Proyecciones Journal of Mathematics
Vol. 42, N^o 3, pp. 757-773, June 2023.
Universidad Católica del Norte
Antofagasta - Chile

10.22199/issn.0717-6279-5421

On uniform-ultimate boundedness and periodicity results of solutions to certain second order non-linear vector differential equations

A. A. Adeyanju
Federal University of Agriculture Abeokuta, Nigeria
Received : April 2022. Accepted : August 2022

Abstract

In this paper, we employ the second method of Lyapunov to examine sufficient conditions for the uniform-ultimate boundedness of solutions and existence of at least one periodic solution to the following second order vector differential equation:

$$\ddot{X} + F(X, \dot{X})\dot{X} + H(X) = P(t, X, \dot{X}),$$

when the non-linear term $H(X)$ is: (i) differentiable, (ii) non-necessarily differentiable. The results contain in this paper are new and complement related ones in the literature.

Subjclass [2010]: *34D05; 34K20.*

Keywords: *Second order; ultimate boundedness; periodicity; second order, Lyapunov function.*

1. Introduction

The goal of this paper is to establish some results on the uniform-ultimate boundedness of solutions and existence of at least one periodic solution to the following second order nonlinear differential equation:

$$\ddot{X} + F(X, \dot{X})\dot{X} + H(X) = P(t, X, \dot{X}). \tag{1.1}$$

Eq. (1.1) can be written as a system of first order differential equations:

$$\dot{X} = Y, \dot{Y} = -F(X, Y)Y - H(X) + P(t, X, Y), \tag{1.2}$$

where $X, Y : \mathbf{R}^+ \to \mathbf{R}^n, \mathbf{R}^+ = [0, \infty)$, $\mathbf{R} = (-\infty, \infty)$; $H : \mathbf{R}^n \to \mathbf{R}^n$; $P : \mathbf{R}^+ \times \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$; F is an $n \times n$ continuous symmetric positive definite matrix function depending on the arguments displayed explicitly and the dots indicate differentiation with respect to variable t. To ensure that solution of Eq. (1.1) exists, we assume the continuity of the functions F, H and P. Furthermore, we assume that functions F, H and P satisfy Lipschitz condition with respect to their respective arguments.

In the past five decades or more, the study of qualitative behaviour of solutions to second order and higher order scalar or vector linear and nonlinear differential equations have been studied by many authors. In literature, some methods such as integral test, frequency domain and direct method of Lyapunov have been employed to study qualitative behaviour of solutions of some differential equations. However, the direct method (also called second method) of Lyapunov has been found and established to be an effective method and mostly used among others. (See, [1] - [40]).

Our findings in literature shows that, Loud [22] gave some conditions for the convergence of solutions of the second order scalar differential equation:

$$\ddot{x} + c\dot{x} + g(x) = p(t), \tag{1.3}$$

where c is a positive constant. Later, Ezeilo [16] considered an n-dimensional form of Eq. (1.3) i.e.

$$\ddot{X} + C\dot{X} + G(X) = P(t, X, \dot{X}), \tag{1.4}$$

where C is a real $n \times n$ constant matrix. The author established some results on the ultimate boundedness and convergence of solutions of (1.4).

Also, Tejumola [31] considered a certain second order matrix differential equation of the form:

$$\ddot{X} + A\dot{X} + H(X) = P(t, X, \dot{X}),$$

with A being a constant $n \times n$ symmetric matrix; X, H and P are $n \times n$ continuous matrices. He established some criteria for the stability of the trivial solution when $H(0) = 0$ and $P \equiv 0$, ultimate boundedness of all solutions and the existence of periodic solution when $P \neq 0$. Later, Afuwape and Omeike[6] considered a more general second order differential equation of the form:

$$\ddot{X} + F(\dot{X}) + G(X) = P(t, X, \dot{X}),$$

and established a convergence result for this equation by imposing certain conditions on vectors $F(\dot{X}), G(X)$ and $P(t, X, Y)$.

Furthermore, Omeike et.al.[25] used an incomplete Lyapunov function supplemented with a signum function to establish the boundedness of solutions of Eq. (1.1). In a recent paper, Adeyanju [5] proved some results on the stability and boundedness of solutions to (1.1) using a complete Lyapunov function.
The works of Omeike et.al.[25], Adeyanju [5] and above listed papers gave us motivation for the present work.

2. Preliminary Results and Definition

In this section, we provide some basic results that are useful in proving our main results.

Lemma 2.1. *([17], [30], [33]) Let A be a real $n \times n$ symmetric matrix, then for any $X \in \mathbf{R}^n$ we have,*

$$\Delta_a \|X\|^2 \geq \langle AX, X \rangle \geq \delta_a \|X\|^2,$$

where δ_a and Δ_a are respectively, the least and greatest eigenvalues of the matrix A.

Lemma 2.2. *([32]). Let $H(0) = 0$ and assume that the matrices A and $J_h(X)$ are symmetric and commute for all $X \in \mathbf{R}^n$. Then,*

(i)

$$\langle H(X), AX \rangle = \int_0^1 X^T A J_h(\sigma X) X d\sigma;$$

(ii)

$$\frac{d}{dt} \int_0^1 \langle H(\sigma X), X \rangle d\sigma = \langle H(X), Y \rangle.$$

Lemma 2.3. *([38], [39])*
Suppose that there exists a Lyapunov function $V(t, X)$ defined on $0 \leq t \leq R$, $\|X\| \geq R$, (where R may be large) which satisfies:

(i) $a(\|X\|) \leq V(t, X) \leq b(\|X\|)$, *where $a(r)$ and $b(r)$ are continuous, monotone increasing functions and;*

(ii) $\dot{V}(t, X) \leq 0$,

then the solutions of equation (1.1) are uniformly bounded.

Lemma 2.4. *([38], [39])*
Under the assumptions of Lemmma 2.3, if $\dot{V}(t, X) \leq -c(\|X\|)$, where $c(r)$ is positive and continuous, then the solutions of systems (1.1) are uniformly ultimately bounded.

Lemma 2.5. *([38], [39])*
If there exists a Lyapunov function satisfying the condition of Lemma 2.4, then the system (1.1) has at least one periodic solution of period ω.

Definition 2.6. *[37] The solutions of equation (1.1) are uniformly ultimately bounded for bound M, if there exists an $M > 0$ and if for any $\alpha > 0$ and $t_0 \in I$ there exists a $T(\alpha) > 0$ such that $X_0 \in S_\alpha$, where $S_\alpha = \{x \in \mathbf{R}^n :\| x \|< \alpha\}$, implies that*

$$\|X(t; t_0, X_0)\| < M$$

for all $t \geq t_0 + T(\alpha)$.

3. Ultimate Boundedness Result

Given that $H(0) = 0$, $H(X) \neq 0$ whenever $X \neq 0$ and $J_h = J_h(X)$ denotes the Jacobian matrix $(\frac{\partial h_i}{\partial x_i})$ of $H(X)$ in Eq. (1.1), then we have the following theorem.

Theorem 3.1. *Suppose that all the basic assumptions imposed on $F(X,Y)$ and $H(X)$ hold and in addition, given that for any arbitrary $X,\ Y \in \mathbf{R}^n$:*

(i) matrix $J_h(X)$ is symmetric and positive definite such that its eigenvalues $\lambda_i(J_h(X))$ $(i = 1, 2, 3, .., n)$ satisfy:

$$0 < \delta_h \leq \lambda_i(J_h(X)) \leq \Delta_h; \tag{3.1}$$

(ii) the eigenvalues $\lambda_i(F(X,Y))$ $(i = 1, 2, 3, .., n)$ of $F(X,Y)$ satisfy:
$\alpha - \epsilon \leq \lambda_i(F(X,Y)) \leq \alpha$,
where α, δ and ϵ are positive constants such that,

$$\delta \geq \frac{\alpha + \epsilon}{\alpha - \epsilon} > 1; \tag{3.2}$$

(iii) there exist some positive finite constants m_1 and m_2 such that vector $P(t,X,Y)$ satisfies:

$$\|P(t,X,Y)\| \leq m_1 + m_2(\| X \| + \| Y \|). \tag{3.3}$$

Then all the solutions of Eq. (1.1) or system 1.2 are uniformly bounded and uniform-ultimately bounded.

Proof. The proof of this theorem rests on the Lyapunov function $V(t) = V(t,X,Y)$ defined as

$$2V(t) = \| \alpha X + Y \|^2 + \delta \| Y \|^2 + 2(\delta + 1) \int_0^1 \langle H(\sigma_1 X), X \rangle d\sigma_1. \tag{3.4}$$

It is clear that when $X = 0$ and $Y = 0$, the function defined by 3.4 vanishes. By applying Lemma 2.1 and Lemma 2.2 to 3.4, we obtain

$$\begin{aligned} 2V(t) &\geq 2\delta_h(\delta + 1) \| X \|^2 + \delta \| Y \|^2 \\ &\geq \delta_1(\| X \|^2 + \| Y \|^2), \end{aligned}$$

where $\delta_1 = \min\{2\delta_h(\delta + 1), \delta\}$.
Similarly, using Lemma 2.1, Lemma 2.2 and the fact that $2 \| X \| \| Y \| \leq \| X \|^2 + \| Y \|^2$ in 3.4, we have

$$\begin{aligned} 2V(t) &\leq 2\Big((\Delta_h(\delta+1)+\alpha^2\Big) \parallel X \parallel^2 + (\delta+2) \parallel Y \parallel^2 \\ &\leq \delta_2(\parallel X \parallel^2 + \parallel Y \parallel^2), \end{aligned}$$

where $\delta_2 = \max\{2\Big((\Delta_h(\delta+1)+\alpha^2\Big), (\delta+2)\}$.

Hence,

$$\delta_1(\parallel X \parallel^2 + \parallel Y \parallel^2) \leq 2V(t) \leq \delta_2(\parallel X \parallel^2 + \parallel Y \parallel^2). \tag{3.5}$$

The time derivative of the function $V(t)$ along the solution path of the equation being studied is given by:

$$\begin{aligned} \frac{d}{dt}V(t) = \dot{V}(t) = &- \alpha\langle X, H(X)\rangle - (1+\delta)\langle Y, FY\rangle + \alpha\langle Y, Y\rangle \\ &+ \alpha\langle X, (F(X,Y) - \alpha I)Y\rangle + \langle \alpha X + (\delta+1)Y, P(t,X,Y)\rangle, \end{aligned}$$

where I is an $n \times n$ identity matrix. This derivative can be written as

$$\dot{V}(t) = -U_1 - U_2 + U_3, \tag{3.6}$$

where

$$U_1 = \frac{\alpha}{2}\langle X, H\rangle - \alpha\langle Y, Y\rangle + \frac{(1+\delta)}{2}\langle Y, F(X,Y)Y\rangle,$$

$$U_2 = \frac{\alpha}{2}\langle X, H\rangle + \alpha\langle X, (F(X,Y) - \alpha I)Y\rangle + \frac{(1+\delta)}{2}\langle Y, F(X,Y)Y\rangle,$$

and

$$U_3 = \langle \alpha X + (\delta+1)Y, P(t,X,Y)\rangle.$$

The second term in U_2 can be expressed as follows

$$\begin{aligned} \langle X, (F(X,Y) - \alpha I)Y\rangle &= \frac{1}{2} \parallel k_1(F - \alpha I)Y + k_1^{-1}X \parallel^2 - \frac{1}{2k_1^2} \parallel X \parallel^2 \\ &- \frac{k_1^2}{2} \parallel (F - \alpha I)Y \parallel^2, \end{aligned}$$

where $k_1 > 0$ is a constant whose value will be given later. By Lemma 2.1 and assumption (ii) of Theorem 3.1, we have

$$\parallel (F - \alpha I)Y \parallel^2 \leq \epsilon^2 \parallel Y \parallel^2 .$$

Therefore,

$$\langle X, (F(X,Y) - \alpha I)Y \rangle \ \geq \ -\frac{1}{2k_1^2} \parallel X \parallel^2 - \frac{\epsilon^2 k_1^2}{2} \parallel Y \parallel^2 .$$

Also using Lemma 2.1, we have

$$\langle X, H(X) \rangle \geq \delta_h \parallel X \parallel^2 .$$

Thus,

$$\begin{aligned} U_1 &\geq \frac{1}{2}\alpha\delta_h \parallel X \parallel^2 + \frac{1}{2}\Big((1+\delta)(\alpha - \epsilon) - 2\alpha\Big) \parallel Y \parallel^2 \\ &\geq \delta_3\{\parallel X \parallel^2 + \parallel Y \parallel^2\}, \end{aligned}$$

where $\delta_3 = \frac{1}{2}\min\{\alpha\delta_h; (1+\delta)(\alpha-\epsilon) - 2\alpha\}$;

$$U_2 \geq \frac{\alpha}{2}(\delta_h - k_1^{-2}) \parallel X \parallel^2 + \frac{1}{2}\Big((\delta+1)(\alpha-\epsilon) - \alpha\epsilon^2 k_1^2\Big) \parallel Y \parallel^2,$$

letting $k_1^2 = \frac{1}{2}\min\left(\frac{1}{\delta_h}, \frac{(\alpha-\epsilon)(1+\delta)}{\alpha\epsilon^2}\right)$, then

$$U_2 \geq 0;$$

and lastly by 3.3, we have

$$\begin{aligned} |U_3| &\leq \{\alpha \parallel X \parallel + (1+\delta) \parallel Y \parallel\} \parallel P(t,X,Y) \parallel \\ &\leq \delta_4\{\parallel X \parallel + \parallel Y \parallel\}\{m_1 + m_2(\parallel X \parallel + \parallel Y \parallel)\} \\ &\leq 2\delta_4 m_2\{\parallel X \parallel^2 + \parallel Y \parallel^2\} + 2^{\frac{1}{2}}\delta_4 m_1\{\parallel X \parallel^2 + \parallel Y \parallel^2\}^{\frac{1}{2}}, \end{aligned}$$

where $\delta_4 = \max\{\alpha; (1+\delta)\}$.

Thus,

$$\begin{aligned} \dot{V}(t) &\leq -\delta_3\{\parallel X \parallel^2 + \parallel Y \parallel^2\} + 2\delta_4 m_2\{\parallel X \parallel^2 + \parallel Y \parallel^2\} + 2^{\frac{1}{2}}\delta_4 m_1\{\parallel X \parallel^2 \\ &+ \parallel Y \parallel^2\}^{\frac{1}{2}} \\ &= -2\delta_5\{\parallel X \parallel^2 + \parallel Y \parallel^2\} + \delta_6\{\parallel X \parallel^2 + \parallel Y \parallel^2\}^{\frac{1}{2}}, \end{aligned} \tag{3.7}$$

where $\delta_5 = \frac{1}{2}(\delta_3 - 2\delta_4 m_2)$, $m_2 < 2^{-1}\delta_3\delta_4^{-1}$, $\delta_6 = 2^{\frac{1}{2}}\delta_4 m_1$.

To conclude the proof of the theorem, we follow the same pattern or argument established in the proof of Theorem 1 of ([7], [23]) or Yoshizawa approach in [37].

So, suppose in the inequality 3.7 we choose $\{\| X \|^2 + \| Y \|^2\}^{\frac{1}{2}} \geq \delta_7 = \delta_6\delta_5^{-1}$, then

$$\dot{V}(t) \leq -\delta_5\{\| X \|^2 + \| Y \|^2\}. \tag{3.8}$$

Now, we prove that there exists a positive constant K such that
$\| X \|^2 + \| Y \|^2 \leq K$, for $t \geq T(T > 0)$,
for any solution $(X(t), Y(t))$ of system 1.2.
However, by 3.8 and 3.5 we have for any solution $(X(t), Y(t))$ of 1.2 that there exists $t_1 > 0$ such that

$$\| X \|^2 + \| Y \|^2 < \delta_7^2, t \geq t_1. \tag{3.9}$$

Precisely, if $\| X \|^2 + \| Y \|^2 \geq \delta_7^2$ for all $t \geq 0$, then, by 3.8, we have

$$\dot{V}(t) \leq -\delta_7^2\delta_5,$$

for all $t \geq 0$, then $V(X(t), Y(t)) \to -\infty$ as $t \to \infty$, this is a contradiction to 3.5.

Therefore, from 3.5 there exists a constant $K_1 > \delta_7$ such that

$$\max_{\|X(t)\|^2+\|Y(t)\|^2=\delta_7^2} V(X(t), Y(t)) < \min_{\|X(t)\|^2+\|Y(t)\|^2=K_1^2} V(X(t), Y(t)). \tag{3.10}$$

In what follows, we establish that the solution $(X(t), Y(t))$ of 1.2 must satisfy the inequality

$$\| X(t) \|^2 + \| Y(t) \|^2 \leq K_1^2, \quad \text{for} \quad t \geq t_1. \tag{3.11}$$

Otherwise, from 3.9 there exist t_2 and t_3, $t_1 < t_2 < t_3$, such that

$$\| X(t_2) \|^2 + \| Y(t_2) \|^2 = \delta_7^2, \tag{3.12}$$

$$\| X(t_3) \|^2 + \| Y(t_3) \|^2 = K_1^2, \tag{3.13}$$

and

$$\delta_7^2 \leq \| X(t) \|^2 + \| Y(t) \|^2 \leq K_1^2, \tag{3.14}$$

for $t_2 \leq t \leq t_3$. By 3.8, the inequality 3.14 means that $V(t_2) > V(t_3)$ and this is a contradiction to the claim that $V(t_2) < V(t_3)$ $(t_2 < t_3)$ that was obtained from 3.10, 3.12 and 3.13.

Thus, $(X(t), Y(t))$ must satisfy 3.11. This completes the proof of the theorem. □

Our next theorem deals with the case where vector $H(X)$ is not necessary differentiable.

Theorem 3.2. *Suppose the basic assumptions of Theorem 3.1 hold but in place of condition (i) and inequality 3.2 of condition (ii) we have,*

(i) there exists an $n \times n$ real continuous operator $A(X, Y)$ for any vectors $X, Y \in \mathbf{R}^n$ such that

$$H(X) = H(Y) + A(X, Y)(X - Y), \tag{3.15}$$

and its eigenvalues $\lambda_i(A(X, Y))(i = 1, 2, 3..., n)$ satisfy

$$0 < a - \epsilon_1 \leq \lambda_i(A(X, Y)) \leq a, \tag{3.16}$$

where a and ϵ_1 are positive constants;

(ii)

$$\alpha > 3\epsilon. \tag{3.17}$$

Then all the solutions of Eq. (1.1) are uniformly bounded and uniform-ultimately bounded.

Proof. It is good to note here that, the proof of this theorem is similar to the proof of Theorem 3.1 except for some little modifications. Hence, we may refer to certain part of the Proof of Theorem 3.1.

First, we define a scalar function $V(t) = V(t, X, Y)$ by

$$2V(t) = \| \alpha X + Y \|^2 + 2a \| X \|^2 + \| Y \|^2, \tag{3.18}$$

where both α and a are as defined above. Obviously, when $X = 0$ and $Y = 0$, $V(t)$ defined by 3.18 becomes zero. As in the proof of Theorem 3.1, one can easily verify that

$$\delta_8(\| X \|^2 + \| Y \|^2) \le 2V(t) \le \delta_9(\| X \|^2 + \| Y \|^2), \tag{3.19}$$

for certain positive constants δ_8 and δ_9.

Differentiating 3.18 with respect to t along 1.2 we obtain

$$\begin{aligned}\frac{d}{dt}V(t) = \dot{V}(t) = & - \alpha\langle X, H(X)\rangle + \alpha\langle Y, Y\rangle \\ & - 2\langle Y, F(X,Y)Y\rangle - \langle \alpha X, (F(X,Y) - \alpha I)Y\rangle \\ & - 2\langle H(X), Y\rangle + 2a\langle X, Y\rangle + \langle \alpha X + 2Y, P(t,X,Y)\rangle.\end{aligned}$$

Setting $Y = 0$ (and note that $H(0) = 0$) in 3.15 and using the result in $\dot{V}(t)$ above, we have

$$\begin{aligned}\frac{d}{dt}V(t) = \dot{V}(t) = & - \alpha\langle X, AX\rangle + \alpha\langle Y, Y\rangle - 2\langle Y, F(X,Y)Y\rangle - \langle X, BY\rangle \\ & + \langle \alpha X + 2Y, P(t,X,Y)\rangle,\end{aligned}$$

where $B = \Big(2(A - aI) + \alpha(F(X,Y) - \alpha I)\Big)$ is a matrix function and I is an $n \times n$ identity matrix. Again like in the proof of Theorem 3.1, we can re-write $\dot{V}(t)$ in the following way

$$\dot{V}(t) = -U_1 - U_2 + U_3, \tag{3.20}$$

where

$$U_1 = \frac{\alpha}{2}\langle X, AX\rangle - \alpha\langle Y, Y\rangle + \frac{3}{2}\langle Y, F(X,Y)Y\rangle,$$

$$U_2 = \frac{\alpha}{2}\langle X, AX\rangle + \langle X, BY\rangle + \frac{1}{2}\langle Y, F(X,Y)Y\rangle,$$

and

$$U_3 = \langle \alpha X + 2Y, P(t,X,Y)\rangle.$$

By Lemma 2.1, assumption (ii) of Theorem 3.1 and 3.16, we have
$\| BY \|^2 \le (2\epsilon_1 + \alpha\epsilon)^2 \| Y \|^2$,
where B is the matrix defined earlier.

Therefore,

$$\langle X, BY\rangle = \frac{1}{2} \| k_2 BY + k_2^{-1} X \|^2 - \frac{1}{2k_2^2} \| X \|^2$$

$$
\begin{aligned}
&- \frac{k_2^2}{2} \parallel BY \parallel^2 \\
&\geq -\frac{1}{2k_2^2} \parallel X \parallel^2 - \frac{(2\epsilon_1+\alpha\epsilon)^2 k_2^2}{2} \parallel Y \parallel^2,
\end{aligned}
$$

where $k_2 > 0$ is also a constant whose value will be determined later.

Also by Lemma 2.1 we have,

$$\langle X, AX\rangle \geq (a-\epsilon_1) \parallel X \parallel^2 .$$

Thus,

$$
\begin{aligned}
U_1 &\geq \frac{1}{2}\alpha(a-\epsilon_1) \parallel X \parallel^2 + \frac{1}{2}(\alpha - 3\epsilon) \parallel Y \parallel^2 \\
&\geq \delta_{10}\{\parallel X \parallel^2 + \parallel Y \parallel^2\},
\end{aligned}
$$

where $\delta_{10} = \frac{1}{2}\min\{\alpha(a-\epsilon_1); (\alpha-3\epsilon)\}$;

$$U_2 \geq \frac{1}{2}(\alpha(a-\epsilon_1) - k_2^{-2}) \parallel X \parallel^2 + \frac{1}{2}\Big((\alpha-\epsilon) - (2\epsilon_1+\alpha\epsilon)^2 k_2^2\Big) \parallel Y \parallel^2 .$$

On setting $k_2^2 = \frac{1}{2}\min\left(\frac{1}{\alpha(a-\epsilon_1)}, \frac{(\alpha-\epsilon)}{(2\epsilon_1+\alpha\epsilon)^2}\right)$, we have

$$U_2 \geq 0.$$

Also, using 3.3 in U_3, we have

$$
\begin{aligned}
|U_3| &\leq \{\alpha \parallel X \parallel + 2 \parallel Y \parallel\} \parallel P(t,X,Y) \parallel \\
&\leq \delta_{11}\{\parallel X \parallel + \parallel Y \parallel\}\{m_1 + m_2(\parallel X \parallel + \parallel Y \parallel)\} \\
&\leq 2\delta_{11} m_2\{\parallel X \parallel^2 + \parallel Y \parallel^2\} + 2^{\frac{1}{2}}\delta_{11} m_1\{\parallel X \parallel^2 + \parallel Y \parallel^2\}^{\frac{1}{2}},
\end{aligned}
$$

where $\delta_{11} = \max\{\alpha; 2\}$.

Therefore,

$$
\begin{aligned}
\dot{V}(t) &\leq -\delta_{10}\{\parallel X \parallel^2 + \parallel Y \parallel^2\} + 2\delta_{11} m_2\{\parallel X \parallel^2 + \parallel Y \parallel^2\} \\
&\quad + 2^{\frac{1}{2}}\delta_{11} m_1\{\parallel X \parallel^2 + \parallel Y \parallel^2\}^{\frac{1}{2}} \\
&\leq -2\delta_{12}\{\parallel X \parallel^2 + \parallel Y \parallel^2\} + \delta_{13}\{\parallel X \parallel^2 + \parallel Y \parallel^2\}^{\frac{1}{2}}, \qquad (3.21)
\end{aligned}
$$

where $\delta_{12} = \frac{1}{2}(\delta_{10} - 2\delta_{11} m_2)$, $m_2 < 2^{-1}\delta_{10}\delta_{11}^{-1}$, $\delta_{13} = 2^{\frac{1}{2}}\delta_{11} m_1$.
The remaining part of the proof follows exactly as in the proof of Theorem 3.1. Thus, it is omitted. □

4. Existence of a periodic solution

Theorem 4.1. *Further to all the conditions of Theorem 3.1, we shall assume that vector $P(t, X, Y)$ is $\omega-$periodic in t, that is,*

$P(t + \omega, X, Y) = P(t, X, Y)$.

Then, there exists at least one ω-periodic solution $(X(t), Y(t))$ of 1.2.

Proof. Suppose $(X(t), Y(t))$ is any solution of (1.1). Then by Theorem 3.1, $V(t)$ defined in Eq. 3.4 satisfied all the conditions of Theorem 4.1 and similarly the conditions of Lemma 2.4. Hence, by Lemma 2.4 and Lemma 2.5 system 1.2 has at least one periodic solution of period ω. This ends the proof of Theorem 4.1. □

Theorem 4.2. *In addition to all the conditions of Theorem 3.2, suppose $P(t, X, Y)$ satisfies:*

$P(t + \omega, X, Y) = P(t, X, Y)$.

Then, there exists at least one ω-periodic solution $(X(t), Y(t))$ of 1.2.

Proof. *The proof of this theorem is similar to the proof of Theorem 4.1.* □

5. Conclusion

By constructing suitable complete Lyapunov functions which serve as basic tools, we are able to establish sufficient conditions that guarantee the uniform-ultimate boundedness of solutions to a certain class of second order vector differential equations when $H(X)$ is differentiable and when it is not necessarily differentiable. Also, conditions for the existence of at least a periodic solution for the equation considered is established for the two cases of $H(X)$.

Statements and Declarations

We declare that there is no competing interests concerning this paper.

References

[1] A. M. A Abou-El-Ela and A. I. Sadek, "A stability result for the solutions of a certain system of fourth order differential equations", *Annals of Differential Equations*, vol. 6, no. 2, pp. 131-141, 1990.

[2] A. T. Ademola, "Boundedness and Stability of Solutions to Certain Second Order Differential Equations", *Differential Equations and Control Processes*, vol. 3, pp. 38-50, 2015.

[3] A. A. Adeyanju, "Existence of a limiting regime in the sense of demidovic for a certain class of second order non-linear vector differential equation", *Differential Equation and Control Processes*, no. 4, pp. 63-79, 2018.

[4] A. A. Adeyanju and D.O. Adams, "Some new results on the stability and boundedness of solutions of certain class of second order vector differential equations", *International Journal of Mathematical Analysis and Optimization: Theory and Applications*, vol. 7, no. 1, pp. 108-115, 2021. doi: 10.52968/28305999

[5] A. A. Adeyanju, "Stability and Boundedness Criteria of Solutions of a Certain System of Second Order Differential Equations", *Annali Dell'Universita' di Ferrara*, vol. 69, no. 1, pp. 81-93, 2023.

[6] A. U. Afuwape and M. O. Omeike, "Convergence of solutions of certain system of third order non-linear ordinary differential equations", *Annals of Differential Equations*, vol. 21, pp. 533-540, 2005.

[7] A. U. Afuwape, "Ultimate Boundedness Results for a Certain System of Third-Order Non-Linear Differential Equations", *Journal of Mathematical Analysis and Applications*, vol. 97, pp. 140-150, 1983. doi: 10.1016/0022-247X(83)90243-3

[8] A. U. Afuwape and M.O. Omeike, Convergence of solutions of some system of second order non-linear ordinary differential equations. In *Appreciating Mathematics in Contemporary World, Proceedings of the International Conference in Honour of Professor E.O. Oshobi and Dr. J. O. Amao*, A. U. Afuwape and Shola Adeyemi (Eds.), 2005.

[9] S. Ahmad and M. Rama Mohana Rao, *Theory of Ordinary Differential Equations. With Applications in Biology and Engineering.* New Delhi: East-West Press Pvt. Ltd., 1999.

[10] J. G. Alaba and B. S. Ogundare, "On stability and boundedness properties of solutions of certain second order non-autonomous non-linear ordinary differential equation", *Kragujevac Journal of Mathematics,* vol. 39, no. 2, pp. 255-266, 2015. doi: 10.5937/KgJMath1502255A

[11] J. Awrejcewicz, *Ordinary Differential Equations and Mechanical Systems.* Cham: Springer, 2014.

[12] A. Baliki, M. Benchohra and J.R. Graef, "Global existence and stability for second order functional evolution equations with infinite delay", *Electronic Journal of Qualitative Theory of Differential Equations,* vol. 2016, no. 23, pp. 10, 2016. doi: 10.14232/ejqtde.2016.1.23

[13] M. L. Cartwright and J. E. Littlewood, "On nonlinear differential equations of the second order", *Annals of Mathematics,* vol. 48, pp. 472-494, 1947. doi: 10.2307/1969181

[14] C. Chicone, *Ordinary Differential Equations with Applications.* Texts in Applied Mathematics, vol. 34. New York: Springer, 1999.

[15] J. O. C. Ezeilo, "On the existence of almost periodic solutions of some dissipative second order differential equations", *Annali di Matematica Pura ed Applicata*, vol. 65, no. 4, pp. 389-406, 1964. doi: 10.1007/BF02416464

[16] J. O. C. Ezeilo, "On the convergence of solutions of certain system of second order differential equations", *Annali di Matematica Pura ed Applicata,* vol. 72, no. 4, pp. 239-252, 1966. doi: 10.1007/BF02414336

[17] J. O. C. Ezeilo and H.O. Tejumola, "Boundedness and Periodicity of Solutions of a Certain Systems of Third Order Nonlinear Differential Equations", *Annali di Matematica Pura ed Applicata,* vol. 74, pp. 283-316, 1966. doi: 10.1007/BF02416460

[18] J. O. C. Ezeilo, "n-Dimensional extensions of boundedness and stability theorem for some third order differential equation", *Journal of Mathematical Analysis and Applications,* vol. 18, pp. 395-416, 1967. doi: 10.1016/0022-247X(67)90035-2

[19] G. A. Grigoryan, "Boundedness and stability criteria for linear ordinary differential equations of the second order", *Russian Mathematics,* vol. 57, no. 12, pp. 8-15, 2013. doi: 10.3103/S1066369X13120025

[20] J. Hale, "Sufficient conditions for stability and instability of autonomous functional-differential equations", *Journal of Differential Equations,* vol. 1, pp. 452-482, 1965. doi: 10.1016/0022-0396(65)90005-7

[21] D. W. Jordan and P. Smith, *Nonlinear Ordinary Differential Equations: Problems and Solutions. A Sourcebook for Scientists and Engineers.* Oxford: Oxford University Press, 2007.

[22] W. S. Loud, "Boundedness and convergence of solutions of $\ddot{x} + c\dot{x} + g(x) = e(t)$", *Duke Mathematical Journal,* vol. 24, pp. 63-72, 1957. doi: 10.1215/S0012-7094-57-02412-2

[23] F. W. Meng, "Ultimate boundedness results for a certain system of third order non-linear differential equations", *Journal of Mathematical Analysis and Applications,* vol. 177, pp. 496-509, 1993.

[24] B. S. Ogundare, A.T. Ademola, M. O. Ogundiran and O.A. Adesina, "On the qualitative behaviour of solutions to certain second order nonlinear differential equation with delay", *Annali dell'Universita' di Ferrara,* 2016. doi: 10.1007/s11565-016-0262-y

[25] M. O. Omeike, O.O. Oyetune and A.L. Olutimo, "Boundedness of solutions of a certain system of second-order ordinary differential equations", *Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica,* vol. 53, pp. 107-115, 2014.

[26] M. O. Omeike, A. A. Adeyanju and D. O. Adams, "Stability and Boundedness of Solutions of Certain Vector Delay Differential Equations", *Journal of the Nigerian Mathematical Society,* vol. 37, no. 2, pp. 77-87, 2018.

[27] M. O. Omeike, A.A. Adeyanju, D.O. Adams and A.L. Olutimo, "Boundedness of Certain System of Second Order Differential Equations", *Kragujevac Journal of Mathematics,* vol. 45, no. 5, pp. 787-796, 2021.

[28] R. Reissig, G. Sansone and R. Conti, *Non-linear Differential Equations of Higher Order*. Leyden: Noordhoff, 1974.

[29] A. I. Sadek, "On the stability of a nonhomogeneous vector differential equations of the fourth-order", *Applied Mathematics and Computation,* vol. 150, pp. 279-289, 2004. doi: 10.1016/S0096-3003(03)00227-3

[30] H. O. Tejumola, “On the Boundedness and periodicity of solutions of certain third-order nonlinear differential equation”, *Annali di Matematica Pura ed Applicata,* vol. 83, no. 4, pp. 195-212, 1969. doi: 10.1007/BF02411167

[31] H. O. Tejumola, “Boundedness Criteria for solutions of some second-order differential equations”, *Atti della Accademia Nazionale dei Lincei, Classe di Scienze Fisiche, Matematiche e Naturali,* vol. 50, no. 8, pp. 432-437, 1971.

[32] H. O. Tejumola, “On a Lienard type matrix differential equations”, *Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti,* vol. 60, no. 8, pp. 100-107, 1976.

[33] C. Tunç, “On the stability and boundedness of solutions of nonlinear vector differential equations of third order”, *Nonlinear Analysis,* vol. 70, no. 6, pp. 2232-2236, 2009. doi: 10.1016/j.na.2008.03.002

[34] C. Tunç and O. Tunç, “A note on the stability and boundedness of solutions to non-linear differential systems of second order”, *Journal of the Association of Arab Universities for Basic and Applied Sciences*, vol. 24, pp. 169-175, 2017. doi: 10.1016/j.jaubas.2016.12.004

[35] C. Tunç and S.A. Mohammed, “On the asymptotic analysis of bounded solutions to nonlinear differential equations of second order”, *Advances in Difference Equations,* vol. 2019, no. 461, 2019. doi: 10.1186/s13662-019-2384-x

[36] T. Yoshizawa, *Stability Theory and the Existence of Periodic Solutions and Almost Periodic Solutions*. New York: Springer-Verlag, 1978.

[37] T. Yoshizawa, *Stability theory by Liapunov’s second method.* Japan: Publications of the Mathematical Society of Japan, 1966.

[38] T. Yoshizawa, “Liapunov’s function and boundedness of solutions”, *Funkcialaj Ekvacioj,* vol. 2, pp. 71-103, 1958.

[39] T. Yoshizawa, *Stability theory by Liapunov’s second method.* The Mathematical Society of Japan, 1966.

[40] M. J. Zainab, “Bounded Solution of the Second Order Differential Equation $\ddot{x} + f(x)\dot{x} + g(x) = u(t)$, *Baghdad Science Journal*, vol. 12, no. 4, pp. 822-825, 2015. doi: 10.21123/bsj.12.4.822-825

Adetunji A. Adeyanju
Department of Mathematics,
Federal University of Agriculture Abeokuta
Nigeria
e-mail: adeyanjuaa@funaab.edu.ng
tjyanju2000@yahoo.com
https://orcid.org/0000-0002-9013-6002.