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Mixed Fuzzy topological space its Hausdorff properties and base

Hari Prasad Chetri
Central Institute of Technology, India
and
Gautam Chandra Ray
Central Institute of Technology, India
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Abstract

In this article, mixed fuzzy topology and its topological properties have been studied. Mixed fuzzy topology is defined with the help of quasi-coincidence and closure of a fuzzy set in one of the fuzzy topologies. Thus, a new fuzzy topology is generated from the given two fuzzy topologies. This new fuzzy topology may or may not contain the topological properties of the parent topologies. This study identifies some topological properties that are carried to the mixed fuzzy topology from the given parent fuzzy topologies and some other properties which are not carried to the mixed fuzzy topology. Here a base for mixed fuzzy topology from the bases of the given parent topologies is constructed. Considering the regularity of one of the parent topologies mixed fuzzy topology is investigated. Hausdorff's properties of mixed fuzzy topological spaces are also discussed. It is now of general interest to know which properties are carried to the mixed topology and which are not. A few of these are being tried to answer here in this paper.

Keywords: *Base, Fuzzy Hausdorff topological space, Fuzzy sets, Fuzzy topological space, Mixed fuzzy topology, Mixed fuzzy topological space, Mixed topology.*

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1. Introduction

The concept of mixing two topologies to obtain a new topology is not new. Mixing two fuzzy topologies with the help of quasi-coincidence in fuzzy sets and closure in one of the fuzzy topologies to get an open set for a fuzzy topology is called mixed fuzzy topology. In 1965, Zadeh[11] introduced a new branch of mathematics namely fuzzy sets and logic. Just after 3 years of introduction of fuzzy concepts, fuzzy sets and logic were applied to study topology and topological properties by Chang[7] and opened the doors to study a new branch of mathematics called fuzzy topological spaces. In 1995, P.C. Baishya and N. R. Das[14] mixed two fuzzy topologies on an underlying given set to get a third fuzzy topology called mixed fuzzy topology. Thereafter, many mathematicians and researchers contributed to the theory of mixed fuzzy topology. B. C. Tripathy and G. C. Ray[2] have generalized the definition of Das et al.[14] and studied mixed fuzzy topology in a more generalized way. A. Vadivel and B. Vijayalakshmi recently, introduced and studied mixed e-fuzzy topological space (2017)[1]. Wadai F. Al-Omeri introduced and studied Mixed $\gamma-$ Fuzzy in Mixed-fuzzy topological spaces and their application (2018)[16] and Mixed b-fuzzy topological spaces (2020)[17]. G. C. Ray and H. P. Chetri recently studied Separation Axioms in Mixed Fuzzy Topological Spaces [9] B. C. Tripathy and G. C. Ray also introduced Fuzzy $\delta-I-$ continuity[3] and Fuzzy δ^* continuous and almost continuous functions[4] in mixed fuzzy ideal topological spaces. G. C. Ray and S. Dey studied Separation axioms in mixed multiset topological space [10]. M. J. Borah and B. Hazarika studied Soft ideal topological space and Mixed fuzzy soft ideal Topological space[13] . B. C. Tripathy and S. Debnath also studied $\gamma-$ open sets and $\gamma-$ continuous mappings in fuzzy bitopological spaces [5] and On fuzzy b-locally open sets in bitopological spaces [6] which involves two fuzzy topologies.

In this paper, a base for mixed fuzzy topology from a base of the given parent fuzzy topology is constructed. Compared interiors, closures, and derived sets in mixed fuzzy topology and given parent fuzzy topologies. Hausdorff's property of mixed fuzzy topology is also investigated. Required basic definitions are included in the preliminary section though most of them are standard.

Definition 1.1. *[15] Let $X \neq \phi$ and $I = [0, 1]$ be the unit interval. Let $\mu_A : X \longrightarrow I$ where μ_A is the membership function of A characterize a function A in X is called a fuzzy set. Here $\mu_A(x)$ represents the membership grade of $x \epsilon A$. The null fuzzy set is defined by $\mu_\phi(x) = 0\ \forall x \in X$. The whole* *space X can be defined as $\mu_X(x) = 1\ \forall x \in X$. Sometimes fuzzy null set and fuzzy whole space are denoted X by $\bar{0}$ and $\bar{1}$ respectively. If the membership functions of two fuzzy sets A and B are equal i.e., $\mu_A(x) = \mu_A(x)$, $\forall x \in X$. then fuzzy sets $A \leq B$. If A and B are two fuzzy sets with $\mu_A(x) \leq \mu_A(x)$, $\forall x \in X$. then it is written as $A \leq B$. The membership function $\mu_{A'} = 1 - \mu_A$ defines the complement. $\mu_{A'}$ or A^c or $co(A)$ is used to represent the fuzzy complement. For a collection $\{A_i | i \in I\}$ of fuzzy sets $\vee_i A_i$ and $\wedge_i A_i$ to represent fuzzy union and fuzzy intersections and are respectively defined as follows:*

$\mu_{\vee_i} A_i = \sup\{\mu_{A_i}(x) | i \in I\}$, $\forall x \in X$. $\mu_{\wedge_i} A_i = \inf\{\mu_{A_i}(x) | i \in I\}$, $\forall x \in X$.

Definition 1.2. *[7] Let$I = [0, 1]$ and $X \neq \phi$. Consider I^X be the collection of all mappings from X into I. A family τ of member of I^X such that*
i) $\bar{0}$ and $\bar{1} \in \tau$.
ii) For any finite sub collection $\beta = (B_i)_{i=1}^n$ of members of τ, $\bigcap_{i=1}^n (B_i) \in \tau$.
iii) For any arbitrary collection Δ of members of τ, $\bigcup_{B \in \Delta} (B) \in \tau$.

Then τ is called fuzzy topology and the pair (X, τ) is called a fuzzy topological space and members of τ are called τ-open fuzzy sets. Fuzzy topological space will be abbreviated as fts.

Definition 1.3. *[15] For a fuzzy set A in an fts (X, τ) the interior is defined as $A^0 = \vee\{B | B \leq A$; B is an open fuzzy set$\}$ and denoted by A^0. If there are more than one fuzzy topology then it is written as A^0_τ.*

Definition 1.4. *[15] Let A be fuzzy set in an fts (X, τ) then closure of A is defined as $\bar{A} = \vee\{F \mid A \leq F$, F is a closed fuzzy set$\}$ and is denoted by $\bar{A}$. If there are more than one fuzzy topology then A_τ will be used.*

Definition 1.5. *[15] A subfamily $\mathcal{B}$ of τ in an fts (X, τ) is called a base for τ iff, (iff is the abbreviation for if and only if) for every $A \in \tau$, there exists $\mathcal{B}_A \subseteq \mathcal{B}$ such that$A = \vee\{B \mid B \in \mathcal{B}_A\}$.*

Definition 1.6. *[10] A fuzzy set is called a fuzzy point iff it takes the value λ $(0 \leq \lambda \leq 1)$ at one point say $x \in X$ and 0 for all other $y \in X$ and it is denoted by x_λ. Where X is the whole space, x is its support and λ its value.*

Definition 1.7. *[15] A fuzzy point $x_\lambda \in A$ iff $\lambda \leq A(x)$.*

Definition 1.8. *[15] Let A and B be two fuzzy sets in X. They are said to be intersecting iff* $\exists$ *a point* $x \in X$ *such that* $(A \wedge B)(x) \neq 0$. *Then it is called A and B intersect at x.*

Definition 1.9. *[3] A fuzzy set A in* (X, τ) *is called a neighborhood of a fuzzy point* $x_\lambda \in A$ *iff there exists* $B \in \tau$ *such that* $x_\lambda \in B \leq A$. *A neighborhood A is called open neighborhood iff* $A \in \tau$.

Definition 1.10. *[15] If* $\lambda > A'(x)$, *or* $\lambda + A(x) > 1$ *then* x_λ *is said to be quasi-coincident with a fuzzy set A. Which is denoted as* $x_\lambda q A$.

Definition 1.11. *[15] Two fuzzy sets A and B are called quasi-coincident iff* $\exists x \in X$ *such that* $A(x) + B(x) > 1$ *and denote it by* AqB.

Definition 1.12. *[15] A fuzzy set A in* (X, τ) *is called a* $Q-$ *neighborhood of* x_λ *iff* $\exists B \in \tau$ *such that* $B \leq A$ *and* $x_\lambda q B$.

Definition 1.13. *[15] A fuzzy point* x_λ *is called an adherence point of a fuzzy set A iff, every* $Q-$*neighborhood B of* x_λ, BqA.

Definition 1.14. *[15] An adherence point* x_λ *of A is called an accumulation point if for every* $Q-$ *neighborhood B of* x_λ, BqA *at some point different from x, whenever* $x_\lambda \in A$. $D = \cup\{x_\lambda | x_\lambda$ *is an accummulation point* $\}$ *is called the derived set of A and is denoted by* A^d. *If there are more than one fuzzy topology then* A^d_τ *is used to represent derived set of A in* τ.

Definition 1.15. *[14] Let* (X, τ_1) *and* (X, τ_2) *be two fts. Then the following collection of fuzzy sets* $\tau_1(\tau_2) = \{A \in I^X :$ *for every* $x_\alpha q A, \exists \tau_2 -Q-$ *neighborhood* A_α *of* x_α *such that* $\overline{(A_1)}_{\tau_1} \leq A\}$. *Then this collection of fuzzy sets is a fuzzy topology and it is called mixed fuzzy topology on X.*

Proposition 1.1. *[15] Let* A_α *be a collection of fuzzy sets in X. Then* $x_\lambda q(\vee A_\alpha)$ *iff* $\exists$ *some* $A_{j_0} \in \{A_\alpha\}$ *such that* $x_\lambda q A_{j_0}$.

Proposition 1.2. *[15] A subfamily* $\mathcal{B}$ *of a fuzzy topology* τ *for X is a base for* τ *iff for each* $x_\lambda \in X$ *and for each open* $Q-$ *neighborhood U of* $x_\lambda, \exists B \in \mathcal{B}$ *such that* $x_\lambda q B < U$.

Definition 1.16. *[12] An fts* (X, τ) *is called Fuzzy-Hausdorff iff for each pair of fuzzy singletons* x_α *and* y_β *with* $x \neq y$, $\exists$ *open fuzzy sets* O_1 *and* O_2 *such that* $x_\alpha \subseteq O_1 \subseteq co(y_\beta)$, $y_\beta \subseteq O_2 \subseteq co(x_\alpha)$ *and* $O_1 \subseteq co(O_2)$.

Definition 1.17. *[12] An fts (X, τ) is called fuzzy regular iff for any fuzzy point x_λ and a closed set B with $x_\lambda \in B^c$ $\exists$ open sets U and V in τ such that $x_\lambda \in U$ and $B \subseteq V$ and $U \subseteq (1 - V)$.*

Proposition 1.3. *[8] An fts (X, τ) is called fuzzy regular space iff for each fuzzy point x_λ with $\lambda \in (0, 1)$ and $A \in \tau$ with $\lambda < A(x)$ there exists $B \in \tau$ such that $\lambda < B(x)$ and $\overline{B} \leq A$.*

Lemma 1.1. *[14] Consider (X, τ_1) and (X, τ_2) be two fuzzy topological spaces such that every $\tau_1 -Q-$ neighborhood of x_λ is $\tau_2 -Q-$ neighborhood of x_λ for all fuzzy point x_λ then τ_1 is coarser than τ_2 " (We filled the gap in the proof given by N.R. Das and P.C. Baishya and included here.)*

Proof: *Let $A \in \tau_1$ and $x_\lambda q A \Rightarrow A$ is a $\tau_1 -Q-$ neighborhood of $x_\lambda \Rightarrow$ According to hypothesis A is a $\tau_2 -Q-$ neighborhood of $x_\lambda \Rightarrow \exists A_2^\lambda \in \tau_2$ such that $x_\lambda q A_2^\lambda$ and $A_2^\lambda \leq A$. We prove that $A(x) = sup A_2^\lambda(x)$, supremum being all over such A_2^λ. Also, $A(x)$ is an upper bound of A_2^λ. But $A(x) + (1 - A(x) + \epsilon) > 1. \Rightarrow A(x) + \lambda \ > 1$, where $\lambda = 1 - A(x) + \epsilon$, $\epsilon \in (0, 1]$ $\Rightarrow x_\lambda q A$ therefore, $\exists A_2^\lambda \in \tau_2$ such that $x_\lambda q A_2^\lambda$ i.e. $(1-A(x)+\epsilon)+A_2^\lambda(x) > 1$ $\Rightarrow A_2^\lambda(x) > A(x) - \epsilon$, since ϵ being arbitrary therefore, $A(x) = sup A_2^\lambda(x)$ and each A_2^λ is $\tau_2 - open \Rightarrow A \in \tau_2$. Therefore τ_1 is coarser than τ_2."*

Definition 1.18. *[2] Let (X, τ_1) and (X, τ_2) be two fts. Then the following collection of fuzzy sets $\tau_1(\tau_2) = \{A \in I^X$: for any $B \in I^X$ $BqA, \exists \tau_2-$ open set A_1 such that $A_1 q B$ and $\overline{(A_1)}_{\tau_1} \leq A\}$ will form a fuzzy topology on X. It will be called generalized mixed fuzzy topology.*

Theorem 1.1. *[14] Let τ_1 and τ_2 be two fts on X. Then mixed fuzzy topology $\tau_1(\tau_2)$ is coarser than τ_2. That is $\tau_1(\tau_2) \subseteq \tau_2$.*

Theorem 1.2. *[14] If τ_1 is fuzzy regular and $\tau_1 \subseteq \tau_2$ then $\tau_1 \subseteq \tau_1(\tau_2)$.*

2. Main Results

In this section, mixed fuzzy topology has been investigated and a base for mixed fuzzy topology is constructed. The definition 1.15. due to N. R. Das and P. C. Baishya[14] is used throughout in this article.

Lemma 2.1. *Let $(X, \tau_1(\tau_2))$ be a mixed fuzzy topology then $A^0_{\tau_1(\tau_2)} \leq A^0_{\tau_2}$.*

Proof: *By definition 1.3. $A^0_{\tau_1(\tau_2)} = \vee\{B|B \leq A$ and $B \in \tau_1(\tau_2)\}$.*
$\leq \vee\{B|B \leq A$ and $B \in \tau_1(\tau_2) \subseteq \tau_2\}$. [since $\tau_1(\tau_2) \subseteq \tau_2$ (Theorem 1.1.)]

$= A^0_{\tau_2}$.
Therefore, $A^0_{\tau_1(\tau_2)} \leq A^0_{\tau_2}$.

Lemma 2.2. *Let* $(X, \tau_1(\tau_2))$ *be a mixed fuzzy topology then* $\overline{A}_{\tau_2} \leq \overline{A}_{\tau_1(\tau_2)}$.

Proof: *By definition 1.4.*
$\overline{A}_{\tau_1(\tau_2)} = \wedge\{B|A \leq B$ *and* B *is a closed set in* $\tau_1(\tau_2)\}$
$\geq \wedge\{B|A \leq B$ *and* B *is a closed set in* $\tau_2 \supseteq \tau_1(\tau_2)\}$
$= \overline{A}_{\tau_2}$ *(By Theorem 1.1).*
Therefore, $\overline{A}_{\tau_2} \leq \overline{A}_{\tau_1(\tau_2)}$.

Lemma 2.3. *Let* A *be a fuzzy set then in a mixed fuzzy topology* $(X, \tau_1(\tau_2))$ *then* $A^d_{\tau_2} \leq A^d_{\tau_1(\tau_2)}$.

Proof: *Let* $x_\lambda \in A^d_{\tau_2}$.
Let B *be a* $Q-$*neighborhood of* x_λ *in* $\tau_1(\tau_2)$.
$\Rightarrow B$ *is also a* $Q-$*neighborhood of* x_λ *in* τ_2. *[*$\tau_1 \subseteq \tau_1(\tau_2)$*, (By Theorem 1.1.)]*
$\Rightarrow$ *If* $x_\lambda \in A$ *then* $\exists y \neq x$*, such that* $B(y) + A(y) > 1$ *or, if* $x_\lambda \notin A$ $B(y) + A(y) > 1$ *for some* y *necessarily* $y \neq x$. *[since* $x_\lambda \in A^d_{\tau_2}$.*]*
$\Rightarrow x_\lambda \in A^d_{\tau_2}$.
Therefore, $A^d_{\tau_2} \leq A^d_{\tau_1(\tau_2)}$.

Lemma 2.4. *If* τ_1 *is regular and* $\tau_1 \subseteq \tau_2$ *then* $A^d_{\tau_2} \leq A^d_{\tau_1(\tau_2)} \leq A^d_{\tau_1}$.

Proof: *From the last result it follows that* $A^d_{\tau_2} \leq A^d_{\tau_1(\tau_2)}$. *(i).*
Also τ_1 *is regular and* $\tau_1 \subseteq \tau_2$*, therefore* $\tau_1 \subseteq \tau_1(\tau_2)$. *(Theorem 2).*
Now let $x_\lambda \in A^d_{\tau_2}$.
Let B *be a* $Q-$*neighborhood of* x_λ *in* τ_1.
$\Rightarrow B$ *is also a* $Q-$ *neighborhood of* x_λ *in* $\tau_1(\tau_2)$. *[*$\tau_1 \subseteq \tau_1(\tau_2)$*] (Theorem 1.2.).*
$\Rightarrow$ *If* $x_\lambda \in A$ *then* $\exists y \neq x$*, such that* $B(y) + A(y) > 1$ *or, if* $x_\lambda \notin A$ $B(y) + A(y) > 1$ *for some* y *necessarily* $y \neq x$. *[since* $x_\lambda \in A^d_{\tau_2}$.*]*
$\Rightarrow x_\lambda \in A^d_{\tau_1}$.
Therefore, $A^d_{\tau_1(\tau_2)} \leq A^d_{\tau_1}$*..........(ii).*
Now Eq. (i) and Eq. (ii) gives $A^d_{\tau_2} \leq A^d_{\tau_1(\tau_2)} \leq A^d_{\tau_1}$.

Theorem 2.1. *If a fuzzy set* A *is open in* τ_2 *and closed in* τ_1 *then* A *is open in* $\tau_1(\tau_2)$.

Proof: *Let the fuzzy set A is open in τ_2 and closed in τ_1. Then for any fuzzy point x_λ with $x_\lambda qA$ and $A \in \tau_2$ therefore, A is an τ_2 $-Q-$ neighborhood of x_λ and $\overline{A}_{\tau_1} = A \subseteq A$. Therefore, by definition 1.15. of $\tau_1(\tau_2)$, $A \in \tau_1(\tau_2)$.*

2.1. Example

Here is an example of mixed fuzzy topology. Theorem 2.1. is used to obtain it. Let $X = [0, 1]$ We define the following fuzzy sets

$$A(x) = \begin{matrix} 3/4 & \text{if } 0 \le x \le 3/4 \\ 0 & \text{if } 3/4 < x \le 1 \end{matrix} \tag{2.1}$$

$$B(x) = \begin{matrix} 0 & \text{if } 0 \le x \le 1/4 \\ 3/4 & \text{if } 1/4 < x \le 1 \end{matrix} \tag{2.2}$$

$$C(x) = \begin{matrix} 3/4 & \text{if } 0 \le x \le 1/4 \\ 0 & \text{if } 1/4 < x \le 1 \end{matrix} \tag{2.3}$$

$$D(x) = \begin{matrix} 1/4 & \text{if } 0 \le x \le 1/4 \\ 1 & \text{if } 1/4 < x \le 1 \end{matrix} \tag{2.4}$$

$$E(x) = \begin{matrix} 1 & \text{if } 0 \le x \le 1/4 \\ 1/4 & \text{if } 1/4 < x \le 1 \end{matrix} \tag{2.5}$$

$\tau_1 = \{\overline{0}, \overline{1}, A, B, A \vee B\}$, $\tau_2 = \{\overline{0}, \overline{1}, C, D, E, C \vee D, C \wedge D, D \wedge E, C \vee E, C \vee (D \wedge E)\}$.

Now the mixed fuzzy topology $\tau_1(\tau_2) \subseteq \tau_2$ by Theorem 1.1. Therefore, the candidates for $\tau_1(\tau_2)$ will be from τ_2. Also, from Theorem 2.1. an open set in τ_2 which is closed in τ_1 is an open set in $\tau_1(\tau_2)$. Therefore, $\tau_1(\tau_2) = \{\overline{0}, \overline{1}, D, E, D \wedge E\}$ which also satisfies the axioms of fuzzy topology.

Theorem 2.2. *If τ_1 and τ_2 are fuzzy topologies on X then $\tau_1(\tau_2) \neq \tau_2(\tau_1)$.*

Proof:*For fuzzy topologies τ_1 and τ_2, $\tau_1(\tau_2) \subseteq (\tau_2)$ and $\tau_1(\tau_2) \subseteq (\tau_1)$(By Theorem 1.1) Therefore, in general $\tau_1(\tau_2) \neq \tau_2(\tau_1)$. Here is a counter example for this. Let X=$\{x, y\}$ and A=$\{(x, 0.3), (y, 0.7)\}$ and B=$\{(x, 0.7), (y, 0.3)\}$ be two fuzzy sets in X. Then $\tau_1 = \{\overline{0}, \overline{1}, A\}$ and $\tau_2 = \{\overline{0}, \overline{1}, B\}$ are two fuzzy topologies on X. Obviously $\overline{0} \in \tau_1(\tau_2)$ and $\overline{1} \in \tau_1(\tau_2)$. To check belongingness of B. For, $x_\lambda qB$*
$\Rightarrow \lambda + B(x) > 1$
$\Rightarrow \lambda + 0.7) > 1$

$\Rightarrow \lambda > 0.3$ *Therefore,for* $\lambda > 0.3$ $x_{\lambda>0.3}qB$ *and* B *is also a* $\tau_2 - Q$ *neighborhood of* $x_{\lambda>0.3}$ and $\overline{B}_{\tau_1} = B \leq B$.
Therefore, $\tau_1(\tau_2) = \{\overline{0}, \overline{1}, B\}$ *and* $\tau_2(\tau_1) = \{\overline{0}, \overline{1}, A\}$.
So, $\tau_1(\tau_2) \neq \tau_2(\tau_1)$.

Definition 2.1. *When* $\tau_1 = \tau_2 = \tau$ *then it is written as* $\tau_1(\tau_2) = \tau^2$.

Theorem 2.3. *Let* τ *be a fuzzy topology such that every open set of* τ *is also a closed set then* $\tau^2 = \tau$

Proof: *Let* $A \in \tau$.
Let $x_\alpha qA$
$\Rightarrow A$ *is a* $\tau - Q$ *neghborhood of* x_α *and* $\overline{A_\tau} = A \leq A$.
$\Rightarrow A \in \tau^2$. *[By definition 1.15.]*
$\Rightarrow \tau \subseteq \tau^2$. *(iii)*
Also, Theorem 1.1. gives $\tau^2 \subseteq \tau$. *(iv)*
Therefore, $\tau^2 = \tau$.

Theorem 2.4. *If mixed fuzzy topological space* $\tau_1(\tau_2)$ *in* X *is Hausdorff then* τ_2 *is also fuzzy Hausdorff space. But if* τ_1 *and* τ_2 *are fuzzy Hausdorff space then* $\tau_1(\tau_2)$ *need not be Hausdorff space.*

Proof: *Let* $x \neq y$ *and* x_α *and* y_β *be two fuzzy points in* A. *As* $\tau_1(\tau_2)$ *is Hausdorff therefore,* $\exists$ *open sets* O_1 *and* O_2 *in* $\tau_1(\tau_2)$ *such that* $x_\alpha \subseteq O_1 \subseteq co(y_\beta)$, $y_(\beta) \subseteq O_2 \subseteq co(x_\alpha)$ *and* $O_1 \subseteq co(O_2)$. *Also* $\tau_1(\tau_2) \subseteq \tau_2$ *(By Theorem 1.1.) therefore the same pair of fuzzy sets* O_1 *and* O_2 *fulfils the requirement of Hausdorff condition for* τ_2. *Therefore* τ_2 *is Hausdorff space.*

Now to show that the Hausdorffness of τ_1 *and* τ_2 *does not imply the Hausdorffness of* $\tau_1(\tau_2)$. *A counter example to show that:*

Let X={x,y} then the collections
$\tau_1 = \{\overline{0}, \overline{1}, \{(x,\alpha),(y,0)\}, \{(x,\alpha),(y,\alpha)\}, \{(x,0),(y,\alpha)\} | 0.6 \leq \alpha < 1\}$ *and* $\tau_2 = \{\overline{0}, \overline{1}, \{(x,1),(y,0)\}, \{(x,0),(y,1)\}\}$ *are fuzzy Hausdorff spaces. Since,* $\tau_1(\tau_2) \subseteq \tau_2$. *Therefore, check the belongingness of {(x,1),(y,0)} and {(x,0),(y,1)} in* $\tau_1(\tau_2)$. *For* $\{(x,1),(y,0)\} = A$ *(say),* $x_{0.1}qA$ *but* $(\overline{A})_{\tau_1} = \overline{1}A \Rightarrow A \notin \tau_1(\tau_2)$. *Similarly* $\{(x,1),(y,0)\} \notin \tau_1(\tau_2)$. *So,*$\tau_1(\tau_2) = \{\overline{0}, \overline{1}\}$ *which is clearly not fuzzy Hausdorff spaces.*

Note: Definition 1.6. is used for a fuzzy point otherwise $\{(x,1),(y,0)\}$, and $\{(x,0),(y,1)\}$ types of fuzzy sets must be open set in a fuzzy topology to be Hausdorff space and inclusion of such set will be sufficient for a fuzzy topology to be Hausdorff fuzzy topological space.

Theorem 2.5. *If $\tau_2 \subseteq \tau_1$ and τ_2 is fuzzy regular and Hausdorff then $\tau_1(\tau_2)$ is fuzzy Hausdorff space.*

Proof: *Let $x \neq y$ and x_α and y_β be two fuzzy points in A. As τ_2 is Hausdorff therefore, $\exists$ open sets O_1 and O_2 in τ_2 such that uch that $x_\alpha \subseteq O_1 \subseteq co(y_\beta)$, $y_(\beta) \subseteq O_2 \subseteq co(x_\alpha)$ and $O_1 \subseteq co(O_2)$. Now to show that the same pair A and B satisfies the Hausdorff condition in $\tau_1(\tau_2)$.*

For $z_\alpha q A$
$\Rightarrow A$ *is a* $\tau_2 - Q-$*neighborhood of* z_α
$\Rightarrow A(z) + \alpha > 1,$
$\Rightarrow A(z) > 1 - \alpha$
$\Rightarrow A(z) > \beta$, *where* $\beta = 1 - \alpha$.
Since τ_2 is fuzzy regular $\exists$ $B \in \tau_2$ such that
$\beta < B(z)$
$\Rightarrow 1 - \alpha < B(z)$
$\Rightarrow B(z) + \alpha > 1.$
Therefore, $z_\alpha q B$ and $(\overline{B})_{\tau_2} \leq A$. (Proposition 1.3.).
Therefore, for $x_\alpha q A$, $\exists \tau_2 - Q-$neighborhood B of x_α such that
$(\overline{B})_{\tau_1} \leq (\overline{B})_{\tau_2} \leq A$. *[since $\tau_2 \subseteq \tau_1 \Rightarrow (\overline{B})_{\tau_1} \leq (\overline{B})_{\tau_2}$]*
Therefore, $A \in \tau_1(\tau_2)$.
Similarly, $B \in \tau_1(\tau_2)$. Hence $\tau_1(\tau_2)$ is fuzzy Hausdorff Space.

Theorem 2.6. *Let $\tau_2 \subseteq \tau_1$ and τ_2 is fuzzy regular then $\tau_1(\tau_2) = \tau_2$.*

Proof: *By Theorem 1, $\tau_1(\tau_2) \subseteq \tau_2$. (v)*
Also $\tau_1 \subseteq \tau_1(\tau_2)$. (vi) (From the previous result)
Therefore Eq. (v) and Eq. (vi) gives $\tau_1 \subseteq \tau_1(\tau_2) \subseteq \tau_2$
But $\tau_1 = \tau_2$. So, $\tau_1(\tau_2) = \tau_1 = \tau_2$.

Theorem 2.7. *If τ_1, τ_2, τ_3 and τ_4 are fuzzy topologies on X such that $\tau_1 \subseteq \tau_3$ and $\tau_2 \subseteq \tau_4$ then,*
$\tau_1(\tau_2) \subseteq \tau_3(\tau_4)$
$\tau_1(\tau_2) \subseteq \tau_3(\tau_2)$
$\tau_1(\tau_2) \subseteq \tau_1(\tau_4)$

Proof: *(a) Let $A \in \tau_1(\tau_2)$. Therefore, for any $x_\alpha q A$, $\exists \tau_2 - Q-$ neighborhood A_α of x_α such that $(\overline{A}_\alpha)_{\tau_1} \leq A$. Since $\tau_2 \subseteq \tau_4$, therefore A_α is* *also a $\tau_4 - Q-$ neighborhood of x_α and $\tau_1 \subseteq \tau_3$ gives $(\overline{A}_\alpha)_{\tau_3} \leq (\overline{A}_\alpha)_{\tau_1}$. Therefore for any fuzzy open set $A \in \tau_1(\tau_2)$ and for any $x_\alpha qA$ $\exists \tau_4 - Q-$ neighborhood A_α such that $(\overline{A}_\alpha)_{\tau_3} \leq (\overline{A}_\alpha)_{\tau_1} \leq A$. Therefore, $A \in \tau_3(\tau_4)$ that is $\tau_1(\tau_2) \subseteq \tau_3(\tau_4)$.*

(b) Let $A \in \tau_1(\tau_2)$. Therefore, for any x_α with $x_\alpha qA$, $\exists \tau_2 - Q-$ neighborhood A_α of x_α such that $(\overline{A}_\alpha)_{\tau_1} \leq A$. Since $\tau_1 \subseteq \tau_3$, therefore $(\overline{A}_\alpha)_{\tau_3} \leq (\overline{A}_\alpha)_{\tau_1}$. Therefore for any fuzzy open set $A \in \tau_1(\tau_2)$ with $x_\alpha qA$ $\Rightarrow \exists \tau_2 - Q-$ neighborhood A_α of x_α such that $(\overline{A}_\alpha)_{\tau_3} \leq A$. Therefore, $A \in \tau_3(\tau_2)$ that is $\tau_1(\tau_2) \subseteq \tau_3(\tau_2)$.

(c) Let $A \in \tau_1(\tau_2)$. Therefore, for any x_α with $x_\alpha qA$, $\exists \tau_2 - Q-$ neighborhood A_α of x_α such that $(\overline{A}_\alpha)_{\tau_1} \leq A$. Since $\tau_2 \subseteq \tau_4$, therefore every $\tau_2 - Q-$ neighborhood of x_α is also a $\tau_4 - Q-$ neighborhood of x_α. Therefore for any fuzzy open set $A \in \tau_1(\tau_2)$ with $x_\alpha qA \Rightarrow \exists \tau_4 - Q-$ neighborhood A_α of x_α such that $(\overline{A}_\alpha)_{\tau_1} \leq A$. Therefore, $A \in \tau_1(\tau_4)$ that is $\tau_1(\tau_2) \subseteq \tau_1(\tau_4)$.

Theorem 2.8. *Consider $\tau_1(\tau_2)$ mixed fuzzy topology on X. Let $\mathcal{B}$ be a base for τ_2. Then $\mathcal{B}_M$=$\{B \in \mathcal{B}|$for any $x_\alpha qB$ $\exists B_\alpha \in \mathcal{B}$ such that $x_\alpha qB_\alpha$ and $(\overline{B}_\alpha)_{\tau_1} \leq B\}$ is a base for the mixed fuzzy topology $\tau_1(\tau_2)$.*

Proof: *Let $B \in \mathcal{B}_M$ then B is an open set in τ_2 so $x_\alpha qB$ means B is a $\tau_2 - Q-$ neighborhood of x_α. Therefore $B \in \mathcal{B}_M$ satisfies the definition for belongingness in $\tau_1(\tau_2)$. So every member of $\mathcal{B}$ is an open set of $\tau_1(\tau_2)$.*

Now let U be any open set in $\tau_1(\tau_2)$. Let $x_\alpha qU$ and assume $B \in \mathcal{B}_M$ such that $x_\alpha qB \leq U$. But $\tau_1(\tau_2) \subseteq \tau_2$ so $U \in \tau_2$ and B be a base for τ_2 therefore $U = \vee B_j$ for $B_j \in \mathcal{Y} \subseteq \mathcal{B}$.

Now for $x_\alpha qU$
$\Rightarrow \alpha + U(x) > 1$
$\Rightarrow \alpha + (\vee B_j)(x) > 1$
$\Rightarrow \alpha + B_{j_0}(x) > 1$, *for some $B_{j_0} \in \mathcal{Y}$. (Proposition 1.1.)*

Then $B_{j_0} \notin \mathcal{B}_M$. Otherwise $B_{j_0} \in \mathcal{B}_M$ such that $x_\alpha qB_{j_0} \leq U$. Which implies $B \in \mathcal{B}$ is a base for $\tau_1(\tau_2)$ and the proof is over.
Therefore, consider $B_{j_0} \notin \mathcal{B}_M$.
$\Rightarrow B_\alpha \in \mathcal{B}_M$ *such that $x_\alpha qB_\alpha$ and $(\overline{B}_\alpha)_{\tau_1} \leq B_{j_0}$*
$\Rightarrow x_\alpha$ *is not quasicoincident with $\vee B_\alpha$ where $B_\alpha \in \mathcal{B}$ such that $(\overline{(\vee B)_\alpha})_{\tau_1} \leq B_{j_0}$. Which is true for any sub collection from $\mathcal{B}$. Since for every $A_\alpha \in \tau_2, \exists B_j \in \mathcal{B}$ such that $\vee B_j = A_\alpha$.*
$\Rightarrow A_\alpha \in \tau_2$ *such that $x_\alpha qA_\alpha$ and $(\overline{A}_\alpha)_{\tau_1} \leq U$.*
$\Rightarrow \tau_2 - Q-$ *neighborhood $-C_\alpha$ such that $x_\alpha qC_\alpha$ and $(\overline{C}_\alpha)_{\tau_1} \leq U$.*

Therefore, $B_{j_0} \in \mathcal{B}_M$ *such that* $x_\alpha q B_{j_0} \leq U$.
So, $\mathcal{B}_M$ *is a base for* $\tau_1(\tau_2)$. *(Proposition 1.2.)*

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Hari Prasad Chetri
Department of Mathematics,
Central Institute of Technology,
Kokrajhar, Assam,
India-783370
India
e-mail: hariprasadchetri1234@gmail.com
Corresponding author

and

Gautam Chandra Ray
Department of Mathematics,
Central Institute of Technology,
Kokrajhar, Assam,
India-783370
India
e-mail: gautomofcit@gmail.com