcal{QE}_M(ID(G)) = \mathcal{LE}_M(G) + \mathcal{QE}_M(G)$. □

The following corollary follows from Theorems 2.8 and 2.12.

Corollary 2.13. *Let G be an r-regular graph. Then $\mathcal{LE}_M(ID(G)) = \mathcal{QE}_M(ID(G)) = 2r\mathcal{E}(G)$.*

Theorem 2.14. *Let G be a graph of order n and DG be its double graph. Then*
(i) Spectrum of $L_M(DG)$ is $\{4\partial_1^L(G), \ldots, 4\partial_n^L(G), 4T_G(v_1), \ldots, 4T_G(v_n)\}$.
(ii) Spectrum of $Q_M(DG)$ is $\{4\partial_1^Q(G), \ldots, 4\partial_n^Q(G), 4T_G(v_1), \ldots, 4T_G(v_n)\}$.

Proof. The graph DG is of order $2n$ with vertex set $V(DG) = V(G) \cup V(G)$ (disjoint union) and for any vertex $v \in V(DG)$, $d_{DG}(v) = 2d_G(v)$. Therefore with suitable labeling of the vertices of DG, we obtain

$$L_M(DG) = \begin{pmatrix} 4D(G) - 2M(G) & -2M(G) \\ \\ -2M(G) & 4D(G) - 2M(G) \end{pmatrix}$$

and

$$Q_M(DG) = \begin{pmatrix} 4D(G) + 2M(G)) & 2M(G) \\ \\ 2M(G) & 4D(G) + 2M(G)) \end{pmatrix}.$$

By Lemma 2.3, the spectrum of $L_M(DG)$ is the union of the spectrum of $4L_M(G)$ and the spectrum of $4D(G)$, and the spectrum of $Q_M(DG)$ is the union of spectrum of $4Q_M(G)$ and the spectrum of $4D(G)$. $\square$

The following corollary can be easily deduced from the above theorem.

Corollary 2.15. *Let G be an r-regular graph. Then* $\mathcal{LE}_M(DG) = \mathcal{QE}_M(DG) = 4r\mathcal{E}(G)$.

3. (Signless) Laplacian spectra of some composition of graphs

In this section, we give the (signless) Laplacian spectra of composition of graphs, namely, the complete product of two regular graphs and corona product of two regular graphs.

Definition 3.1. [4] *The complete product $G_1 \bigtriangledown G_2$ of two graphs G_1 and G_2 is the graph obtained by joining every vertex of G_1 with every vertex of G_2*

Definition 3.2. [9] *The corona product of two graphs G_1 and G_2 is obtained by taking $|V(G_1)|$ copies of G_2 and joining each vertex of G_1 with every vertices of the corresponding copy of G_2 .*

In the following lemma we give an upper bound for the graph parameter $T(G)$ in terms of first Zagreb index.

Lemma 3.3. *Let G be a graph on n vertices with m edges. If $d_1 \geq d_2 \geq \ldots \geq d_n$ is the degree sequence of G. Then*

$$M_1(G) \leq T(G) \leq M_1(G) + 2mn - n^2\delta.$$

Proof. We have

$$
\begin{aligned}
T(G) &= \sum_{i=1}^{n}\sum_{v_i\sim v_j}\max\{d_i,d_j\} \\
&= \textstyle\sum_{i=1}^{n}\left(d_i^2+\sum_{i>k v_i\sim v_k}(d_k-d_i)\right) \qquad (3.1)\\
&\le \sum_{i=1}^{n}\left(d_i^2+\sum_{k=1}^{i-1}(d_k-d_i)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} d_i^2+\sum_{i=1}^{n}\sum_{k=1}^{i-1}(d_k-d_i)\\
&= M_1(G)+n\textstyle\sum_{i=1}^{n} d_i-\sum_{i=1}^{n}(2i-1)d_i\\
&= M_1(G)+2nm-\textstyle\sum_{i=1}^{n}(2i-1)d_i\\
&\le M_1(G)+2nm-\delta\textstyle\sum_{i=1}^{n}(2i-1)\\
&= M_1(G)+2nm-n^2\delta.
\end{aligned}
$$

Thus, $T(G)\le M_1(G)+2nm-n^2\delta$. Proving the right inequality. From equation (3.1) the left inequality follows directly. □

In the following theorem we give an upper bound for the maximum degree (signless) Laplacian energy of disjoint union of graphs in terms of maximum degree (signless) Laplacian energy of parent graphs, first Zagreb index, order and size of the parent graphs.

Theorem 3.4. *For $1\le i\le k$, let G_i be a graph of order n_i and size m_i. Suppose G is a disjoint union of graphs G_i, $1\le i\le k$. Then*

(i) $\mathcal{LE}_M(G)\le\sum_{i=1}^{k}\left(\mathcal{LE}_M(G_i)+2[M_1(G_i)+2n_im_i-n_i^2\delta(G_i)]\right)$.

(ii) $\mathcal{QE}_M(G)\le\sum_{i=1}^{k}\left(\mathcal{QE}_M(G_i)+2[M_1(G_i)+2n_im_i-n_i^2\delta(G_i)]\right)$.

Proof. Let $n=n_1+n_2+\cdots+n_k$. Since $G=\bigcup_{i=1}^{k}G_i$, we have,

$$
LE_M(G)=\begin{pmatrix}
L_M(G_1) & \mathbf{0} & \cdots & \mathbf{0}\\
\mathbf{0} & L_M(G_2) & \cdots & \mathbf{0}\\
\vdots & \vdots & \ddots & \mathbf{0}\\
\mathbf{0} & \mathbf{0} & \cdots & L_M(G_k)
\end{pmatrix}
$$

.

Therefore the spectrum of $LE_M(G)$ is the disjoint union of the spectrum of each $L_M(G_i)$. Let μ_{ij} $(1 \leq j \leq n_i)$ be the eigenvalues of $L_M(G_i)$. Suppose $T(G) = T$ and $T(G_i) = T_i$ for $1 \leq i \leq k$. Then

$$\begin{aligned}
\mathcal{LE}_M(G) &= \sum_{j=1}^{n_1}\left|\mu_{1j} - \frac{T}{n}\right| + \sum_{j=1}^{n_2}\left|\mu_{2j} - \frac{T}{n}\right| + \cdots + \sum_{j=1}^{n_k}\left|\mu_{kj} - \frac{T}{n}\right| \\
&= \sum_{j=1}^{n_1}\left|\mu_{1j} - \frac{T_1}{n_1} + \frac{T_1}{n_1} - \frac{T}{n}\right| + \cdots + \sum_{j=1}^{n_k}\left|\mu_{kj} - \frac{T_k}{n_k} + \frac{T_k}{n_k} - \frac{T}{n}\right| \\
&\leq \sum_{i=1}^{k}\mathcal{LE}_M(G_i) + T + (n_1 + n_2 + \cdots + n_k)\frac{T}{n} \\
&\quad \text{(by triangular inequality)} \\
&= \sum_{i=1}^{k}\mathcal{LE}_M(G_i) + 2T = \sum_{i=1}^{k}\Big(\mathcal{LE}_M(G_i) + 2T_i\Big) \\
&\leq \sum_{i=1}^{k}\Big(\mathcal{LE}_M(G_i) + 2[M_1(G_i) + 2n_i m_i - n_i^2\delta(G_i)]\Big) \\
&\quad \text{(by Lemma 3.3).}
\end{aligned}$$

By a similar argument, the upper bound for $\mathcal{QE}_M(G)$ also follows. □

From the proof of the above theorem and from Theorem 2.8, we obtain the following corollary.

Corollary 3.5. *If G_i is an r_i regular on n_i vertices for $1 \leq i \leq k$. Let $G = \bigcup_{i=1}^{k} G_i$. Then $\mathcal{LE}_M(G) \leq \sum_{i=1}^{k} r_i\Big(2n_i r_i + \mathcal{E}(G_i)\Big)$ and $\mathcal{QE}_M(G) \leq \sum_{i=1}^{k} r_i\Big(2n_i r_i + \mathcal{E}(G_i)\Big)$.*

A matrix of order $n_1 \times n_2$ with all its entries equal to 1 is denoted by $J_{n_1\times n_2}$ or simply, J_{n_1} if $n_1 = n_2$. The column vector of size n with all its entries equal to 1 is denoted by 1_n.

Lemma 3.6. [2] *For $i = 1, 2$, let M_i be a normal matrix of order n_i having all its row sums equal to r_i. Suppose $r_i, \theta_{i2}, \theta_{i3}, \ldots, \theta_{in_i}$ are the eigenvalues of M_i, then for any two constants a and b, the eigenvalues of $M := \begin{pmatrix} M_1 & aJ_{n_1\times n_2} \\ bJ_{n_2\times n_1} & M_2 \end{pmatrix}$ are θ_{ij} for $i = 1, 2,$ $j = 2, 3, \ldots, n_i$ and the two roots of the quadratic equation $(x - r_1)(x - r_2) - abn_1n_2 = 0$.*

The following theorem gives the maximum degree (signless) Laplacian spectrum of the complete product of two regular graphs.

Theorem 3.7. *Let G_i be an r_i regular graph on n_i vertices for $i = 1, 2$. Suppose $r_1 + n_2 \geq r_2 + n_1$. Then*
(i) the spectrum of $L_M(G_1 \bigtriangledown G_2)$ consists of $(r_1 + n_2)(n_1 + n_2)$; 0; $(r_1 + n_2)(r_1 + n_2 - \lambda_i(G_1))$ for $2 \leq i \leq n_1$; $n_1(r_1+n_2)+(r_2+n_1)(r_2-\lambda_i(G_2))$ for $2 \leq i \leq n_2$.

(ii) the spectrum of $Q_M(G_1 \bigtriangledown G_2)$ consists of $(r_1+n_2)(r_1+n_2+\lambda_i(G_1))$ for $2 \leq i \leq n_1$; $n_1(r_1+n_2)+(r_2+n_1)(r_2+\lambda_i(G_2))$ for $2 \leq i \leq n_2$ and the two roots of the polynomial $(x - (r_1 + n_2)(2r_1 + n_2))(x - n_1(r_1 + n_2) - 2r_2(r_2 + n_1)) - (r_1 + n_2)^2 n_1 n_2$.

Proof. Let $V(G_1 \bigtriangledown G_2) = \{v_1, v_2, \ldots, v_{n_1}, u_1, u_2, \ldots, u_{n_2}\}$ such that $V(G_1) = \{v_1, v_2, \ldots, v_{n_1}\}$ and $V(G_2) = \{u_1, u_2, \ldots, u_{n_2}\}$. Then $d_{G_1 \bigtriangledown G_2}(v_i) = r_1 + n_2$ and $d_{G_1 \bigtriangledown G_2}(u_j) = r_2 + n_1$ for $1 \leq i \leq n_1$ and $1 \leq j \leq n_2$. Therefore the matrix $L_M(G_1 \bigtriangledown G_2)$ is equal to

$$\left(\begin{array}{c|c} (r_1+n_2)\Big((r_1+n_2)I_{n_1} - A(G_1)\Big) & -(r_1+n_2)J_{n_1\times n_2} \\ \hline -(r_1+n_2)J_{n_2\times n_1} & \Big(n_1(r_1+n_2)+r_2(r_2+n_1)\Big)I_{n_2} - (r_2+n_1)A(G_2) \end{array}\right).$$

Now, set $M_1 = (r_1 + n_2)((r_1 + n_2)I_{n_1} - A(G_1))$, $M_2 = (n_1(r_1 + n_2) + r_2(r_2+n_1))I_{n_2} - (r_2+n_1)A(G_2)$ and $a = b = -(r_1+n_2)$ in Lemma 3.6. Since the spectrum of M_1 and M_2 are respectively, $\{n_2(r_1 + n_2), (r_1 + n_2)(r_1 + n_2 - \lambda_i(G_1)), 1 \leq i \leq n_1\}$ and $\{n_1(r_1 + n_2), n_1(r_1 + n_2) + (r_2 + n_1)(r_2 - \lambda_i(G_2)), 1 \leq i \leq n_2\}$, the theorem for $L_M(G_1 \bigtriangledown G_2)$ follows. Similarly, we obtain the spectrum of $Q_M(G_1 \bigtriangledown G_2)$. □

Corollary 3.8. *Let G_1 and G_2 be two integral regular graphs. Then the maximum degree Laplacian spectrum of $G_1 \bigtriangledown G_2$ is integral.*

Let $P = (p_{ij})$ and Q be two matrices. The Kronecker product of P and Q is denoted by $P \otimes Q$, and is obtained from P by replacing each entry p_{ij} by $p_{ij}Q$. For matrices P, Q, R and S, $(P \otimes Q)(R \otimes S) = PR \otimes QS$. More details on Kronecker product of matrices can be found in [16].

Let $A = (a_{ij})$ be a real symmetric matrix of order p. Let B_i $(1 \leq i \leq p)$ be a real symmetric matrix of order n such that $B_i 1_n = r 1_n$. Define $\mathcal{S} = \mathcal{S}[A, B, a, b] = \begin{bmatrix} A \otimes aJ_n + B & I_p \otimes b1_n \\ I_p \otimes b1_n^T & A \end{bmatrix}$, where B is a bock diagonal matrix given by $B = diag(B_1, B_2, \ldots, B_p)$ and a, b are real constants. Let $\theta_{i1} = r, \theta_{i2}, \ldots, \theta_{in}$ be the eigenvalues of B_i $(1 \leq i \leq p)$ and let

$\gamma_1, \gamma_2, \ldots, \gamma_p$ be the eigenvalues of A.

In the following lemma we give the spectrum of the matrix $\mathcal{S} = \mathcal{S}[A, B, a, b]$ as defined above.

Lemma 3.9. *The spectrum of the matrix $\mathcal{S}$ consists of θ_{ij} for $1 \le i \le p$ and $2 \le j \le n$; $r + an\gamma_i + b\alpha_{i1}$ and $r + an\gamma_i + b\alpha_{i2}$ $(1 \le i \le p)$, where α_{i1} and α_{i2} are the two roots of the polynomial $bx^2 + (an\gamma_i - \gamma_i + r)x - bn$.*

Proof. Let $X_{i1} = \frac{1}{\sqrt{n}} 1_n, X_{i2}, \ldots, X_{in}$ be a set of orthonormal eigenvectors of the matrix B_i corresponding to the eigenvalues $\theta_{i1} = r, \theta_{i2}, \ldots, \theta_{in}$, respectively. Let $Y_1, Y_2, \ldots, Y_p$ be a set of orthonormal eigenvectors of the matrix A corresponding to the eigenvalues $\gamma_1, \gamma_2, \ldots, \gamma_p$, respectively. For $1 \le i \le p$ and $2 \le j \le n$, define

$Z_{ij} = \begin{bmatrix} e_i \otimes X_{ij} \\ \mathbf{0}_p \end{bmatrix}$, where e_i is the i-th vector of the canonical basis of R^p. Then $\mathcal{S}Z_{ij} = \theta_{ij} Z_{ij}$. Thus, θ_{ij} is an eigenvalue of $\mathcal{S}$ corresponding to the eigenvector Z_{ij}. Let

$Z_{i1} = \begin{bmatrix} Y_i \otimes 1_n \\ \alpha_i Y_i \end{bmatrix}$, where α_i is any non-zero real number. Then

$\mathcal{S}Z_{i1} = \begin{bmatrix} (r + an\gamma_i + b\alpha_i) Y_i \otimes 1_n \\ (bn + \gamma_i \alpha_i) Y_i \end{bmatrix}$. Therefore Z_{i1} is an eigenvector of $\mathcal{S}$ corresponding to the eigenvalue $r + an\gamma_i + b\alpha_i$ if and only if $bn + \gamma_i\alpha_i = \alpha_i(r + an\gamma_i + b\alpha_i)$. Thus, $r + an\gamma_i + b\alpha_i$ is an eigenvalue of $\mathcal{S}$ if and only if α_i is a root of the polynomial $f_i(x) := bx^2 + (an\gamma_i - \gamma_i + r)x - bn$. Hence if α_{i1} and α_{i2} are the two roots of the polynomial $f_i(x)$, then $r + an\gamma_i + b\alpha_{i1}$ and $r + an\gamma_i + b\alpha_{i2}$ are the eigenvalues of $\mathcal{S}$. Thus we have listed all the eigenvalues of $\mathcal{S}$. This completes the proof. $\square$

Theorem 3.10. *Let G_1 be an r_1 regular graph on p vertices and G_2 be an r_2 regular graph on n vertices. Then the maximum degree Laplacian spectrum of the corona product $G_1 \circ G_2$ consists of $(r_2 + 1)(r_2 - \lambda_i(G_2)) + r_1 + n$ with multiplicity p for $2 \le i \le n$; 0; $n^2 + nr_1 + n + r_1$; $(r_1 + n)(\alpha_{i1} + 1)$ and $(r_1 + n)(\alpha_{i2} + 1)$, where α_{i1} and α_{i2} are the roots of the polynomial $(r_1 + n)x^2 + (r_1 + n - (r_1 + n)(r_1 - \lambda_i(G_1)) - n(r_1 + n))x - (r_1 + n)n$ for $2 \le i \le p$.*

Proof. The maximum degree Laplacian matrix $L_M(G_1 \circ G_2)$ of graph $G_1 \circ G_2$ is

$$\begin{bmatrix} I_p \otimes \left[(r_2+1)(r_2 I_n - A(G_2)) + (r_1+n)I_n\right] & -I_p \otimes (r_1+n)1_n \\ -I_p \otimes (r_1+n)1_n^T & (r_1+n)(r_1 I_p - A(G_1)) + n(r_1+n)I_p \end{bmatrix}.$$

Let $B_i = (r_2+1)(r_2 I_n - A(G_2)) + (r_1+n)I_n$ for $1 \leq i \leq p$, $A = (r_1+n)(r_1 I_p - A(G_1)) + n(r_1+n)I_p$, $a = 0$ and $b = r_1+n$. Then $L_M(G_1 \circ G_2) = \mathcal{S}[A, B, a, b]$. Therefore the theorem follows from Lemma 3.9. $\square$

Theorem 3.11. *Let G_1 be an r_1 regular graph on p vertices and G_2 be an r_2 regular graph on n vertices. Then the maximum degree signless Laplacian spectrum of the corona product $G_1 \circ G_2$ consists of $(r_2+1)(r_2+\lambda_i(G_2)) + r_1 + n$ with multiplicity p for $2 \leq i \leq n$; $(r_1+n)(\alpha_{i1}+1)$ and $(r_1+n)(\alpha_{i2}+1)$, where α_{i1} and α_{i2} are the roots of the polynomial $(r_1+n)x^2 + (r_1+n-(r_1+n)(r_1+\lambda_i(G_1)) - n(r_1+n))x - (r_1+n)n$ for $1 \leq i \leq p$.*

Proof. The maximum degree signless Laplacian matrix $Q_M(G_1 \circ G_2)$ of graph $G_1 \circ G_2$ is

$$\begin{bmatrix} I_p \otimes (r_2+1)(r_2 I_n + A(G_2)) + (r_1+n)I_n & I_p \otimes (r_1+n)1_n \\ I_p \otimes (r_1+n)1_n^T & (r_1+n)(r_1 I_p + A(G_1)) + n(r_1+n)I_p \end{bmatrix}.$$

Let $B_i = (r_2+1)(r_2 I_n + A(G_2)) + (r_1+n)I_n$ for $1 \leq i \leq p$, $A = (r_1+n)(r_1 I_p + A(G_1)) + n(r_1+n)I_p$, $a = 0$ and $b = r_1+n$. Then $Q_M(G_1 \circ G_2) = \mathcal{S}[A, B, a, b]$. Therefore the theorem follows from Lemma 3.9. $\square$

4. Bounds for the eigenvalue and energy of maximum degree (signless) Laplacian matrix

In this section, we give some bounds for the eigenvalues of maximum degree (signless) Laplacian matrix. Also some bounds for the maximum degree (signless) Laplacian energy are presented. We denote the ith largest eigenvalue of an Hermitian matrix H by $\theta_i(H)$.

4.1. Eigenvalue bounds

Let $A = (a_{ij})$ and $B = (b_{ij})$ be real matrices of same order. We denote by A^+ the matrix obtained from A by taking the absolute value of each entries in A. The notation $A \leq B$ implies that $a_{ij} \leq b_{ij}$ for all i and j. We need the following lemmas to achieve our bounds.

Lemma 4.1. [16] *Let $M = N + P$, where N and P are Hermitian matrices of order n. Then for $1 \leq i, j \leq n$, we have (i) $\theta_i(N) + \theta_j(P) \leq \theta_{i+j-n}(M)$ $(i + j > n)$ and (ii) $\theta_{i+j-1}(M) \leq \theta_i(N) + \theta_j(P)$ $(i + j - 1 \leq n)$.*

Lemma 4.2. [20] *Let H be a Hermitian matrix of order n with diagonal elements $d_1, d_2, \ldots, d_n$ and eigenvalues $\theta_1 \geq \theta_2 \geq \cdots \geq \theta_n$. Then $\sum_{i=1}^{k} d_i \leq \sum_{i=1}^{k} \theta_i$.*

Lemma 4.3. [16] *Let A and B be two real matrices of order n such that $A^+ \leq B$. Let $\rho(A)$ and $\rho(B)$ be the spectral radius of A and B. Then $\rho(A) \leq \rho(B)$.*

Lemma 4.4. [16] *Let $M = (m_{ij})$ be an $n \times n$ irreducible non-negative matrix with spectral radius θ_1. Let $R_i(M) = \sum_{j=1}^{n} m_{ij}$. Then*

$$\min\{R_i(M) : 1 \leq i \leq n\} \leq \theta_1 \leq \max\{R_i(M) : 1 \leq i \leq n\}.$$

Lemma 4.5. [16] *Let M be a symmetric matrix of order n and let $\theta_1 \geq \theta_2 \geq \ldots \geq \theta_n$ be the eigenvalues of M. Then for any non zero vector X in R^n, $\theta_n \leq \dfrac{X^T M X}{X^T X}$. Moreover the equality holds if and only if X is an eigenvector corresponding to the eigenvalue θ_n.*

Theorem 4.6. *Let G be a graph of order n. Then $\dfrac{2M_1(G)}{n} \leq \partial_1^Q(G) \leq 2\Delta^2(G)$.*

Proof. Left inequality: Let X be a non-zero column vector of order n. Then by Rayleigh principle, $\partial_1^Q(G) \geq \dfrac{X^T Q_M X}{X^T X}$. Set $X = (1, 1, \ldots, 1)^T$. Then $\partial_1^Q(G) \geq \dfrac{2T(G)}{n}$. Therefore, $\partial_1^Q(G) \geq \dfrac{2M_1(G)}{n}$ by Lemma 3.3.

Right inequality: By Lemma 4.4, we get $\partial_1^Q(G) \leq \max\limits_{v_i \in V(G)} \{2T(v_i)\} = 2\Delta^2(G)$. $\square$

In the following theorem we give an upper and lower bound for largest maximum degree Laplacian eigenvalue in terms of maximum degree only.

Theorem 4.7. *Let G be a graph on n vertices. Then $\Delta^2(G) \leq \partial_1^L(G) \leq 2\Delta^2(G)$.*

Proof. Note that $\Delta^2(G) \geq T(u)$ for all $u \in V(G)$. Now let v be a vertex of G such that $d_G(v) = \Delta$. Then $T(v) = \Delta^2(G)$. Therefore from Lemma 4.2, we must $\Delta^2(G) \leq \partial_1^L(G)$. Since $L_M(G)^+ = Q_M(G)$, the upper bound follows from Lemma 4.3 and Theorem 4.6. □

The following theorem gives an upper bound for the second smallest eigenvalue of the maximum degree Laplacian matrix in terms of vertex connectivity.

Theorem 4.8. *Let G be a graph on n vertices and let k be the vertex connectivity of G. Then $\partial_{n-1}^L(G) \leq k(n-1)$.*

Proof. Let $V_k = \{v_1, v_2, \ldots, v_k\}$ be a vertex cut set of G. Let $G[V_k]$ and $G[V_k^c]$ be the subgraphs of G induced by the vertex sets V_k and $V_k^c = V \backslash V_k$, respectively. The maximum degree Laplacian matrix of the graph $G_1 = K_k \bigtriangledown G[V_k^c]$ is

$$L_M(G_1) = \begin{bmatrix} (n-1)((nI_k - J_k) & -(n-1)J_{k\times(n-k)} \\ -(n-1)J_{(n-k)\times k} & L_M(G[V_k^c]) + kL(G[V_k^c]) + (n-1)kI_{n-k} \end{bmatrix}.$$

Since $G[V_k^c]$ is disconnected, 0 is an eigenvalue of $L_M(G[V_k^c]) + kL(G[V_k^c])$ with multiplicity at least 2. Thus there exists an eigenvector X corresponding to the eigenvalue 0 such that $1_{n-k}^T X = 0$, because $L_M(G[V_k^c]) + kL(G[V_k^c])$ has $n-k$ orthogonal eigenvectors and 1_{n-k} is an eigenvector of $L_M(G[V_k^c]) + kL(G[V_k^c])$ corresponding to the eigenvalue 0. Let $Y = \begin{bmatrix} \mathbf{0}_k \\ X \end{bmatrix}$. Then $L_M(G_1)Y = (n-1)kY$. Therefore, $(n-1)k$ is an eigenvalue of $L_M(G_1)$. Since $L_M(G_1)$ is positive semidefinite having 0 as one of its eigenvalue, we have $\partial_{n-1}^L(G_1) \leq (n-1)k$. Now from Lemma 4.1, we get

$$\theta_n(L_M(G_1) - L_M(G)) + \partial_{n-1}^L(L_M(G)) \leq \partial_{n-1}^L(L_M(G_1)).$$

Since $L_M(G_1) - L_M(G)$ is positive semidefinite with 0 as its eigenvalue, we have $\theta_n(L_M(G_1) - L_M(G)) = 0$. Thus, $\partial_{n-1}^L(L_M(G)) \leq (n-1)k$. This completes the proof. □

Since the vertex connectivity of a graph G is less than or equal to $\delta(G)$, the following corollary follows.

Corollary 4.9. *Let G be a graph on n vertices. Then $\partial_{n-1}^L(G) \leq \delta(G)(n-1)$.*

Proposition 4.10. *Let G be a graph on n vertices and $V(G) = \{v_1, v_2, \ldots, v_n\}$. Suppose v_k is a vertex in G such that $N_d(v_k) = \min\{N_d(v_i) : v_i \in V(G)$ and $d_G(v) = \delta(G)\}$. Then $\partial_n^Q(G) \leq N_d(v_k)$.*

Proof. Let $X = (x_1, x_2, \ldots, x_n)$ be a unit vector of size n. Then from Lemma 4.5, we get $\partial_n^Q(G) \leq X^T Q_M(G) X = \sum_{i \sim j}(x_i + x_j)^2 \max\{d_i, d_j\}$. Let $x_i = 0$ for all $i \neq k$ and $x_k = 1$. Then $\sum_{i\sim j}(x_i + x_j)^2 \max\{d_i, d_j\} = \sum_{i \sim k} d_i = N_d(v_k)$. Therefore, $\partial_n^Q(G) \leq X^T Q_M(G) X = N_d(v_k)$. □

4.2. Bounds for maximum degree (signless) Laplacian energy

Let $\Theta_i(G) = \partial_i^L(G) - \dfrac{T(G)}{n}$. The following lemma will be used to give an upper bound for the maximum degree (signless) Laplacian energy.

Lemma 4.11. *Let G be a graph of order n. Then*
(i) $\sum_{i=1}^{n} \left(\partial_i^L(G)\right)^2 = T(G)^2 - 2a_2(G)$.
(ii) $\sum_{i=1}^{n} \Theta_i^2(G) = \left(\dfrac{n-1}{n}\right) T(G)^2 - 2a_2(G)$
where $a_2(G)$ is the coefficient of x^{n-2} in $det(xI_n - L_M(G))$.

Proof. We have
$\left(\sum_{i=1}^{n} \partial_i^L(G)\right)^2 = \sum_{i=1}^{n} \left(\partial_i^L(G)\right)^2 + 2 \sum_{1 \leq i < j \leq n} \partial_i^L(G) \partial_j^L(G) = \sum_{i=1}^{n} \left(\partial_i^L(G)\right)^2 + 2a_2(G)$. Therefore, $T(G)^2 = \sum_{i=1}^{n} \left(\partial_i^L(G)\right)^2 + 2a_2(G)$. Proving (i).

Also,

$$\sum_{i=1}^{n} \Theta_i^2(G) = \sum_{i=1}^{n} \left(\partial_i^L(G) - \frac{T(G)}{n}\right)^2 = \sum_{i=1}^{n} \left(\partial_i^L(G)\right)^2 - \frac{1}{n} T(G)^2.$$

Therefore, $\sum_{i=1}^{n} \Theta_i^2(G) = \left(\dfrac{n-1}{n}\right) T(G)^2 - 2a_2(G)$. Proving (ii). □

In the following theorem we give an upper bound for the maximum degree Laplacian energy.

Theorem 4.12. *Let G be a graph of order n with m edges. Then*

$$\mathcal{LE}_M(G) \leq \sqrt{(n-1)(M_1(G)+2mn-n^2\delta)^2 - n\left(M_1^2(G) - \sum_{i=1}^{n} d_i^4 - 2\sum_{i=1}^{n} x_i d_i^2\right)},$$

where x_i is the number of vertices in the neighborhood of v_i whose degree is less than or equal to d_i.

Proof. By Cauchy-Schwarz inequality and from Lemma 4.11,

$$\begin{aligned}\mathcal{LE}_M(G)^2 &= \left(\sum_{i=1}^{n} \mid \Theta_i(G) \mid\right)^2 \leq n\left(\sum_{i=1}^{n} \Theta_i^2(G)\right)\\ &= (n-1)T(G)^2 - 2na_2(G),\end{aligned} \tag{4.1}$$

where $a_2(G)$ is the coefficient of x^{n-2} in $det(xI_n - L_M(G))$.

Since $a_2(G)$ is the sum of all principal minors of order 2 in $L_M(G)$, we have

$$\begin{aligned} a_2(G) &= \sum_{1\leq i<j\leq n} \gamma_i(G)\gamma_j(G)\\ &= \sum_{1\leq i<j\leq n} det\begin{pmatrix} T(v_i) & -\max\{d_i,d_j\}\\ -\max\{d_i,d_j\} & T(v_j)\end{pmatrix}\\ &= \sum_{1\leq i<j\leq n} T(v_i)T(v_j) - \sum_{i=1}^{n}(x_i)d_i^2,\end{aligned}$$

and

$$\begin{aligned}\sum_{1\leq i<j\leq n} T(v_i)T(v_j) &\geq \sum_{1\leq i<j\leq n} d_i^2 d_j^2\\ &= \frac{1}{2}\sum_{i=1}^{n}\sum_{i\neq j j=1}^{n} d_i^2 d_j^2\\ &= \frac{1}{2}\sum_{i=1}^{n} d_i^2(M_1(G) - d_i^2)\\ &= \frac{1}{2}\left(M_1^2(G) - \sum_{i=1}^{n} d_i^4\right).\end{aligned}$$

Thus,

$$a_2 \geq \frac{1}{2}\left(M_1^2(G) - \sum_{i=1}^{n} d_i^4\right) - \sum_{i=1}^{n} x_i d_i^2. \tag{4.2}$$

Using equation (4.2) and Lemma 3.3 in (4.1), we get the desired result. □

Note that the above upper bound for the maximum degree Laplacian energy also holds for the maximum degree signless Laplacian energy. The following theorem gives a lower bound for the maximum degree Laplacian energy.

Theorem 4.13. *Let G be a graph on n vertices. Then* $\mathcal{LE}_M(G) \geq \dfrac{2M_1(G)}{n}$.

Proof. We have

$$\begin{aligned}\mathcal{LE}_M(G) &= \textstyle\sum_{i=1}^{n} |\partial_i^L(G) - \dfrac{T(G)}{n}| \\ &\geq \partial_1^L(G) + \textstyle\sum_{i=2}^{n-1} |\partial_i^L(G) - \dfrac{T(G)}{n}| \\ &\geq \partial_1^L(G) + \Big|\textstyle\sum_{i=2}^{n-1} (\partial_i^L(G) - \dfrac{T(G)}{n})\Big| \text{(by triangular inequality)} \\ &= \partial_1^L(G) + \Big|T(G) - \partial_1^L(G) - \dfrac{(n-2)T(G)}{n}\Big| \\ &\geq \dfrac{2T(G)}{n} \\ &\geq \dfrac{2M_1(G)}{n} \text{(by Lemma 3.3).}\end{aligned}$$

Thus, $\mathcal{LE}_M(G) \geq \dfrac{2M_1(G)}{n}$. □

Theorem 4.14. *Let G be a graph on n vertices. Then* $\mathcal{QE}_M(G) \geq \dfrac{2M_1(G)}{n}$.

Proof. Let $\gamma_i = \partial_i^Q(G) - \dfrac{T(G)}{n}$. Then $\sum_{i=1}^{n} \gamma_i = 0$ and so $\mathcal{QE}_M(G) = \displaystyle\sum_{i=1}^{n} |\gamma_i| = 2\sum_{i=1}^{s} \gamma_i$, where s is the largest integer such that $\gamma_s \geq 0$. Therefore, $\mathcal{QE}_M(G) \geq 2(\partial_1^Q(G) - \dfrac{T(G)}{n})$. From the proof of Theorem 4.6, we get $\partial_1^Q(G) \geq \dfrac{2T(G)}{n}$. Therefore from Lemma 3.3, we get, $\mathcal{QE}_M(G) \geq \dfrac{2M_1(G)}{n}$.
□

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