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Total absolute difference edge irregularity strength of T_p-tree graphs

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Abstract

A total labeling ξ is defined to be an edge irregular total absolute difference k-labeling of the graph G if for every two different edges e and f of G there is $wt(e) \neq wt(f)$ where weight of an edge $e = xy$ is defined as $wt(e) = |\xi(e) - \xi(x) - \xi(y)|$. The minimum k for which the graph G has an edge irregular total absolute difference labeling is called the total absolute difference edge irregularity strength of the graph G, $tades(G)$. In this paper, we determine the total absolute difference edge irregularity strength of the precise values for T_p-tree related graphs.

AMS Subject Classification 2010: *05C78.*

Keywords: *Edge irregularity strength, total absolute difference edge irregularity strength, T_p-tree.*

1. Introduction

Here, we consider a simple graph G with vertex set V and the edge set E. The total edge irregular strength of graphs was introduced by Baca et al. [2]. The basic idea came from irregular assignments and the irregular strength of graphs introduced by Chartrand et al. [3]. The total edge irregular k-labeling of a graph $G = (V, E)$ namely the labeling $\xi : V \bigcup E \rightarrow \{1, 2, \ldots, k\}$ such that all edge weights are distinct. The weight $wt(uv)$ of an edge uv is defined as $wt_\xi(uv) = \xi(u) + \xi(uv) + \xi(v)$. The total edge irregularity strength G denoted by $tes(G)$, is the smallest k for which G has a total edge irregular k-labeling. In the year 2006, Ivanco and Jendrol stated a conjecture that ,

$$tes(G) = max\left\{\left\lceil\frac{E(G)+2}{3}\right\rceil, \left\lceil\frac{\Delta(G)+1}{2}\right\rceil\right\}$$

for an arbitrary graph G different from K_5.

This conjecture has been verified for all trees in [5]. The Ivanco and Jendrol's conjecture has been verified for K_n and $K_{m,n}$ in [6], for cartesian product of two paths in [8], for the corona product of a path with certain graphs in [9], for categorical product of cycle and path in [11], for a subdivision of stars in [12] and for hexagonal grid in [1].

In [10] we find the details for total absolute difference edge irregularity strength which we described here:"Ramalakshmi and Kathiresan introduced the concept of total absolute difference edge irregularity strength of graphs to reduce the edge weights. For a graph $G = (V(G), E(G))$, the weight of $e = xy$ under a total labeling ξ is $wt(e) = |\xi(e) - \xi(x) - \xi(y)|$. We define a labeling $\xi : V(G) \bigcup E(G) \rightarrow \{1, 2, \ldots, k\}$ as an edge irregular total absolute difference k-labeling of a graph G if for every two different edges $e = xy$ and $f = x_0y_0$ we have $wt(e) \neq wt(f)$. The total absolute difference edge irregular strength, $tades(G)$, is the minimum k such that G posses an edge irregular total absolute difference k-labeling". In [10], we find the following conjectures,

1. $tades(T) = max\left\{\frac{p}{2}, \frac{\Delta+1}{2}\right\}$ for a tree T on p vertices,
2. $tes(G) \leq tades(G)$.

Theorem 1.1. *Let $G = (V, E)$ be a graph. Then* $\left\lceil\frac{|E|}{2}\right\rceil \leq tades(G) \leq |E| + 1$.

Lourdusamy et al. [7] have computed the $tades(G)$ for snake related graphs, wheel related graphs, lotus inside the circle and double fan graph. Here, we investigate the total absolute difference edge irregularity strength of T_P-tree related graphs.

Definition 1.2. *[4] Let T be a tree and u_0 and v_0 be two adjacent vertices in T. Let there be two pendant vertices u and v in T such that the length of $u_0 - u$ path is equal to the length of $v_0 - v$ path. If the edge u_0v_0 is deleted from T and u, v are joined by an edge uv, then such a transformation of T is called an elementary parallel transformation (or an ept) and the edge u_0v_0 is called transformable edge.*

If by the sequence of ept's, T can be reduced to a path, then T is called a T_p-tree (transformed tree) and such a sequence regarded as a composition of mappings (ept's) denoted by P, is called a parallel transformation of T. The path, the image of T under P is denoted as $P(T)$. We use the notation $d(u, v)$ to denote the distance between the vertices of u and v.

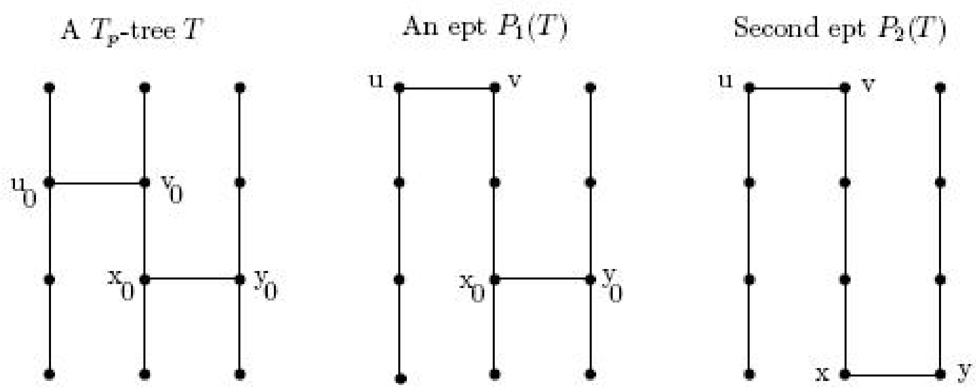

Figure 1. A T_p-tree and a sequence of two ept's reducing it to a path

Definition 1.3. *Let G_1 be a graph with p vertices and G_2 be any graph. A graph $G_1\widehat{O}G_2$ is obtained from G_1 and p copies of G_2 by identifying one vertex of i^{th} copy of G_2 with i^{th} vertex of G_1.*

Definition 1.4. *The corona $G_1 \odot G_2$ of two graphs $G_1(p_1, q_1)$ and $G_2(p_2, q_2)$ is defined as the graph obtained by taking one copy of G_1 and p_1 copies of G_2 and joining the i^{th} vertex of G_1 with an edge to every vertex in the i^{th} copy of G_2.*

2. Main Results

Theorem 2.1. *Let T be a T_p-tree on m vertices. Then* $tades(T\widehat{O}P_n) = \left\lceil \frac{mn-1}{2} \right\rceil$.

Proof. Consider T be a T_p-tree with m vertices. Applying the definition of a transformed tree we can find a parallel transformation P of T which will satisfy (i) $V(P(T)) = V(T)$ and (ii) $E(P(T)) = (E(T) - E_d) \bigcup E_p$, where $P(T)$ is the path, E_d is the set of edges removed from T and E_p is the set of edges newly introduced using the sequence $P = (P_1, P_2, \cdots, P_k)$ of the *epts* P used to arrive at the path $P(T)$. Obviously, we have the same number of edges for E_d and E_p. We use the label $\alpha_1, \alpha_2, \cdots, \alpha_m$ successively starting from one pendant vertex of $P(T)$ and proceed in the right direction up to the other pendant vertex to denote the vertices of $P(T)$. Let $\beta_1^s, \beta_2^s, \cdots, \beta_n^s (1 \leq s \leq m)$ be the vertices in s^{th} copy of P_n with $\beta_1^s = \alpha_s$. Then $V(T\widehat{O}P_n) = \{\alpha_s, \beta_r^s : 1 \leq r \leq n, 1 \leq s \leq m$ with $\beta_1^s = \alpha_s\}$ and $E(T\widehat{O}P_n) = E(T) \bigcup \{\beta_r^s \beta_{r+1}^s : 1 \leq r \leq n-1, 1 \leq s \leq m\}$.

By Theorem 1.1, we have $tades(T\widehat{O}P_n) \geq \left\lceil \frac{mn-1}{2} \right\rceil$. To prove the reverse inequality, we define the labeling $\xi : V \bigcup E \to \{1, 2, 3, \ldots, \left\lceil \frac{mn-1}{2} \right\rceil\}$ as follows:

Case 1. m is even, n is odd or even and m is odd, n is even.
When s is odd, $1 \leq s \leq m$ and $1 \leq r \leq n$,

$$\xi(\beta_r^s) = \begin{cases} \frac{n(s-1)}{2} + \frac{r+1}{2} & \text{if } r \text{ is odd} \\ \frac{n(s-1)}{2} + \frac{r}{2} & \text{if } r \text{ is even ;} \end{cases}$$

When s is even, $1 \leq s \leq m$ and $1 \leq r \leq n$,

$$\xi(\beta_r^s) = \begin{cases} \frac{ns}{2} - \frac{r-1}{2} & \text{if } r \text{ is odd} \\ \frac{ns}{2} - \frac{r}{2} + 1 & \text{if } r \text{ is even ;} \end{cases}$$

and

$$\xi(\beta_r^s \beta_{r+1}^s) = 2,\ 1 \leq s \leq m,\ 1 \leq r \leq n-1;$$
$$\xi(\alpha_s \alpha_{s+1}) = 2,\ 1 \leq s \leq m-1.$$

Case 2. m and n are odd.
When s is odd,

$$\xi(\beta_r^s) = \begin{cases} \frac{(s-1)n}{2} + \frac{r+1}{2} & \text{if } r \text{ is odd and } 1 \leq s \leq m-1,\ 1 \leq r \leq n \\ \frac{(s-1)n}{2} + \frac{r}{2} & \text{if } r \text{ is even and } 1 \leq s \leq m-1,\ 1 \leq r \leq n\ ; \end{cases}$$

$$\xi(\beta_r^m) = \begin{cases} \frac{(m-1)n}{2} + \frac{r+1}{2} & \text{if } r \text{ is odd and } 1 \leq r \leq n-1 \\ \frac{(m-1)n}{2} + \frac{r}{2} & \text{if } r \text{ is even and } 1 \leq r \leq n-1 \\ \frac{nm-1}{2} & \text{if } r = n\ ; \end{cases}$$

$$\xi(\beta_r^s\beta_{r+1}^s) = 2,\ 1 \leq s \leq m-1,\ 1 \leq r \leq n-1;$$

$$\xi(\beta_r^m\beta_{r+1}^m) = \begin{cases} 2 & \text{if } 1 \leq r \leq n-2 \\ 1 & \text{if } r = n-1\ ; \end{cases}$$

when s is even, $1 \leq s \leq m$ and $1 \leq r \leq n$,

$$\xi(\beta_r^s) = \begin{cases} \frac{nj}{2} - \frac{r-1}{2} & \text{if } r \text{ is odd} \\ \frac{ns}{2} - \frac{r}{2} + 1 & \text{if } r \text{ is even}\ ; \end{cases}$$

$$\xi(\beta_r^s\beta_{r+1}^s) = 2,\ 1 \leq r \leq n-1;$$

and

$$\xi(\alpha_s\alpha_{s+1}) = 2,\ 1 \leq s \leq m-1.$$

Take $\alpha_r\alpha_s$ be a transformed edge in T, $1 \leq r < s \leq m$ and take P_1 be the *ept* derived by removing the edge $\alpha_r\alpha_s$ and introducing the edge $\alpha_{r+t}\alpha_{s-t}$ where $t = d(\alpha_r, \alpha_{r+t}) = d(\alpha_s, \alpha_{s-t})$. Consider P to be a parallel transformation of T that contains P_1 as one of the constituent *epts*.

Note that $\alpha_{r+t}\alpha_{s-t}$ is an edge in the path $P(T)$. So $r+t+1 = s-t$ which implies $s = r+2t+1$. Therefore, r and s are of opposite parity.
The weight of edge $\alpha_r\alpha_s$ is defined as

$$\begin{aligned} wt(\alpha_r\alpha_s) &= wt(\alpha_r\alpha_{r+2t+1}) \\ &= |\xi(\alpha_r\alpha_{r+2t+1}) - \xi(\alpha_r) - \xi(\alpha_{r+2t+1})| \\ &= n(r+t) - 1. \end{aligned}$$

The weight of edge $\alpha_{r+t}\alpha_{s-t}$ is defined as

$$\begin{aligned} wt(\alpha_{r+t}\alpha_{s-t}) &= wt(\alpha_{r+t}\alpha_{r+t+1}) \\ &= |\xi(\alpha_{r+t}\alpha_{r+t+1}) - \xi(\alpha_{r+t}) - \xi(\alpha_{r+t+1})| \\ &= n(r+t) - 1. \end{aligned}$$

Therefore, $wt(\alpha_r\alpha_s) = wt(\alpha_{r+t}\alpha_{s-t})$.
The edge weights are as follows:

$$wt(\alpha_s\alpha_{s+1}) = ns - 1,\ 1 \leq s \leq m-1;$$

for $1 \leq r \leq n-1$ and $1 \leq s \leq m$,

$$wt(\beta_r^s\beta_{r+1}^s) = (s-1)n + r - 1,\ s \text{ is odd};$$

$$wt(\beta_r^s\beta_{r+1}^s) = sn - r - 1,\ s \text{ is even}.$$

It is a routine matter to verify that all vertex and edge labels are at most $\left\lceil \frac{mn-1}{2} \right\rceil$ and the edge weights are distinct. Hence $tades(T\widehat{O}P_n) = \left\lceil \frac{mn-1}{2} \right\rceil$.
□

Figure 2 illustrates a total absolute difference edge irregularity strength of $T\widehat{O}P_5$, for a T_p-tree T with 11 vertices.

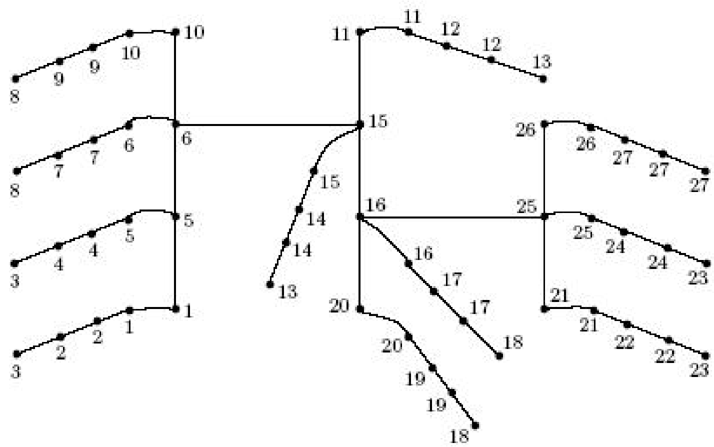

Figure 2.

Theorem 2.2. *For a T_p-tree T on m vertices, $tades(T\widehat{O}K_{1,n}) = \left\lceil \frac{(n+1)m-1}{2} \right\rceil$.*

Proof. By hypothesis, T is a T_p-tree with m vertices. Then there is a parallel transformation P of T with the property that for the path $P(T)$, the following are true

1. $V(P(T)) = V(T)$ and
2. $E(P(T)) = (E(T) - E_d) \bigcup E_p$, where E_d is the set of edges deleted from T and E_p stands for edges newly added through the sequence $P = (P_1, P_2, \cdots, P_k)$ of the *epts* P which is used to arrive at the path $P(T)$.

Clearly, $|E_d| = |E_p|$. Let us denote the successive vertices of $P(T)$ to be $\alpha_1, \alpha_2, \cdots, \alpha_m$ starting from one pendant vertex of $P(T)$ right up to the other. Let $\beta_0^s, \beta_1^s, \cdots, \beta_n^s (1 \leq s \leq m)$ to be the vertices of r^{th} copy of $K_{1,n}$ with $\beta_1^s = \alpha_s$. The vertex set of $V(T\widehat{O}K_{1,n}) = \{\alpha_s, \beta_0^s, \beta_r^s : 1 \leq r \leq n, 1 \leq s \leq m \text{ with } \alpha_s = \beta_1^s\}$. The edge set of $E(T\widehat{O}K_{1,n}) = E(T) \bigcup \{\beta_0^s \beta_r^s : 1 \leq s \leq m, 1 \leq r \leq n\}$.

By Theorem 1.1, we have $tades(T\widehat{O}K_{1,n}) \geq \left\lceil \frac{m(n+1)-1}{2} \right\rceil$. We now prove that $tades(T\widehat{O}K_{1,n}) \leq \left\lceil \frac{m(n+1)-1}{2} \right\rceil$. Define
$\xi : V \bigcup E \to \left\{1, 2, 3, \ldots, \left\lceil \frac{(n+1)m-1}{2} \right\rceil \right\}$ as follows:

Case 1. m is even; n is odd or even.

$$\xi(\alpha_s) = \begin{array}{ll} \frac{(s-1)(n+1)}{2} + 1 & \text{if } s \text{ is odd and } 1 \leq s \leq m \\ \frac{s(n+1)}{2} & \text{if } s \text{ is even and } 1 \leq s \leq m \text{ ;} \end{array}$$

$$\xi(\beta_0^s) = \begin{array}{ll} \frac{(s-1)(n+1)}{2} + 1 & \text{if } s \text{ is odd and } 1 \leq s \leq m \\ \frac{s(n+1)}{2} & \text{if } s \text{ is even and } 1 \leq s \leq m \text{ ;} \end{array}$$

$$\xi(\beta_r^s) = \begin{array}{ll} \frac{(s-1)(n+1)}{2} + r & \text{if } s \text{ is odd and } 1 \leq s \leq m,\ 2 \leq r \leq n \\ \frac{s(n+1)}{2} - r + 1 & \text{if } s \text{ is even and } 1 \leq s \leq m,\ 2 \leq r \leq n; \end{array}$$

$\xi(\alpha_s\alpha_{s+1}) = 2,\ 1 \leq s \leq m-1;$
$\xi(\alpha_s\beta_0^s) = 2,\ 1 \leq s \leq m;$
$\xi(\beta_0^s\beta_r^s) = 2,\ 1 \leq s \leq m,\ 2 \leq r \leq n.$

Case 2. m is odd; n is odd or even.

$$\xi(\alpha_s) = \begin{cases} \frac{(s-1)(n+1)}{2} + 1 & \text{if } s \text{ is odd and } 1 \leq s \leq m \\ \frac{s(n+1)}{2} & \text{if } s \text{ is even and } 1 \leq s \leq m \text{ ;} \end{cases}$$

$$\xi(\beta_0^s) = \begin{cases} \frac{(s-1)(n+1)}{2} + 1 & \text{if } s \text{ is odd and } 1 \leq s \leq m-1 \\ \frac{s(n+1)}{2} & \text{if } s \text{ is even and } 1 \leq s \leq m-1 \\ \frac{(n+1)(m-1)}{2} + \left\lceil \frac{n}{2} \right\rceil & \text{if } s = m; \end{cases}$$

$$\xi(\beta_r^s) = \begin{cases} \frac{(s-1)(n+1)}{2} + r & \text{if } s \text{ is odd and } 1 \leq s \leq m-1,\ 2 \leq r \leq n \\ \frac{s(n+1)}{2} - r + 1 & \text{if } s \text{ is even and } 1 \leq s \leq m-1,\ 2 \leq r \leq n; \end{cases}$$

$$\xi(\beta_r^m) = \begin{cases} \frac{(m-1)(n+1)}{2} + \frac{r}{2} & \text{if } r \text{ is even and } 2 \leq r \leq n, \\ \frac{(m-1)(n+1)}{2} + \frac{r+1}{2} & \text{if } r \text{ is odd and } 2 \leq r \leq n; \end{cases}$$

$\xi(\alpha_s\alpha_{s+1}) = 2,\ 1 \leq s \leq m-1;$
$\xi(\alpha_s\beta_0^s) = 2,\ 1 \leq s \leq m-1;$
$\xi(\alpha_m\beta_0^m) = \left\lceil \frac{n}{2} \right\rceil + 1;$
$\xi(\beta_0^s\beta_r^s) = 2,\ 1 \leq s \leq m-1,\ 2 \leq r \leq n;$

$$\xi(\beta_0^m\beta_r^m) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil - \frac{r-2}{2} & \text{if } r \text{ is even and } 2 \leq r \leq n \\ \left\lceil \frac{n}{2} \right\rceil - \frac{r-3}{2} & \text{if } r \text{ is odd and } 2 \leq r \leq n. \end{cases}$$

Let $\alpha_r\alpha_s$ be a transformed edge in T, $1 \leq r < s \leq m$ and let P_1 be the *ept* obtained by removing the edge $\alpha_r\alpha_s$ and including the edge $\alpha_{r+t}\alpha_{s-t}$ where $t = d(\alpha_r, \alpha_{r+t}) = d(\alpha_s, \alpha_{s-t})$. Take P to be a parallel transformation

of T containing P_1 as one of the constituent *epts*. Obviously $\alpha_{r+t}\alpha_{s-t}$ is an edge in the path $P(T)$. So $r+t+1=s-t$ and thus $s=r+2t+1$. Clearly, s and t are of opposite parity.
The weight of the edge $\alpha_r\alpha_s$ is

$$\begin{aligned} wt(\alpha_r\alpha_s) &= wt(\alpha_r\alpha_{r+2t+1}) \\ &= |\xi(\alpha_r\alpha_{r+2t+1}) - \xi(\alpha_r) - \xi(\alpha_{r+2t+1})| \\ &= (n+1)(r+t)-1. \end{aligned}$$

The weight of the edge $\alpha_{r+t}\alpha_{s-t}$ is

$$\begin{aligned} wt(\alpha_{r+t}\alpha_{s-t}) &= wt(\alpha_{r+t}\alpha_{r+t+1}) \\ &= |\xi(\alpha_{r+t}\alpha_{r+t+1}) - \xi(\alpha_{r+t}) - \xi(\alpha_{r+t+1})| \\ &= (n+1)(r+t)-1. \end{aligned}$$

The above argument implies that $wt(\alpha_r\alpha_s) = wt(\alpha_{r+t}\alpha_{s-t})$.
The edge weights are:

$$wt(\alpha_s\alpha_{s+1}) = s(n+1)-1, \ 1 \le s \le m-1;$$

for $1 \le s \le m$,

$$wt(\alpha_s\beta_0^s) = \begin{cases} (s-1)(n+1) & \text{if } s \text{ is odd} \\ s(n+1)-2 & \text{if } s \text{ is even;} \end{cases}$$

$$wt(\beta_0^s\beta_r^s) = \begin{cases} (s-1)(n+1)+r-1 & \text{if } s \text{ is odd and } 2 \le r \le n \\ s(n+1)-r-1 & \text{if } s \text{ is even and } 2 \le r \le n. \end{cases}$$

Clearly, $tades(T\widehat{O}K_{1,n}) \le \left\lceil \frac{(n+1)m-1}{2} \right\rceil$. Note that the edge weights are distinct. Hence $tades(T\widehat{O}K_{1,n}) = \left\lceil \frac{(n+1)m-1}{2} \right\rceil$. □

Figure 3 illustrates a total absolute difference edge irregularity strength of $T\widehat{O}K_{1,3}$ for a T_p-tree T with 11 vertices.

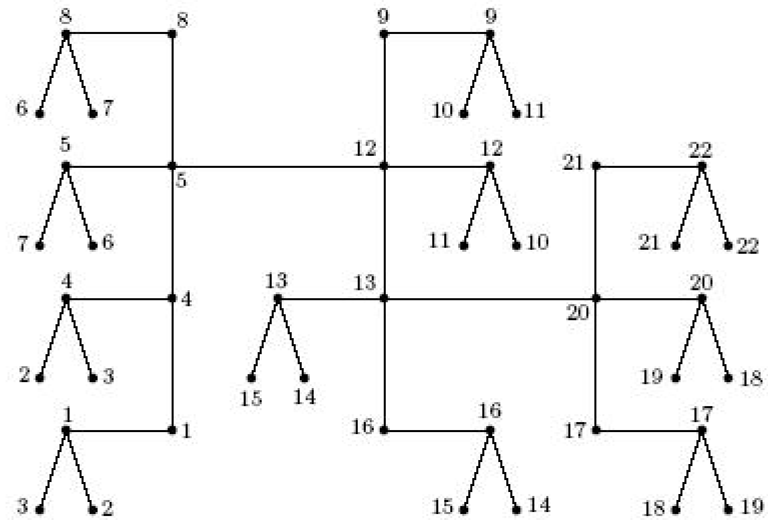

Figure 3.

Theorem 2.3. *Let T be a T_p-tree on m vertices, $tades(T\widehat{O}C_n) = \left\lceil \frac{mn+m-1}{2} \right\rceil$.*

Proof. Consider T to be a T_p-tree with m vertices. By the definition of a transformed tree there exists a parallel transformation P of T for which for the path $P(T)$, the following two results will hold,

1. $V(P(T)) = V(T)$ and
2. $E(P(T)) = (E(T) - E_d) \bigcup E_p$, where E_d is the set of edges removed from T and E_p is the set of edges newly introduced through the sequence $P = (P_1, P_2, \cdots, P_k)$ of the *epts* P used to arrive at the path $P(T)$.

Clearly, $|E_d| = |E_p|$. We denote the vertices of $P(T)$ successively as $\alpha_1, \alpha_2, \cdots, \alpha_m$ starting from one pendant vertex of $P(T)$ right up to the other. Let $\beta_1^s, \beta_2^s, \cdots, \beta_n^s$ $(1 \leq s \leq m)$ be the vertices in the s^{th} copy of C_n with $\beta_1^s = \alpha_s$. Then $V(T\widehat{O}C_n) = \{\beta_r^s : 1 \leq r \leq n, 1 \leq s \leq m\}$ and $E(T\widehat{O}C_n) = E(T) \bigcup E(C_n)$.

By Theorem 1.1, we have $tades(T\widehat{O}C_n) \geq \left\lceil \frac{mn+m-1}{2} \right\rceil$. For the reverse inequality, it is enough to show that $tades(T\widehat{O}C_n) \leq \left\lceil \frac{mn+m-1}{2} \right\rceil$.

Case 1. Let $n \equiv 0$ $(mod\ 4)$.
Choose s is odd and $1 \leq s \leq m$,

$\xi(\beta_1^s) = \frac{s(n+1)-1}{2}$;

$$\xi(\beta_r^s) = \begin{cases} \frac{(s-1)(n+1)}{2} + \frac{r}{2} & \text{if } r \equiv 0\ (mod\ 2) \text{ and } 2 \leq r \leq n \\ \frac{(s-1)(n+1)}{2} + \frac{r-1}{2} & \text{if } r \equiv 1\ (mod\ 2) \text{ and } 2 \leq r \leq n\ ; \end{cases}$$

$\xi(\beta_1^s\beta_2^s) = \frac{n}{2}$; $\quad \xi(\beta_2^s\beta_3^s) = 2$;

$\xi(\beta_n^s\beta_1^s) = 1 = \xi(\beta_r^s\beta_{r+1}^s),\ 3 \leq r \leq n-1$;

choose s is even and $1 \leq s \leq m$,

$\xi(\beta_1^s) = \frac{(n+1)(s-1)+1}{2}$;

$$\xi(\beta_r^s) = \begin{cases} \frac{(s-1)(n+1)-1}{2} + \frac{r}{2} & \text{if } r \equiv 0\ (mod\ 2) \text{ and } 2 \leq r \leq \frac{n}{2} \\ \frac{(s-1)(n+1)+1}{2} + \frac{r}{2} & \text{if } r \equiv 0\ (mod\ 2) \text{ and } \frac{n}{2}+1 \leq r \leq n \\ \frac{(s-1)(n+1)+1}{2} + \frac{r-1}{2} & \text{if } r \equiv 1\ (mod\ 2) \text{ and } 2 \leq r \leq n\ ; \end{cases}$$

$\xi(\beta_n^s\beta_1^s) = 1 = \xi(\beta_r^s\beta_{r+1}^s),\ 1 \leq r \leq n-1$;

and $\xi(\alpha_s\alpha_{s+1}) = 1,\ 1 \leq s \leq m-1$.

Let $\alpha_r\alpha_s$ be a transformed edge in T, $1 \leq r < s \leq m$ and let P_1 be the *ept* derived by deleting the edge $\alpha_r\alpha_s$ and including the edge $\alpha_{r+t}\alpha_{s-t}$ where $t = d(\alpha_r, \alpha_{r+t}) = d(\alpha_s, \alpha_{s-t})$. Consider P to be a parallel transformation of T that contains P_1 as one of the constituent *epts*.

Note that $\alpha_{r+t}\alpha_{s-t}$ is an edge in the path $P(T)$, it follows that $r+t+1 = s-t$ which implies $s = r+2t+1$. Therefore, s and r are of opposite parity.

The weight of the edge $\alpha_r\alpha_s$ is

$$\begin{aligned} wt(\alpha_r\alpha_s) &= wt(\alpha_r\alpha_{r+2t+1}) \\ &= (n+1)(r+t)-1. \end{aligned}$$

The weight of the edge $\alpha_{r+t}\alpha_{s-t}$ is

$$\begin{aligned} wt(\alpha_{r+t}\alpha_{s-t}) &= wt(\alpha_{r+t}\alpha_{r+t+1}) \\ &= (n+1)(r+t)-1. \end{aligned}$$

Therefore, $wt(\alpha_r\alpha_s) = wt(\alpha_{r+t}\alpha_{s-t})$.

The edge weights are caculated as follows:

$$wt(\alpha_s\alpha_{s+1}) = s(n+1)-1,\ 1 \leq s \leq m-1;$$

for $1 \leq s \leq m$ and $1 \leq r \leq n-1$,

$$wt(\beta_r^s\beta_{r+1}^s) = \begin{cases} (s-1)(n+1)+r & \text{if } r = 1 \text{ and } s \text{ is odd} \\ (s-1)(n+1)+r-2 & \text{if } r = 2 \text{ and } s \text{ is odd} \\ (s-1)(n+1)+r-1 & \text{if } r \geq 3 \text{ and } s \text{ is odd}\ ; \end{cases}$$

$$wt(\beta_r^s\beta_{r+1}^s) = \begin{cases} (s-1)(n+1)+r-1 & \text{if } 1 \leq r \leq \frac{n}{2} \text{ and } s \text{ is even} \\ (s-1)(n+1)+r & \text{if } \frac{n}{2}+1 \leq r \leq n-1 \text{ and } s \text{ is even}\ ; \end{cases}$$

$$wt(\beta_n^s\beta_1^s) = \begin{cases} (s-1)(n+1)+n-1 & \text{if } s \text{ is odd} \\ (s-1)(n+1)+\frac{n}{2} & \text{if } s \text{ is even.} \end{cases}$$

Case 2. Let $n \equiv 2\ (mod\ 4)$.
Choose s to be odd and $1 \leq s \leq m$,

$$\xi(\beta_1^s) = \tfrac{s(n+1)-1}{2};$$
$$\xi(\beta_r^s) = \begin{cases} \frac{(s-1)(n+1)}{2}+\frac{r}{2} & \text{if } r \equiv 0\ (mod\ 2) \text{ and } 2 \leq r \leq n \\ \frac{(s-1)(n+1)}{2}+\frac{r-1}{2} & \text{if } r \equiv 1\ (mod\ 2) \text{ and } 2 \leq r \leq n\ ; \end{cases}$$
$$\xi(\beta_1^s\beta_2^s) = \tfrac{n}{2}; \quad \xi(\beta_2^s\beta_3^s) = 2;$$
$$\xi(\beta_n^s\beta_1^s) = 1 = \xi(\beta_r^s\beta_{r+1}^s),\ 3 \leq r \leq n-1;$$
choose s to be even and $1 \leq s \leq m$,
$$\xi(\beta_1^s) = \tfrac{(s-1)(n+1)+1}{2};$$
$$\xi(\beta_r^s) = \begin{cases} \frac{(s-1)(n+1)-1}{2}+\frac{r}{2} & \text{if } r \equiv 0\ (mod\ 2) \text{ and } 2 \leq r \leq \frac{n}{2}-1 \\ \frac{(s-1)(n+1)+1}{2}+\frac{r}{2} & \text{if } r \equiv 0\ (mod\ 2) \text{ and } \frac{n}{2} \leq r \leq n \\ \frac{(s-1)(n+1)+1}{2}+\frac{r-1}{2} & \text{if } r \equiv 1\ (mod\ 2) \text{ and } 2 \leq r \leq n\ ; \end{cases}$$
$$\xi(\beta_n^s\beta_1^s) = 2;\ \xi(\beta_r^s\beta_{r+1}^s) = 1,\ 1 \leq r \leq n-1;$$
and $\xi(\alpha_s\alpha_{s+1}) = 1,\ 1 \leq s \leq m-1$.

Let $\alpha_r\alpha_s$ be a transformed edge in T, $1 \leq r < s \leq m$ and let P_1 be the *ept* obtained by removing the edge $\alpha_r\alpha_s$ and including the edge $\alpha_{r+t}\alpha_{s-t}$ where $t = d(\alpha_r, \alpha_{r+t}) = d(\alpha_s, \alpha_{s-t})$. Take P to be a parallel transformation of T that contains P_1 as one of the constituent *epts*.

Since $\alpha_{r+t}v_{s-t}$ is an edge in the path $P(T)$. Clearly, $r+t+1 = s-t$ gives $s = r+2t+1$. Therefore, r and s are of opposite parity.

The weight of edge $\alpha_r\alpha_s$ is
$$\begin{aligned} wt(\alpha_r\alpha_s) &= wt(\alpha_r\alpha_{r+2t+1}) \\ &= (n+1)(r+t)-1. \end{aligned}$$
The weight of edge $\alpha_{r+t}\alpha_{s-t}$ is
$$\begin{aligned} wt(\alpha_{r+t}\alpha_{s-t}) &= wt(\alpha_{r+t}\alpha_{r+t+1}) \\ &= (n+1)(r+t)-1. \end{aligned}$$
Therefore, $wt(\alpha_r\alpha_s) = wt(\alpha_{r+t}\alpha_{s-t})$.
The edge weights are obtained as follows:

$$wt(\alpha_s\alpha_{s+1}) = s(n+1)-1,\ 1 \leq s \leq m-1;$$
for $1 \leq s \leq m$ and $1 \leq r \leq n-1$,

$$wt(\beta_r^s\beta_{r+1}^s) = \begin{cases} (s-1)(n+1)+r & \text{if } r=1 \text{ and } s \text{ is odd} \\ (s-1)(n+1)+r-2 & \text{if } r=2 \text{ and } s \text{ is odd} \\ (s-1)(n+1)+r-1 & \text{if } r\geq 3 \text{ and } s \text{ is odd ;} \end{cases}$$

$$wt(\beta_r^s\beta_{r+1}^s) = \begin{cases} (s-1)(n+1)+r-1 & \text{if } 1\leq r\leq \frac{n}{2}-1 \text{ and } s \text{ is even} \\ (s-1)(n+1)+r & \text{if } \frac{n}{2}\leq r\leq n-1 \text{ and } s \text{ is even ;} \end{cases}$$

$$wt(\beta_n^s\beta_1^s) = \begin{cases} (s-1)(n+1)+n-1 & \text{if } s \text{ is odd} \\ (s-1)(n+1)+\frac{n}{2}-1 & \text{if } s \text{ is even .} \end{cases}$$

Case 3. Let $n \equiv 3$ $(mod\ 4)$.
Choose s is odd and $1 \leq s \leq m$,

$\xi(\beta_1^s) = \lceil \frac{n}{2} \rceil s$;

$$\xi(\beta_r^s) = \begin{cases} \lceil \frac{n}{2} \rceil (s-1)+\frac{r}{2} & \text{if } r\equiv 0\ (mod\ 2) \text{ and } 2\leq r\leq n \\ \lceil \frac{n}{2} \rceil (s-1)+\frac{r-1}{2} & \text{if } r\equiv 1\ (mod\ 2) \text{ and } 2\leq r\leq n \text{ ;} \end{cases}$$

$\xi(\beta_1^s\beta_2^s) = \lceil \frac{n}{2} \rceil$; $\quad \xi(\beta_2^s\beta_3^s) = 2$;

$\xi(\beta_n^s\beta_1^s) = 1 = \xi(\beta_r^s\beta_{r+1}^s),\ 3 \leq r \leq n-1$;

choose s is even and $1 \leq s \leq m$,

$\xi(\beta_1^s) = \lceil \frac{n}{2} \rceil (s-1)$;

$$\xi(\beta_1^s) = \begin{cases} \lceil \frac{n}{2} \rceil (s-1)+\frac{r}{2} & \text{if } r\equiv 0\ (mod\ 2) \text{ and } 2\leq r\leq n \\ \lceil \frac{n}{2} \rceil (s-1)+\frac{r-1}{2} & \text{if } r\equiv 1\ (mod\ 2) \text{ and } 2\leq r\leq \lceil \frac{n}{2} \rceil \\ \lceil \frac{n}{2} \rceil (s-1)+\frac{r-1}{2}+1 & \text{if } r\equiv 1\ (mod\ 2) \text{ and } \lceil \frac{n}{2} \rceil+1\leq r\leq n \text{ ;} \end{cases}$$

$\xi(\beta_n^s\beta_1^s) = 1;\ \xi(\beta_r^s\beta_{r+1}^s) = 1,\ 1 \leq r \leq n-1$;

and $\xi(\alpha_s\alpha_{s+1}) = 1,\ 1 \leq s \leq m-1$.

Let $\alpha_r\alpha_s$ be a transformed edge in T, $1 \leq r < s \leq m$ and let P_1 be the *ept* obtained by removing the edge $\alpha_r\alpha_s$ and including the edge $\alpha_{r+t}\alpha_{s-t}$ where $t = d(\alpha_r, \alpha_{r+t}) = d(\alpha_s, \alpha_{s-t})$. Let P be a parallel transformation of T that contains P_1 as one of the constituent *epts*.

Clearly, $\alpha_{r+t}\alpha_{s-t}$ is an edge in the path $P(T)$. So $r+t+1 = s-t$ which implies $s = r+2t+1$. Therefore, r and s are of opposite parity.

The weight of edge $\alpha_r\alpha_s$ is given by

$$\begin{aligned} wt(\alpha_r\alpha_s) &= wt(\alpha_r\alpha_{r+2t+1}) \\ &= (n+1)(r+t)-1. \end{aligned}$$

The weight of edge $\alpha_{r+t}\alpha_{s-t}$ is given by

$$\begin{aligned} wt(\alpha_{r+t}\alpha_{s-t}) &= wt(\alpha_{r+t}\alpha_{r+t+1}) \\ &= (n+1)(r+t)-1. \end{aligned}$$

Therefore, $wt(\alpha_r\alpha_s) = wt(\alpha_{r+t}\alpha_{s-t})$.
The edge weights are computed as follows:

$$wt(\alpha_s\alpha_{s+1}) = s(n+1)-1,\ 1 \le s \le m-1;$$

for $1 \le s \le m$ and $1 \le r \le n-1$,

$$wt(\beta_r^s\beta_{r+1}^s) = \begin{cases} (s-1)(n+1)+r & \text{if } r=1 \text{ and } s \text{ is odd} \\ (s-1)(n+1)+r-2 & \text{if } r=2 \text{ and } s \text{ is odd} \\ (s-1)(n+1)+r-1 & \text{if } r \ge 3 \text{ and } s \text{ is odd }; \end{cases}$$

$$wt(\beta_r^s\beta_{r+1}^s) = \begin{cases} (s-1)(n+1)+r-1 & \text{if } 1 \le r \le \frac{n-1}{2} \text{ and } s \text{ is even} \\ (s-1)(n+1)+r & \text{if } \frac{n+1}{2} \le r \le n-1 \text{ and } s \text{ is even }; \end{cases}$$

$$wt(\beta_n^s\beta_1^s) = \begin{cases} (s-1)(n+1)+n-1 & \text{if } s \text{ is odd} \\ (s-1)(n+1)+\frac{n-1}{2} & \text{if } s \text{ is even }. \end{cases}$$

Case 4. Let $n \equiv 1\ (mod\ 4)$.
Choose s is odd and $1 \le s \le m$,

$$\xi(\beta_1^s) = \left\lceil \tfrac{n}{2} \right\rceil s;$$

$$\xi(\beta_r^s) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil (s-1) + \frac{r}{2} & \text{if } r \equiv 0\ (mod\ 2) \text{ and } 2 \le r \le n \\ \left\lceil \frac{n}{2} \right\rceil (s-1) + \frac{r-1}{2} & \text{if } r \equiv 1\ (mod\ 2) \text{ and } 2 \le r \le n\ ; \end{cases}$$

$$\xi(\beta_1^s\beta_2^s) = \left\lceil \tfrac{n}{2} \right\rceil; \quad \xi(\beta_2^s\beta_3^s) = 2;$$

$$\xi(\beta_n^s\beta_1^s) = 1 = \xi(\beta_r^s\beta_{r+1}^s),\ 3 \le r \le n-1;$$

choose s is even and $1 \le s \le m$,

$$\xi(\beta_1^s) = \left\lceil \tfrac{n}{2} \right\rceil (s-1);$$

$$\xi(\beta_r^s) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil (s-1) + \frac{r}{2} & \text{if } r \equiv 0\ (mod\ 2) \text{ and } 2 \le r \le n \\ \left\lceil \frac{n}{2} \right\rceil (s-1) + \frac{r-1}{2} & \text{if } r \equiv 1\ (mod\ 2) \text{ and } 2 \le r \le \left\lfloor \frac{n}{2} \right\rfloor \\ \left\lceil \frac{n}{2} \right\rceil (s-1) + \frac{r-1}{2} + 1 & \text{if } r \equiv 1\ (mod\ 2) \text{ and } \left\lfloor \frac{n}{2} \right\rfloor + 1 \le r \le n\ ; \end{cases}$$

$$\xi(\beta_n^s\beta_1^s) = 2;\ \xi(\beta_r^s\beta_{r+1}^s) = 1,\ 1 \le r \le n-1;$$

and $\xi(\alpha_s\alpha_{s+1}) = 1,\ 1 \le s \le m-1$.

Let $\alpha_r\alpha_s$ be a transformed edge in T, $1 \le r < s \le m$ and let P_1 be the *ept* obtained by removing the edge $\alpha_r v = \alpha_s$ and including the edge $\alpha_{r+t}\alpha_{s-t}$ where $t = d(\alpha_r, \alpha_{r+t}) = d(\alpha s, \alpha_{s-t})$. Take P to be a parallel transformation of T containing P_1 as one of the constituent *epts*.

Since $\alpha_{r+t}\alpha_{s-t}$ is an edge in the path $P(T)$. So $r+t+1 = s-t$ and thus $s = r+2t+1$. Therefore, r and s are of opposite parity.

The weight of edge $\alpha_r\alpha_s$ is

$$\begin{aligned} wt(\alpha_r\alpha_s) &= wt(\alpha_r\alpha_{r+2t+1}) \\ &= (n+1)(r+t)-1. \end{aligned}$$

The weight of edge $\alpha_{r+t}\alpha_{s-t}$ is

$$\begin{aligned} wt(\alpha_{r+t}\alpha_{s-t}) &= wt(\alpha_{r+t}\alpha_{r+t+1}) \\ &= (n+1)(r+t)-1. \end{aligned}$$

Therefore, $wt(\alpha_r\alpha_s) = wt(\alpha_{r+t}\alpha_{s-t})$.

The edge weights are determined as follows:

$$wt(\alpha_s\alpha_{s+1}) = s(n+1)-1,\ 1 \le s \le m-1;$$

for $1 \le s \le m$ and $1 \le r \le n-1$,

$$wt(\beta_r^s\beta_{r+1}^s) = \begin{cases} (s-1)(n+1)+r & \text{if } r=1 \text{ and } s \text{ is odd} \\ (s-1)(n+1)+r-2 & \text{if } r=2 \text{ and } s \text{ is odd} \\ (s-1)(n+1)+r-1 & \text{if } r \ge 3 \text{ and } s \text{ is odd}\ ; \end{cases}$$

$$wt(\beta_r^s\beta_{r+1}^s) = \begin{cases} (s-1)(n+1)+r-1 & \text{if } 1 \le r \le \frac{n-3}{2} \text{ and } s \text{ is even} \\ (s-1)(n+1)+r & \text{if } \frac{n-1}{2} \le r \le n-1 \text{ and } s \text{ is even}\ ; \end{cases}$$

$$wt(\beta_n^s\beta_1^s) = \begin{cases} (s-1)(n+1)+n-1 & \text{if } s \text{ is odd} \\ (s-1)(n+1)+\frac{n-3}{2} & \text{if } s \text{ is even}\ . \end{cases}$$

Clearly, $tades(T\widehat{O}C_n) \le \left\lceil \frac{mn+m-1}{2} \right\rceil$ and the edge weights are distinct. Hence $tades(T\widehat{O}C_n) = \left\lceil \frac{mn+m-1}{2} \right\rceil$. □

A total absolute difference edge irregularity strength of $T\widehat{O}C_7$ where T is a T_p-tree with 8 vertices is shown in Figure 4.

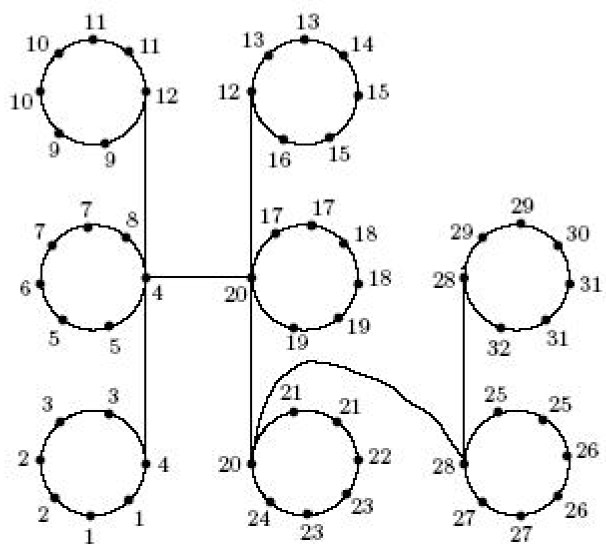

Figure 4.

Theorem 2.4. *For a T_p-tree T with even number of vertices, $tades(T \odot nK_1) = \left\lceil \frac{mn+m-1}{2} \right\rceil$.*

Proof. By hypothesis, the T_p-tree T has m vertices where m is even. Applying the definition of T_p-tree there is a parallel transformation P of T with the property that for the path $P(T)$, we have

1. $V(P(T)) = V(T)$ and
2. $E(P(T)) = (E(T) - E_d) \bigcup E_p$, where E_d is the set of edges removed from T and E_p is the set of edges newly introduced through the sequence $P = (P_1, P_2, \cdots, P_k)$ of the *epts* P used to arrive at the path $P(T)$.

Clearly, $|E_d| = |E_p|$. Let us denote the vertices of $P(T)$ successively as $\alpha_1, \alpha_2, \cdots, \alpha_m$ starting from one pendant vertex of $P(T)$ right up to the other. Let $\beta_1^s, \beta_2^s, \cdots, \beta_n^s (1 \leq s \leq m)$ be the pendant vertices joined with $\alpha_s (1 \leq s \leq m)$ by an edge. Then $V(T \odot nK_1) = \{\alpha_s, \beta_r^s : 1 \leq r \leq n, 1 \leq s \leq m\}$ and $E(T \odot nK_1) = E(T) \bigcup \{\alpha_s \beta_r^s :\ 1 \leq s \leq m,\ 1 \leq r \leq n\}$.

By Theorem 1.1, we have $tades(T \odot nK_1) \geq \left\lceil \frac{mn+m-1}{2} \right\rceil$. For the reverse inequality, we define the labeling $\xi : V \bigcup E \to \{1, 2, 3, \ldots, \left\lceil \frac{mn+m-1}{2} \right\rceil\}$ as follows:

For $1 \leq s \leq m$,

$$\xi(\alpha_s) = \begin{cases} \frac{(s-1)(n+1)}{2} + 1 & \text{if } s \text{ is odd} \\ \frac{s(n+1)}{2} & \text{if } s \text{ is even ;} \end{cases}$$
$$\xi(\beta_r^s) = \begin{cases} \frac{(s-1)(n+1)}{2} + r & \text{if } s \text{ is odd } 1 \leq r \leq n \\ \frac{(s-2)(n+1)}{2} + r + 1 & \text{if } s \text{ is even } 1 \leq r \leq n \text{ ;} \end{cases}$$
$$\xi(\alpha_s \alpha_{s+1}) = 2,\ 1 \leq s \leq m-1;$$
$$\xi(\beta_r^s \alpha_s) = 2,\ 1 \leq s \leq m,\ 1 \leq r \leq n.$$

Let $\alpha_r \alpha_s$ be a transformed edge in T, $1 \leq r < s \leq m$ and let P_1 be the *ept* obtained by removing the edge $\alpha_r \alpha_s$ and including the edge $\alpha_{r+t} \alpha_{s-t}$ where t is the distance of α_r from α_{r+t} and the distance of α_s from α_{s-t}. Take P to be a parallel transformation of T that contains P_1 as one of the constituent *epts*.

Note that $\alpha_{r+t}\alpha_{s-t}$ is an edge in the path $P(T)$. So $r+t+1 = s-t$ which implies $s = r+2t+1$. Therefore, r and s are of opposite parity.

The weight of edge $\alpha_r\alpha_s$ is

$$\begin{aligned} wt(\alpha_r\alpha_s) &= wt(\alpha_r\alpha_{r+2t+1}) \\ &= |\xi(\alpha_r\alpha_{r+2t+1}) - \xi(\alpha_r) - \xi(\alpha_{r+2t+1})| \\ &= (n+1)(r+t) - 1. \end{aligned}$$

The weight of edge $\alpha_{r+t}\alpha_{s-t}$ is

$$\begin{aligned} wt(\alpha_{r+t}\alpha_{s-t}) &= wt(\alpha_{r+t}\alpha_{r+t+1}) \\ &= |\xi(\alpha_{r+t}\alpha_{r+t+1}) - \xi(\alpha_{r+t}) - \xi(\alpha_{r+t+1})| \\ &= (n+1)(r+t) - 1. \end{aligned}$$

Therefore, $wt(\alpha_r\alpha_s) = wt(\alpha_{r+t}\alpha_{s-t})$.
The edge weights are obtained as follows:

$$wt(\alpha_s\alpha_{s+1}) = s(n+1) - 1, \ 1 \le s \le m-1;$$

for $1 \le s \le m$,

$$wt(\alpha_s\beta_r^s) = (s-1)(n+1) + r - 1, \ 1 \le r \le n.$$

Thus the edge weights are distinct. Hence $tades(T \odot nK_1) = \left\lceil \frac{mn+m-1}{2} \right\rceil$.
□

Figure 5 illustrates a total absolute difference edge irregularity strength of $T \odot 5K_1$ where T is a T_p-tree with 10 vertices .

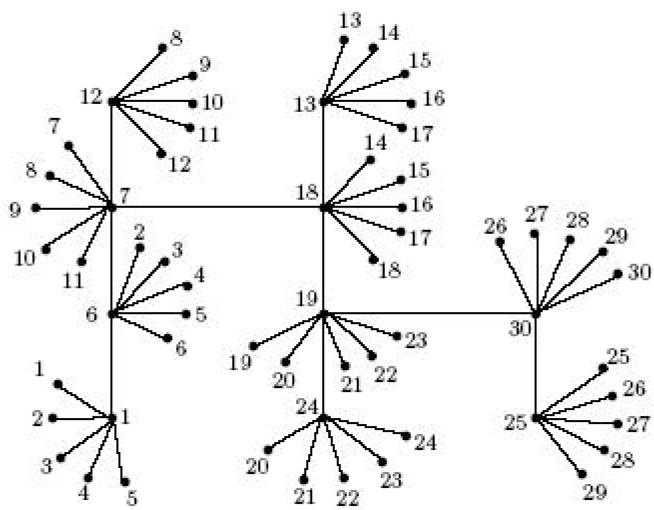

Figure 5.

References

[1] O. Al-Mushayt, A. Ahmad, and M. K. Siddiqui, "On the total edge irregularity strength of Hexagonal Grid graphs", *Austtralas. J. Combin.*, vol. 53, pp. 263-271, 2012.

[2] M. Baca, S. Jendrol, M. Miller and J. Ryan, "On irregular total labeling", *Discrete Math.*, vol. 307, pp. 1378-1388, 2007. doi: 10.1016/j.disc.2005.11.075

[3] G. Chartrand, M. S. Jacobson, J. Lehel, O. R. Oellermann, S. Ruiz and F. Saba, "Irregular networks", *Congr. Numer.*, vol. 64, pp. 187-192, 1988.

[4] S. M. Hegde and Sudhakar Shetty, "On Graceful Trees", *Applied Mathematics E-Notes*, vol. 2, pp. 192-197, 2002.

[5] J. Ivanco and S. Jendrol, "Total edge irregularity strength for trees", *Discuss. Math. Graph Theory*, vol. 26, pp. 449-456, 2006. doi: 10.7151/dmgt.1337

[6] S. Jendrol, J. Miskuf and R. Sotak, "Total edge irregularity strength of complete graphs and complete bipartite graphs", *Discrete Math.*, vol. 310, pp. 400-407, 2010. doi: 10.1016/j.disc.2009.03.006

[7] A. Lourdusamy and F. Joy Beaula, "Total absolute difference edge irregularity strength of some families of graphs", *TWMS Journal of Applied and Engineering Mathematics*, vol. 13, no. 3, pp. 1005-1012, 2023.

[8] J. Miskuf and S. Jendrol, "On the total edge irregularity strength of the grids", *Tatra Mt. Math Publ.*, vol. 36, pp. 147-151, 2007.

[9] Nurdin, A. N. M. Salman and E.T. Baskaro, "The total edge irregularity strength of the corona product of paths with some graphs", J.*Combin. Math. Combin. Comput.*, vol. 65, pp. 163-175, 2008.

[10] M. Ramalakshmi and K. M. Kathiresan, "Total absolute difference edge irregularity strength of the graph", *Kragujevac Journal of Mathematics*, vol. 46, no. 6, pp. 895-903, 2022. doi: 10.1016/j.disc.2009.03.006

[11] M. K. Siddiqui, "On total edge irregularity strength of categorical product of cycle and path", *AKCE Int. J. Graphs Comb.*, vol. 9, no. 1, pp. 43-52, 2012. doi: 10.1080/09728600.2012.12088948.

[12] M. K. Siddiqui, "On Tes of subdivision of star", *Int. J. of Math and Soft Compu.*, vol. 2, no. 1, pp. 75-82, 2012.

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