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On approximation of double Fourier series and its conjugate series for functions in mixed Lebesgue space $L_{\vec{p}}$, $\vec{p} \in [1, \infty]^2$

Yogeshkumar K. Patel
Government Science College, India
and
Rajendra G. Vyas
The Maharaja Sayajirao University of Baroda, India
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Abstract

In this paper, we study the approximation of double Fourier series and its conjugate series for functions in mixed Lebesgue space $L_{\vec{p}}$, $\vec{p} \in [1, \infty]^2$ using double Karamata $K^{\lambda,\mu}$ means.

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1. Introduction

Various summability methods are used in approximation theory, operator theory and the theory of orthogonal Fourier series to get the better rate of convergence as well as to minimize the error in the degree of approximation. By using the Karamata K^{λ} means of Fourier series and conjugate Fourier series, Sadangi [5] obtained the degree of approximation of functions of H_{α} class in the Hölder metric. Recently, Landon et al. extended these results for functions in $H_{\alpha,p}$ class [3]. By using double Karamata $K^{\lambda,\mu}$ means of double Fourier series Nigam and Hadish [4] approximated two variables functions of Hölder class $H_{\alpha,\beta}$. Lal and Mishra studied the approximation of double Fourier series for functions of generalized Lipschitz class by (N, p_m, q_n) method [2].

In this paper, we have estimated the degree of approximation of double Fourier series and its conjugate series for functions in the mixed Lebesgue space $L_{\vec{p}}$, $\vec{p} \in [1,\infty]^2$ using double Karamata $K^{\lambda,\mu}$ means. The paper is organised in the following manner. In section 2, we discuss the preliminaries required for the development of paper. In section 3, we give some additional notations and prove some lemmas required in the proof of main theorem. In section 4, we discuss the main theorems and aim of this work. In consecutive sections, we prove our main theorems.

2. Preliminaries

For $0 \leq k \leq n$, define the numbers $\begin{bmatrix} n \\ k \end{bmatrix}$ by

$$\prod_{v=0}^{n-1}(x+v) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k, x > 0,$$

where $\prod_{v=0}^{n-1}(x+v) = x(x+1)(x+2).....(x+n-1) = \frac{\Gamma(x+n)}{\Gamma(x)}$. The numbers $\begin{bmatrix} n \\ k \end{bmatrix}$ are called Stirling's number of first kind. We shall use the convention that $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 1$.

Let f be a Lebesgue integrable function over T^2, where $T = [-\pi, \pi]$, periodic with period 2π in each variable. Then the double Fourier series for function f is defined as

$$\begin{aligned}(2.1)\qquad &\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\lambda_{mn}[\alpha_{mn}\cos(mx)\cos(ny)+\beta_{mn}\sin(mx)\cos(ny)\\ &\qquad +\gamma_{mn}\cos(mx)\sin(ny)+\delta_{mn}\sin(mx)\sin(ny)]\\ &=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(\alpha,\beta,\gamma,\delta;x,y)_{m,n}=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\lambda_{mn}A_{mn}(x,y),\end{aligned}$$

where

$$\alpha_{mn}=\frac{1}{\pi^2}\int\int_{T^2}f(x,y)\cos(mx)\cos(ny)dx\,dy,\ \ m,n=0,1,2,\ldots,$$

with similar expressions for β_{mn}, γ_{mn} and δ_{mn}, and $\lambda_{m,n}=1, \lambda_{m,0}=\lambda_{0,n}=\frac{1}{2}, m,n\geq 1, \lambda_{0,0}=\frac{1}{4}$. The three conjugate series of double Fourier series (2.1) are given as follows:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(-\beta,\alpha,-\delta,\gamma;x,y)_{m,n}, \tag{2.2}$$

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(-\gamma,-\delta,\alpha,\beta;x,y)_{m,n}, \tag{2.3}$$

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(\delta,-\gamma,-\beta,\alpha;x,y)_{m,n}. \tag{2.4}$$

Let $\{S_{mn}\}$ be the sequence of partial sums of double infinite series $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}a_{mn}$, and define the $K^{\lambda\mu}$- mean of the sequence $\{S_{mn}\}$ of order $\lambda>0,\mu>0$ as follows:

$$S_{mn}^{\lambda\mu}=\frac{\Gamma(\lambda)}{\Gamma(\lambda+m)}\frac{\Gamma(\mu)}{\Gamma(\mu+n)}\sum_{p=0}^{m}\sum_{q=0}^{n}\left[\begin{array}{c} m\\ p\end{array}\right]\left[\begin{array}{c} n\\ q\end{array}\right]\lambda^p\mu^q S_{pq}.$$

If $S_{mn}^{\lambda\mu}\to s$ as $m,n\to\infty$, then the series $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}a_{mn}$ is said to be summable to s by double Karamata $K^{\lambda\mu}$ method of order $\lambda>0,\mu>0$.

In [7], the mixed Lebesgue space $L_{\vec{p}}[T^2]$ is defined as the set of all measurable function f such that

$$\|f\|_{\vec{p}}=\left(\int_{-\pi}^{\pi}\left(\int_{-\pi}^{\pi}|f(x,y)|^{p_1}dt\right)^{\frac{p_2}{p_1}}ds\right)^{\frac{1}{p_2}}<\infty.$$

Note that the Banach space space $L_{\vec{p}}[T^2]$ ([1], also see [7]) is finer then the classical structure of Banach space $L_p[T^2]$, replacing the constant exponent p of the L^p-norm by an exponent vector $\vec{p} = (p_1, p_2) \in [1, \infty]^2$.

3. Additional Notation and Lemmas

In the sequel, following notations are used :

$$\phi_{xy}(s,t) = \frac{f(x+s,y+t)+f(x-s,y+t)+f(x+s,y-t)+f(x-s,y-t)-4f(x,y)}{4},$$

$$\psi_{xy}(s,t) = \frac{f(x+s,y+t)-f(x-s,y+t)-f(x+s,y-t)+f(x-s,y-t)}{4},$$

$$K_m^{\lambda}(s) = \frac{\Gamma(\lambda)}{\Gamma(\lambda+m)} \sum_{p=0}^{m} \begin{bmatrix} m \\ p \end{bmatrix} \lambda^p \frac{\sin(p+\frac{1}{2})s}{\sin(\frac{s}{2})}, \tag{3.1}$$

$$K_n^{\mu}(t) = \frac{\Gamma(\mu)}{\Gamma(\mu+n)} \sum_{q=0}^{n} \begin{bmatrix} n \\ q \end{bmatrix} \mu^q \frac{\sin(q+\frac{1}{2})t}{\sin(\frac{t}{2})}, \tag{3.2}$$

$$\tilde{D_m^{\lambda}}(s) = \frac{\Gamma(\lambda)}{\Gamma(\lambda+m)} \sum_{p=0}^{m} \begin{bmatrix} m \\ p \end{bmatrix} \lambda^p \frac{\cos(\frac{s}{2})-\cos(p+\frac{1}{2})s}{\sin(\frac{s}{2})}, \tag{3.3}$$

$$\tilde{D_n^{\mu}}(t) = \frac{\Gamma(\mu)}{\Gamma(\mu+n)} \sum_{q=0}^{n} \begin{bmatrix} n \\ q \end{bmatrix} \mu^q \frac{\cos(\frac{t}{2})-\cos(q+\frac{1}{2})t}{\sin(\frac{t}{2})}, \tag{3.4}$$

$$\tilde{K_m^{\lambda}}(s) = \frac{\Gamma(\lambda)}{\Gamma(\lambda+m)} \sum_{p=0}^{m} \begin{bmatrix} m \\ p \end{bmatrix} \lambda^p \frac{\cos(p+\frac{1}{2})s}{\sin(\frac{s}{2})}, \tag{3.5}$$

$$\tilde{K_n^{\mu}}(t) = \frac{\Gamma(\mu)}{\Gamma(\mu+n)} \sum_{q=0}^{n} \begin{bmatrix} n \\ q \end{bmatrix} \mu^q \frac{\cos(q+\frac{1}{2})t}{\sin(\frac{t}{2})}. \tag{3.6}$$

For convenience, we write $\frac{1}{m} = h$ and $\frac{1}{n} = k$. For the proofs of the main theorems, we need following order relation lemmas.

Lemma 3.1. *Let $K_m^{\lambda}(s), K_n^{\mu}(t)$ be defined as above. Then the following order relations holds:*

(i) *For $0 \le s \le h$, $K_m^{\lambda}(s) = O(\lambda \log(m) + 1)$.*

Similarly, for $0 \le t \le k$, $K_n^{\mu}(t) = O(\mu \log(n) + 1)$.

(ii) *For* $h \leq s \leq \pi$, $K_m^\lambda(s) = O\left(\frac{1}{s}\right)$.

Similarly, for $k \leq t \leq \pi$, $K_n^\mu(t) = O\left(\frac{1}{t}\right)$.

Proof. **(i):** For $0 \leq s \leq h$,

$$\mathrm{K}_m^\lambda(s) = \frac{\Gamma(\lambda)}{\Gamma(\lambda+m)} \sum_{p=0}^{m} \begin{bmatrix} m \\ p \end{bmatrix} \lambda^p \frac{\sin(p+\frac{1}{2})s}{\sin(\frac{s}{2})} = \frac{\Gamma(\lambda)\Im\left(e^{\frac{is}{2}}\frac{\Gamma(\lambda e^{is}+n)}{\Gamma(\lambda e^{is})}\right)}{\sin(\frac{s}{2})\Gamma(\lambda+m)},$$

where $\Im$ means the imaginary part. Using the arguments given in the proof of the theorem [6, p.194], it is easy to check that,
$K_m^\lambda(s) = O\left(\lambda \log(m) + 1\right)$.

Similarly, for $0 \leq t \leq k$, we get $K_n^\mu(t) = O\left(\mu \log(n) + 1\right)$.

(ii): For $h \leq s \leq \pi, \sin(\frac{s}{2}) \geq \frac{s}{\pi}$. Thus,

$$\begin{aligned}
\mathrm{K}_m^\lambda(s) &\leq \left|\frac{\Gamma(\lambda)}{\Gamma(\lambda+m)} \sum_{p=0}^{m} \begin{bmatrix} m \\ p \end{bmatrix} \lambda^p \frac{\sin(p+\frac{1}{2})s}{\sin(\frac{s}{2})}\right| \\
&\leq \frac{\Gamma(\lambda)}{\Gamma(\lambda+m)} \sum_{p=0}^{m} \begin{bmatrix} m \\ p \end{bmatrix} \lambda^p \frac{|\sin(p+\frac{1}{2})s|}{|\sin(\frac{s}{2})|} \\
&\leq \frac{\Gamma(\lambda)}{\Gamma(\lambda+m)} \sum_{p=0}^{m} \begin{bmatrix} m \\ p \end{bmatrix} \lambda^p \frac{1}{\frac{s}{\pi}} \\
&= O\left(\frac{1}{s}\right).
\end{aligned}$$

Similarly, for $k \leq t \leq \pi$, we get $K_n^\mu(t) = O\left(\frac{1}{t}\right)$.
□

Lemma 3.2. *Let* $\tilde{D}_m^\lambda(s), \tilde{D}_n^\mu(t)\tilde{K}_m^\lambda(s), \tilde{K}_n^\mu(t)$ *be defined as above. Then the following order relations holds:*

(i) *For* $0 \leq s \leq h$, $\tilde{D}_m^\lambda(s) = O(m)$.

Similarly, for $0 \leq t \leq k$, $\tilde{D}_n^\mu(t) = O(n)$.

(ii) *For* $h \leq s \leq \pi$, $\tilde{K}_m^\lambda(s) = O\left(\frac{1}{s}\right)$.

Similarly, for $k \leq t \leq \pi$, $\tilde{K}_n^\mu(t) = O\left(\frac{1}{t}\right)$.

Proof. **(i):** For $0 \le s \le h$,

$$\begin{aligned}
\tilde{D_m^{\lambda}}(s) &\le \left| \frac{\Gamma(\lambda)}{\Gamma(\lambda+m)} \sum_{p=0}^{m} \begin{bmatrix} m \\ p \end{bmatrix} \lambda^p \frac{\cos(\frac{s}{2}) - \cos(p+\frac{1}{2})s}{\sin(\frac{s}{2})} \right| \\
&= \left| \frac{\Gamma(\lambda)}{\Gamma(\lambda+m)} \sum_{p=0}^{m} \begin{bmatrix} m \\ p \end{bmatrix} \lambda^p \sum_{r=0}^{p} 2\sin(r\, s) \right| \\
&\le \left| \frac{\Gamma(\lambda)}{\Gamma(\lambda+m)} \sum_{p=0}^{m} \begin{bmatrix} m \\ p \end{bmatrix} \lambda^p\, p \right| \\
&= O(m).
\end{aligned}$$

Similarly, for $0 \le t \le k$, $\tilde{D_n^{\mu}}(t) = O(n)$.

(ii): For $h \le s \le \pi$, $\sin(\frac{s}{2}) \ge \frac{s}{\pi}$. Thus,

$$\begin{aligned}
\tilde{K_m^{\lambda}}(s) &\le \left| \frac{\Gamma(\lambda)}{\Gamma(\lambda+m)} \sum_{p=0}^{m} \begin{bmatrix} m \\ p \end{bmatrix} \lambda^p \frac{\cos(p+\frac{1}{2})s}{\sin(\frac{s}{2})} \right| \\
&\le \frac{\Gamma(\lambda)}{\Gamma(\lambda+m)} \sum_{p=0}^{m} \begin{bmatrix} m \\ p \end{bmatrix} \lambda^p \frac{|\cos(p+\frac{1}{2})s|}{|\sin(\frac{s}{2})|} \\
&\le \frac{\Gamma(\lambda)}{\Gamma(\lambda+m)} \sum_{p=0}^{m} \begin{bmatrix} m \\ p \end{bmatrix} \lambda^p \frac{\pi}{s} \\
&= O\left(\frac{1}{s}\right).
\end{aligned}$$

Similarly, for $k \le t \le \pi$, $\tilde{K_n^{\mu}}(t) = O\left(\frac{1}{t}\right)$. □

4. Main results

The aim of this paper is to estimate the degree of approximation for functions from mixed Lebesgue space $L_{\vec{p}}$, $\vec{p} \in [1,\infty]^2$ using double Karamata $K^{\lambda,\mu}$ means of its double Fourier series. Essentially we prove the following result.

Theorem 4.1. *Let $S_{m,n}^{\lambda,\mu}(x,y)$ be the $K^{\lambda,\mu}$ mean of double Fourier series of a 2π- periodic function $f \in L_{\vec{p}}[T^2]$ such that $\|f(x+s,y+t) - f(x,y)\|_{\vec{p}} = O(\xi_1(s)\xi_2(t))$, where ξ_1 and ξ_2 are positive monotonic non decreasing functions satisfying that $\left(\frac{\xi_1(s)}{s}\right)$, $\left(\frac{\xi_2(t)}{t}\right)$ are monotonic decreasing, and $\frac{\xi_1(h)}{h} \to 0$ as $h \to 0$, $\frac{\xi_2(k)}{k} \to 0$ as $k \to 0$, then*

$$\|S_{m,n}^{\lambda,\mu}(x,y) - f(x,y)\|_{\vec{p}} = O\left(\frac{\xi_1(h)}{h}\frac{\xi_2(k)}{k}\right).$$

Also, the approximation of double conjugate Fourier series for functions from mixed Lebesgue space $L_{\vec{p}}$, $\vec{p} \in [1, \infty]^2$ using double Karamata $K^{\lambda,\mu}$ means is derived using truncated conjugate function in the following form:

Theorem 4.2. *Let $\tilde{S}^{\lambda,\mu}_{m,n}(f; x, y)$ be the $K^{\lambda,\mu}$-mean of the double conjugate Fourier series (2.4) of a 2π- periodic function $f \in L_{\vec{p}}[T^2]$ and ξ_1, ξ_2 be defined as in Theorem 4.1, then*

$$\|4\pi^2 \tilde{S}^{\lambda,\mu}_{m,n}(f; x, y) - \tilde{f}(x, y; h, k)\|_{\vec{p}} = O\left(\frac{\xi_1(h)}{h}\frac{\xi_2(k)}{k}\right),$$

where

$$\tilde{f}(x, y) = \lim_{h\to 0}\lim_{k\to 0} \tilde{f}(x, y; h, k)$$

and

$$\tilde{f}(x, y; h, k) = \frac{1}{4\pi^2}\int_h^\pi \int_k^\pi \psi_{xy}(s, t) \cot(\frac{s}{2}) \cot(\frac{t}{2}) ds\, dt.$$

5. Proof of Theorem 4.1

Recall that, the partial sum of double Fourier series is

$$S_{m,n}(f; x, y) - f(x, y) = \frac{1}{\pi^2}\int_0^\pi \int_0^\pi \phi_{xy}(s, t)\frac{\sin(m + \frac{1}{2})s}{\sin(\frac{s}{2})}\frac{\sin(n + \frac{1}{2})t}{\sin(\frac{t}{2})}dsdt.$$

$K^{\lambda\mu}$-means of order $\lambda > 0, \mu > 0$ is

$$\begin{aligned} S^{\lambda\mu}_{m,n}(f; x, y) - f(x, y)&\frac{\Gamma(\lambda)}{\Gamma(\lambda + m)}\frac{\Gamma(\mu)}{\Gamma(\mu + n)}\sum_{p=0}^{m}\sum_{q=0}^{n}\binom{m}{p}\binom{n}{q}\lambda^p\mu^q \\ &= \frac{\Gamma(\lambda)}{\Gamma(\lambda + m)}\frac{\Gamma(\mu)}{\Gamma(\mu + n)}\sum_{p=0}^{m}\sum_{q=0}^{n}\binom{m}{p}\binom{n}{q}\lambda^p\mu^q \\ &\times\frac{1}{\pi^2}\int_0^\pi \int_0^\pi \phi_{xy}(s, t)\frac{\sin(m + \frac{1}{2})s}{\sin(\frac{s}{2})}\frac{\sin(n + \frac{1}{2})t}{\sin(\frac{t}{2})}dsdt. \end{aligned}$$

Therefore,

$$\begin{aligned} S^{\lambda\mu}_{m,n}(f; x, y) - f(x, y) &= \tfrac{1}{\pi^2}\textstyle\int_0^\pi\int_0^\pi \phi_{xy}(s, t)\frac{\Gamma(\lambda)}{\Gamma(\lambda+m)}\sum_{p=0}^{m}\binom{m}{p}\lambda^p\displaystyle\frac{\sin(p + \frac{1}{2})s}{\sin(\frac{s}{2})} \\ &\times\tfrac{\Gamma(\mu)}{\Gamma(\mu+n)}\textstyle\sum_{q=0}^{n}\binom{n}{q}\mu^q\displaystyle\frac{\sin(q + \frac{1}{2})t}{\sin(\frac{t}{2})}dsdt. \\ &= \tfrac{1}{\pi^2}\textstyle\int_0^\pi\int_0^\pi \phi_{xy}(s, t)K^\lambda_m(s)K^\mu_n(t)dsdt \\ &= I_{m,n}(x, y). \end{aligned}$$

Now,

$$
\begin{aligned}
I_{m,n}(x,y) &= \tfrac{1}{\pi^2}\int_0^\pi\int_0^\pi \phi_{xy}(s,t)K_m^\lambda(s)K_n^\mu(t)dsdt\\
&= \tfrac{1}{\pi^2}\Big(\int_0^h\int_0^k \phi_{xy}(s,t)K_m^\lambda(s)K_n^\mu(t)dsdt\\
&+\int_0^h\int_k^\pi \phi_{xy}(s,t)K_m^\lambda(s)K_n^\mu(t)dsdt\\
&+\int_h^\pi\int_0^k \phi_{xy}(s,t)K_m^\lambda(s)K_n^\mu(t)dsdt\\
&+\int_h^\pi\int_k^\pi \phi_{xy}(s,t)K_m^\lambda(s)K_n^\mu(t)dsdt\\
&= \tfrac{1}{\pi^2}(I_1+I_2+I_3+I_4).
\end{aligned}
$$

By Minkowski's inequality, we have

$$\|I_{m,n}(x,y)\|_{\vec{p}} \le \frac{1}{\pi^2}(\|I_1\|_{\vec{p}}+\|I_2\|_{\vec{p}}+\|I_3\|_{\vec{p}}+\|I_4\|_{\vec{p}}),$$

where

$$\|I_1\|_{\vec{p}} = \int_0^h\int_0^k \|\phi_{xy}(s,t)\|_{\vec{p}}|K_m^\lambda(s)||K_n^\mu(t)|dsdt,$$

$$\|I_2\|_{\vec{p}} = \int_0^h\int_k^\pi \|\phi_{xy}(s,t)\|_{\vec{p}}|K_m^\lambda(s)||K_n^\mu(t)|dsdt,$$

$$\|I_3\|_{\vec{p}} = \int_h^\pi\int_0^k \|\phi_{xy}(s,t)\|_{\vec{p}}|K_m^\lambda(s)||K_n^\mu(t)|dsdt,$$

$$\|I_4\|_{\vec{p}} = \int_h^\pi\int_k^\pi \|\phi_{xy}(s,t)\|_{\vec{p}}|K_m^\lambda(s)||K_n^\mu(t)|dsdt.$$

From Lemma 3.1 (i),

$$
\begin{aligned}
\|I_1\|_{\vec{p}} &= \int_0^h\int_0^k \|\phi_{xy}(s,t)\|_{\vec{p}}|K_m^\lambda(s)||K_n^\mu(t)|dsdt\\
&= O\left((\lambda\log(m)+1)(\mu\log(n)+1)\int_0^h\int_0^k(\xi_1(s)\xi_2(t))dsdt\right)\\
&= O\left((\lambda\log(m)+1)(\mu\log(n)+1)hk(\xi_1(h)\xi_2(k))\right)\\
&= O\left(\xi_1(h)\xi_2(k)\right).
\end{aligned}
\tag{5.1}
$$

Using Lemma 3.1,

$$
\begin{aligned}
\|I_2\|_{\vec{p}} &= \int_0^h\int_k^\pi \|\phi_{xy}(s,t)\|_{\vec{p}}|K_m^\lambda(s)||K_n^\mu(t)|dsdt\\
&= O\left((\lambda\log(m)+1)\int_0^h\int_k^\pi \frac{1}{t}(\xi_1(s)\xi_2(t))dsdt\right)\\
&= O\left((\lambda\log(m)+1)\int_0^h\int_k^\pi \frac{1}{t}(\xi_1(s)\xi_2(t))dsdt\right)\\
&= O\left((\lambda\log(m)+1)\int_0^h\xi_1(s)ds\int_k^\pi \frac{1}{t}\xi_2(t)dt\right)\\
&= O\left((\lambda\log(m)+1)\frac{\xi_1(h)\xi_2(k)h}{k}\right)\\
&= O\left(\xi_1(h)\tfrac{\xi_2(k)}{k}\right).
\end{aligned}
\tag{5.2}
$$

Similarly,

$$\|I_3\|_{\vec{p}} = O\left(\frac{\xi_1(h)}{h}\xi_2(k)\right). \tag{5.3}$$

Now, from Lemma 3.1 (ii),

$$\begin{aligned} \|I_4\|_{\vec{p}} &= \int_h^\pi \int_k^\pi \|\phi_{xy}(s,t)\|_{\vec{p}} |K_m^\lambda(s)||K_n^\mu(t)| ds dt \\ &= O\left(\int_h^\pi \int_k^\pi \frac{1}{st}(\xi_1(s)\xi_2(t)) ds dt\right) \\ &= O\left(\int_h^\pi \frac{\xi_1(s)}{s} ds \int_k^\pi \frac{\xi_2(t)}{t} dt\right) \\ &= O\left(\frac{\xi_1(h)}{h}\frac{\xi_2(k)}{k}\right). \end{aligned} \tag{5.4}$$

Thus, from equations (5.1) to (5.4) we arrive at

$$\|I_{m,n}(x,y)\|_{\vec{p}} = \|S_{m,n}^{\lambda\mu}(f;x,y) - f(x,y)\|_{\vec{p}} = O\left(\frac{\xi_1(h)}{h}\frac{\xi_2(k)}{k}\right).$$

6. Proof of Theorem 4.2

The $(m,n)^{th}$ partial sum of the double conjugate Fourier series (2.4) is

$$\tilde{S}_{m,n}(f;x,y)$$

$$= \frac{1}{4\pi^2}\int_0^\pi \int_0^\pi \psi_{xy}(s,t) \frac{\cos(\frac{s}{2}) - \cos(m+\frac{1}{2})s}{\sin(\frac{s}{2})} \times \frac{\cos(\frac{t}{2}) - \cos(n+\frac{1}{2})t}{\sin(\frac{t}{2})} ds dt.$$

Taking $\tilde{S}_{m,n}^{\lambda,\mu}(f;x,y)$ as the $K^{\lambda\mu}$-mean of $\tilde{S}_{m,n}(f;x,y)$ for $\lambda > 0, \mu > 0$,

$$\begin{aligned} 4\pi^2 \tilde{S}_{m,n}^{\lambda,\mu} &(f;x,y) - \tilde{f}(x,y;h,k) \\ &= \int_0^h \int_0^k \psi_{xy}(s,t)\tilde{D_m^\lambda}(s)\tilde{D_n^\mu}(t) ds\, dt - \int_h^\pi \int_0^k \psi_{xy}(s,t)\tilde{K_m^\lambda}(s)\tilde{D_n^\mu}(t) ds\, dt \\ &- \int_0^h \int_k^\pi \psi_{xy}(s,t)\tilde{D_m^\lambda}(s)\tilde{K_n^\mu}(t) ds\, dt + \int_h^\pi \int_k^\pi \psi_{xy}(s,t)\tilde{K_m^\lambda}(s)\tilde{K_n^\mu}(t) ds\, dt \\ &= L_1 + L_2 + L_3 + L_4. \end{aligned}$$

By Minkowski's inequality, we have

$$\|4\pi^2 \tilde{S}_{m,n}^{\lambda,\mu}(f;x,y) - \tilde{f}(x,y;h,k)\|_{\vec{p}} = \|L_1\|_{\vec{p}} + \|L_2\|_{\vec{p}} + \|L_3\|_{\vec{p}} + \|L_4\|_{\vec{p}},$$

where

$$\|L_1\|_{\vec{p}} = \int_0^h \int_0^k \|\psi_{xy}(s,t)\|_{\vec{p}} |\tilde{D_m^\lambda}(s)|\, |\tilde{D_n^\mu}(t)| ds\, dt,$$
$$\|L_2\|_{\vec{p}} = \int_h^\pi \int_0^k \|\psi_{xy}(s,t)\|_{\vec{p}} |\tilde{K_m^\lambda}(s)|\, |\tilde{D_n^\mu}(t)| ds\, dt,$$
$$\|L_3\|_{\vec{p}} = \int_0^h \int_k^\pi \|\psi_{xy}(s,t)\|_{\vec{p}} |\tilde{D_m^\lambda}(s)|\, |\tilde{K_n^\mu}(t)| ds\, dt,$$
$$\|L_4\|_{\vec{p}} = \int_h^\pi \int_k^\pi \|\psi_{xy}(s,t)\|_{\vec{p}} |\tilde{K_m^\lambda}(s)|\, |\tilde{K_n^\mu}(t)| ds\, dt.$$

From Lemma 3.2 (i),

$$\begin{aligned}
\|L_1\|_{\vec{p}} &= \textstyle\int_0^h \int_0^k \|\psi_{xy}(s,t)\|_{\vec{p}} |\tilde{D_m^\lambda}(s)|\, |\tilde{D_n^\mu}(t)| dsdt \\
&= O\left(mn \textstyle\int_0^h \xi_1(s)ds \int_0^k \xi_2(t)dt\right) \\
&= O\left(mn\xi_1(h)\xi_2(k) \textstyle\int_0^h ds \int_0^k dt\right) \\
&= O\left(mnh\xi_1(h)k\xi_2(k)\right) \\
&= O\left(\xi_1(h)\xi_2(k)\right).
\end{aligned} \tag{6.1}$$

Using Lemma 3.2,

$$\begin{aligned}
\|L_2\|_{\vec{p}} &= \textstyle\int_h^\pi \int_0^k \|\psi_{xy}(s,t)\|_{\vec{p}} |\tilde{K_m^\lambda}(s)| |\tilde{D_n^\mu}(t)| dsdt \\
&= O\left(n \textstyle\int_h^\pi \int_0^k \frac{\|\psi_{xy}(s,t)\|_{\vec{p}}}{s} dsdt\right) \\
&= O\left(n \textstyle\int_h^\pi \int_0^k \frac{\xi_1(s)\xi_2(t)}{s} dsdt\right) \\
&= O\left(n \textstyle\int_h^\pi \frac{\xi_1(s)}{s} ds \int_0^k \xi_2(t)dt\right) \\
&= O\left(nk\frac{\xi_1(h)}{h}\xi_2(k)\right) \\
&= O\left(\frac{\xi_1(h)}{h}\xi_2(k)\right).
\end{aligned} \tag{6.2}$$

Similarly,

$$\|L_3\|_{\vec{p}} = O\left(\xi_1(h)\frac{\xi_2(k)}{k}\right). \tag{6.3}$$

By Lemma 3.2 (ii),

$$
\begin{aligned}
\|L_4\|_{\vec{p}} &= \textstyle\int_h^\pi \int_k^\pi \|\psi_{xy}(s,t)\|_{\vec{p}} |\tilde{K_m}^\lambda(s)|\, |\tilde{K_n}^\mu(t)| ds\, dt \\
&= O\left(\textstyle\int_h^\pi \int_k^\pi \xi_1(s)\xi_2(t) \frac{1}{s}\frac{1}{t} ds\, dt\right) \\
&= O\left(\textstyle\int_h^\pi \frac{\xi_1(s)}{s} ds \int_k^\pi \frac{\xi_2(t)}{t} dt\right) \\
&= O\left(\textstyle\frac{\xi_1(h)}{h}\frac{\xi_2(k)}{k} \int_h^\pi ds \int_k^\pi dt\right) \\
&= O\left(\textstyle\frac{\xi_1(h)}{h}\frac{\xi_2(k)}{k}\right).
\end{aligned} \tag{6.4}
$$

Thus, from equations 6.1 to 6.4, we arrive at

$$\|4\pi^2 \tilde{S}_{m,n}^{\lambda,\mu}(f;x,y) - \tilde{f}(x,y;h,k)\|_{\vec{p}} = O\left(\frac{\xi_1(h)}{h}\frac{\xi_2(k)}{k}\right).$$

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Yogeshkumar K. Patel
Department of Mathematics,
Government Science College,
Pardi, Gujarat,
India 396125
India
e-mail: ykpmaths@gmail.com
Corresponding author

and

Rajendra G. Vyas
Department of Mathematics,
Faculty of Science
The Maharaja Sayajirao University of Baroda Vadodara,
Gujarat, India 390002
India
e-mail: vyas.rajendra@gmail.com