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A theoretical approach on intuitionistic Fuzzy Hausdorff space

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Abstract

The object of this paper is to introduce a new definition for intuitionistic fuzzy Hausdorff space (IFHS). We investigate some of its characterizations and discuss it with some necessary counter examples. In addition, we compared the new notion with the existing notions. Finally we point out the significance of Hausdorffness in digital image processing.

Keywords: *Intuitionistic fuzzy sets, Intutionistic fuzzy topological space, Nearly intuitionistic fuzzy Hausdorff space, Intuitionistic fuzzy Hausdorff space, Intuitionistic fuzzy closed.*

Mathematics Subject Classification: *03B52, 03E72, 54A40.*

1. Introduction

The concept of fuzziness exists almost everywhere in our daily life. To confront the difficulty due to ambiguity, Zadeh [16] proposed the fuzzy theory in 1965 and it was generalised into notion of intuitionistic fuzzy sets (IFS) by Atanassov [1] in 1986. The research in fuzzy theory grew rapidly day by day and got its credit in all the branches of Mathematics. Fuzzy topological space (FTS) was studied by several authors like Chang [4] and Lowen [10].

The notion of intutionistic fuzzy topological space (IFTS) was introduced and studied by Coker [5],[6]. Since every concept in topology is defined in terms of open sets, separation axioms especially T_2 axiom developed by Hausdorff plays a vital role in making non trivial and interesting statements. Though different versions of fuzzy Hausdorff spaces are available in literature, the notion of nearly fuzzy Hausdorff space (NFHS) developed by Ramakrishnan and Lakshmana Gomathi nayagam [14] was a generalised one.

A few definitions for IFHS were introduced and studied by several authors like Cooker [5], Gallego Lupianez [11], Lakshmana and Muralikrishnan [13] and A.k. Singh and R.Srivastava [15]. Later on Md Sadadat Hossain[7] has given seven definitions for IFT_2 spaces out of which $IFT_2(iv)$ is the most generalised one. Saiful Islam [8] has also presented eight types of IFT_2 spaces and concluded that $IFT_2(viii)$ is the most generalised version. Recently Md. Aman Mahbub [2] worked on seperation axioms in intutionistic fuzzy compact topological spaces.

Though the definition [8] generalises all other existing definitions, it had some drawbacks. Let $T = \{a, b\}$ and $A = (\mathcal{M}_{a^{.2}b^{.8}}, \mathcal{N}_{a^{.8}b^{.1}})$. Here A is an IFS which can be viewed according to [8], as a set to which a belongs and b does not belong, though the membership of $a(\mathcal{M}_A(a) = 0.2)$ to lie in A is much lesser than the membership of $b(\mathcal{M}_A(b) = 0.8)$ to lie in A. Also the membership of a to lie in A is much lesser than the non membership of a to lie in A. But while generalising the concept of "an element x to belong a crisp set A" to the concept of "an element belongs to a IFS A" it has to be logically assumed that the membership degree of x to lie in A is greater than the non membership degree of x to lie in A. So $\mathcal{M}_A(a) > \mathcal{N}_A(a)$. Since $\mathcal{M}_A(a) + \mathcal{N}_A(a) \leq 1$ we get $\mathcal{N}_A(a) < \frac{1}{2}$ and $\mathcal{M}_A(a) > \frac{1}{2}$. Hence we can think of as a point a belongs to a IFS A if $\mathcal{M}_A(a) > \frac{1}{2}$. This concept

[14] generalises the crisp concept while greater care is taken intuitively to meet the logical requirements for belongingness.
This paper proposes a new definition of IFHS by adopting the concept [14], in order to rectify the above mentioned illogicality. Further we describe a definition for intuitionistic fuzzy closedness of a singleton and explore some of its characteristics. Some results in crisp topology spaces are discussed in intuitionistic fuzzy set up. We have also compared the proposed definition with the definitions available in literature.

2. Preliminaries

This section follows some elementary definitions.

Definition 2.1. *[1] Let T be a nonempty set. An IFS $R = \{\langle t, \mathcal{M}_R(t), \mathcal{N}_R(t)\rangle : t \in T\}$ where the functions $\mathcal{M}_R : T \to I$ and $\mathcal{N}_R : T \to I$ denote the degree of membership and the degree of non membership of $t \in T$ to the set R respectively, and $0 \leq \mathcal{M}_R(t) + \mathcal{N}_R(t) \leq 1$ for each $t \in T$.*

Definition 2.2. *[1] Let T be a nonempty set and the IFSs, R and S be in the form $R = \{\langle t, \mathcal{M}_R(t), \mathcal{N}_R(t)\rangle : t \in T\}$, $S = \{\langle t, \mathcal{M}_S(t), \mathcal{N}_S(t)\rangle : t \in T\}$. Then*

1. *$R \subseteq S$ if and only if $\mathcal{M}_R(t) \leq \mathcal{M}_S(t)$ and $\mathcal{N}_R(t) \geq \mathcal{N}_S(t)$, for all $t \in T$;*
2. *$R = S$ if and only if $R \subseteq S$ and $S \subseteq R$;*
3. *$R^c = \{\langle t, \mathcal{N}_R(t), \mathcal{M}_R(t)\rangle, t \in T\}$;*
4. *$R \cap S = \{\langle t, \min(\mathcal{M}_R(t), \mathcal{M}_S(t)), \max(\mathcal{N}_R(t), \mathcal{N}_S(t))\rangle\}$;*
5. *$R \cup S = \{\langle t, \max(\mathcal{M}_R(t), \mathcal{M}_S(t)), \min(\mathcal{N}_R(t), \mathcal{N}_S(t))\rangle\}$.*

Definition 2.3. *[5] Let T be a universal set. We define $0_\sim$ and $1_\sim$ as follows :*
$0_\sim = \{\langle t, 0, 1\rangle : t \in T\}$ *and* $1_\sim = \{\langle t, 1, 0\rangle : t \in T\}$

Definition 2.4. *[4] An intuitionistic fuzzy topology (IFT) on a non empty set T is a family δ of IFSs in T satisfying the following axioms:*

A_1 $0_\sim, 1_\sim \in \delta$

A_2 $G_1 \cap G_2 \in \delta$ *for any* $G_1, G_2 \in \delta$.

A_3 $\bigcup G_i \in \delta$ *for any arbitrary family* $\{G_i : i \in I\} \subseteq \delta$.

The pair (T, δ) *is called an IFTS and any IFS in* δ *is known as an intuitionistic fuzzy open set (IFOS).*

Definition 2.5. *[15] Let* $(\alpha, \beta) \in (0, 1)$ *and* $\alpha + \beta \leq 1$. *An intuitionistic fuzzy point* (IFP) $P^t_{(\alpha,\beta)}$ *in* T *is an IFS of* T *defined by* $P^t_{(\alpha,\beta)} = \langle t, \mathcal{M}_p, \mathcal{N}_p\rangle$ *where for* $y \in T$

$$\mathcal{M}_p(y) = \begin{cases} \alpha & \text{if} \quad y = t; \\ 0 & \text{if} \quad \text{otherwise} \end{cases}$$

$$\mathcal{N}_p(y) = \begin{cases} \beta & \text{if} \quad y = t; \\ 1 & \text{if} \quad \text{otherwise} \end{cases}$$

In this case, t *is called the support of* $P^t_{(\alpha,\beta)}$.

Definition 2.6. *[15] Let* $P^t_{(\alpha,\beta)}$ *be an IFP in* T *and* $R = \langle t, \mathcal{M}_R, \mathcal{N}_R\rangle$ *be an IFS in* T. *Then* $P^t_{(\alpha,\beta)}$ *is said to be properly contained in* R, ($P^t_{(\alpha,\beta)} \in R$) *if* $\alpha < \mathcal{M}_R(t)$ *and* $\beta > \mathcal{N}_R(t)$.

Definition 2.7. *[9] Let* (T, δ) *be an IFTS on* T *and* N *be an IFS in* T. *Then* N *is said to be an* ε*-neighbourhood of an IFP* $P^t_{(\alpha,\beta)}$ *in* T *if there exist an IFOS,* G *in* T *such that* $P^t_{(\alpha,\beta)} \in G \subseteq N$

Definition 2.8. *[5] Let* $f : T \to Y$ *be a map. Let* $R \in I^T$ *be an intuitionistic fuzzy (IF) subset of* T. *Then the IFS,* $f(R)$ *is defined as* $f(R) = \{\langle y, \mathcal{M}_{f(R)}(y), \mathcal{N}_{f(R)}(y)\rangle : y \in Y\}$, *where*

$$\mathcal{M}_{f(R)}(y) = \begin{cases} \sup_{t \in f^{-1}(y)} \mathcal{M}_R(t)) & \text{if} \quad f^{-1}(y) \neq 0; \\ 0 & \text{if} \quad \text{otherwise} \end{cases}$$

$$\mathcal{N}_{f(R)}(y) = \begin{cases} \inf_{t \in f^{-1}(y)} \mathcal{N}_R(t)) & \text{if} \quad f^{-1}(y) \neq 0; \\ 1 & \text{if} \quad \text{otherwise} \end{cases} \quad \textit{Let}$$

$S \in I^Y$ *be an IF subset of* Y. *Then the IFS* $f^{-1}(S)$ *is defined as* $f^{-1}(S) = \{\langle t, \mathcal{M}_{f^{-1}(S)}(t), \mathcal{N}_{f^{-1}(S)}(t)\rangle : t \in T\}$, *where* $\mathcal{M}_{f^{-1}(S)}(t) = \mathcal{M}_S(f(t))$, $\mathcal{N}_{f^{-1}(S)}(t) = \mathcal{N}_S(f(t))$.

Definition 2.9. *[5] Let* (T, δ) *and* (Y, τ) *be two IFTSs and let* $f : T \to Y$ *be a map. Then* f *is said to be IF continuous if and only if the pre-image of each IFS open in* Y *is an IFS open in* T.

Definition 2.10. *[5] Let (T, δ) and (Y, τ) be two IFTSs and let $f : T \to Y$ be a map. Then f is said to be IF open if and only if the image of each IFS in δ is an IFS in τ.*

Definition 2.11. *[3] Let $\{(T_i, \delta_i)\}_{i \in I}$ be any indexed family of IFTSs. Then the product intuitionistic fuzzy topology (PIFT) $\Pi\delta_i$ on ΠT_i is the IFT generated by $\sigma = \{\theta_i^{-1}(U_i) \mid U_i \in \delta_i, i \in I\}$ as subbasis. The pair $(\Pi T_i, \Pi\delta i)$ is called the product intuitionistic fuzzy topological space (PIFTS).*

Definition 2.12. *[14] A FTS (T, δ) is said to be a NFHS if for every pair of elements $x \neq y$ of T, there exist disjoint fuzzy open sets $\mathcal{M}, \mathcal{N} \in \delta$ such that $\mathcal{M}(x) > \frac{1}{2}$ and $\mathcal{N}(y) > \frac{1}{2}$. Equivalently, if there exist $\mathcal{M}, \mathcal{N} \in \delta$ such that $\mathcal{M}(x) > \frac{1}{2}$ and $\mathcal{N}(y) > \frac{1}{2}$ and $\mathcal{M}(z) + \mathcal{N}(z) \leq 1$, for every $z \in T$.*

Definition 2.13. *[5] An IFTS (T, δ) is called Hausdorff if and only if for every t_1, $t_2 \in T$ and $t_1 \neq t_2$, there exist $G_1 = \langle t, \mathcal{M}_{G_1}, \mathcal{N}_{G_1} \rangle$, $G_2 = \langle t, \mathcal{M}_{G_2}, \mathcal{N}_{G_2} \rangle \in \delta$ with $\mathcal{M}_{G_1}(t_1) = 1, \mathcal{N}_{G_1}(t_1) = 0, \mathcal{M}_{G_2}(t_2) = 1, \mathcal{N}_{G_2}(t_2) = 0$ and $G_1 \cap G_2 = 0_\sim$.*

Definition 2.14. *[11] An IFTS (T, δ) called $q{-}T_2$ if for every distinct IFPS p, q in T, there exists ε-neighbourhood M and N of p and q respectively such that $\mathcal{M}_M \leq \mathcal{M}'_N$ and $\mathcal{N}_M \geq \mathcal{N}'_N$.*

Definition 2.15. *[13] An IFTS (T, δ) is said to be a nearly intuitionistic fuzzy Hausdorff space (NIFHS) if for every pair of elements $x \neq y$ of T, there exist non zero disjoint IFOS R and S of δ such that $\mathcal{M}_R(x) > \frac{1}{2}$, $\mathcal{M}_S(y) > \frac{1}{2}$, that is, there exist IFOSs $R \neq 0$, $S \neq 0$ in δ such that $\mathcal{M}_R(x) > \frac{1}{2}$, $\mathcal{M}_S(y) > \frac{1}{2}$, $\mathcal{N}_S(t) \geq \mathcal{M}_R(t)$ and $\mathcal{N}_R(t) \geq \mathcal{M}_S(t)$ for every $t \in T$.*

Definition 2.16. *[8] An IFTS (T, δ) is said to be IFT2 if for every pair of elements $x \neq y$ of T, there exist two IFOS R and S of δ such that $\mathcal{M}_R(x) > 0, \mathcal{N}_R(x) < 1, \mathcal{M}_R(y) < 1, \mathcal{N}_R(y) > 0$, and $\mathcal{M}_S(y) > 0, \mathcal{N}_S(y) < 1, \mathcal{M}_S(x) < 1, \mathcal{N}_S(x) > 0$ with $(\mathcal{M}_R \cap \mathcal{M}_S) \subset (\mathcal{N}_R \cup \mathcal{N}_S)$.*

Definition 2.17. *[13] A sequence of points t_n of T is said to converge intuitionistic fuzzily to $t \in T$ in (T, δ) denoted as t_ni-ft if for every $R \in \delta$ such that $\mathcal{M}_R(t) > \frac{1}{2}$, there exist N such that $\mathcal{M}_R(t_n) > \frac{1}{2}$, $\forall n \geq N$.*

3. Nearly intuitionistic fuzzy Hausdorff spaces

In this section a new notion of NIFHS is introduced and its properties are studied. This notion is compared with the existing notions.

Definition 3.1. *Two IFSs R and S of T are said to intersect at t if $\mathcal{N}_R(t) + \mathcal{N}_S(t) < 1$.*

Definition 3.2. *Two IFSs R and S of T are said to be disjoint if they do not intersect at any point of T that is, $\mathcal{N}_R(t) + \mathcal{N}_S(t) \geq 1$, for all $t \in T$.*

Definition 3.3. *An IFTS (T, δ) is said to be NIFHS if for every pair of elements $x \neq y$ of T, there exist non zero disjoint IFOSs R and S of δ such that $\mathcal{M}_R(x) > \frac{1}{2}$, $\mathcal{M}_S(y) > \frac{1}{2}$. That is, there exist nonempty IFOSs R, S in δ such that $\mathcal{M}_R(x) > \frac{1}{2}$, $\mathcal{M}_S(y) > \frac{1}{2}$, and $\mathcal{N}_R(t) + \mathcal{N}_S(t) \geq 1$, for every $t \in T$.*

Theorem 3.4. *Let (T, δ) be a NIFHS. Then a subspace of a NIFHS is a NIFHS.*

Proof. Let (T, δ) be a NIFHS and $Y \subseteq T$. To prove that $(Y, \delta \mid Y)$ is a NIFHS, let two distinct points $y_1, y_2 \in Y$. Since $Y \subseteq T$ and (T, δ) is a NIFHS, there exist two non zero open sets R and $S \in \delta$ such that $\mathcal{M}_R(y_1) > \frac{1}{2}$, $\mathcal{M}_S(y_2) > \frac{1}{2}$, and $\mathcal{N}_R(t) + \mathcal{N}_S(t) \geq 1$, for every $t \in T$. Now $R \mid Y$ and $S \mid Y \in \delta \mid Y$ such that $(\mathcal{M}_R \mid Y)(y_1) > \frac{1}{2}$, $(\mathcal{M}_S \mid Y)(y_2) > \frac{1}{2}$ and $(\mathcal{N}_R \mid Y)(t) + (\mathcal{N}_S \mid Y)(t)1$, for all $t \in Y$. Hence $(Y, \delta \mid Y)$ is a NIFHS. □

Theorem 3.5. *Let $(T_i, \delta_i)_{i \in I}$ be a family of NIFHS. Then arbitrary product of NIFHS is a NIFHS.*

Proof. Let I be an indexed set and $(T_i, \delta_i)_{i \in I}$ be a family of NIFHS. Let $(T = \Pi T_i, \delta = \Pi \delta_i)$ be the PIFT in which each projection mapping $\theta_i : (T, \delta) \to (T_i, \delta_i)$ is IF continuous. We know that $\sigma = \{\theta_i^{-1}(R_i) \mid R_i \in \delta_i, i \in I\}$ forms sub-base for PIFT. To prove that (T, δ) is a NIFHS, consider $x \neq y \in \Pi T_i$. So there exist atleast one $j \in I$ such that $x_j \neq y_j$. Since (T_j, δ_j) is a NIFHS, there exist two open sets R_j, S_j such that $\mathcal{M}_{R_j}(x_j) > \frac{1}{2}$, $\mathcal{M}_{S_j}(y_j) > \frac{1}{2}$ and $\mathcal{N}_{R_j}(t_j) + \mathcal{N}_{S_j}(t_j) \geq 1$, for all $t_j \in T_j$.
Clearly $\theta_j^{-1}(R_j)$ and $\theta_j^{-1}(S_j)$ are the members of σ and hence elements of δ.

Consider

$$\begin{aligned}
(3.1)\qquad \theta_j^{-1}(R_j) &= \langle x, \mathcal{M}_{\theta_j^{-1}(R_j)}(x), \mathcal{N}_{\theta_j^{-1}(R_j)}(x)\rangle \\
(3.2)\qquad &= \langle x, \mathcal{M}_{R_j}(\theta_j(x)), \mathcal{N}_{R_j}(\theta_j(x))\rangle \\
(3.3)\qquad &= \langle x, \mathcal{M}_{R_j}(x_j)), \mathcal{N}_{R_j}(x_j))\rangle
\end{aligned}$$

Hence $\mathcal{M}_{\theta_j^{-1}(R_j)}(x) > \frac{1}{2}, \mathcal{M}_{\theta_j^{-1}(S_j)}(y) > \frac{1}{2}$.
Now we claim that $\mathcal{N}_{\theta_j^{-1}(R_j)}(t) + \mathcal{N}_{\theta_j^{-1}(S_j)}(t) \geq 1$, for all $t \in T$. Assume that $\mathcal{N}_{\theta_j^{-1}(R_j)}(t) + \mathcal{N}_{\theta_j^{-1}(S_j)}(t) < 1$ for some $t \in T$.
By definition

$$\begin{aligned}
(3.4)\qquad \mathcal{N}_{\theta_j^{-1}(R_j)}(t) + \mathcal{N}_{\theta_j^{-1}(S_j)}(t) &= \mathcal{N}_{R_j}(\theta_j(t)) + \mathcal{N}_{S_j}(\theta_j(t)) \\
(3.5)\qquad &= \mathcal{N}_{R_j}(t_j) + \mathcal{N}_{S_j}(t_j) < 1, t_j \in T_j.
\end{aligned}$$

This is a contradiction. Hence (T, δ) is a NIFHS. □

Theorem 3.6. *In a NIFHS (T, δ), any sequence of points of T converges intuitionistic fuzzily to unique point, if it converges.*

Proof. Assume that $\{t_n\}$ converges intuitionistic fuzzily to distinct points x and y. Since (T, δ) is a NIFHS, there exist two open sets R and S such that $\mathcal{M}_R(x) > \frac{1}{2}$, $\mathcal{M}_S(y) > \frac{1}{2}$, and $\mathcal{N}_R(t) + \mathcal{N}_S(t) \geq 1$, for every $t \in T$. As t_ni-fx and $\mathcal{M}_R(x) > \frac{1}{2}$, there exist m_1 such that $\mathcal{M}_R(t_n) > \frac{1}{2}$, $\forall n \geq m_1$. Similarly there exist m_2 such that $\mathcal{M}_S(t_n) > \frac{1}{2}$, $\forall n \geq m_2$. Clearly $\forall n \geq \max\{m_1, m_2\}$, $\mathcal{N}_R(t_n) + \mathcal{N}_S(t_n) < 1$, a contradiction. Hence the theorem. □

Remark 3.7. *The following example reveals that converse of the above theorem need not be true.*

Example 3.8. *Let T be an uncountable set and $\delta = \{\langle \mathcal{M}, \mathcal{N}\rangle \in I^T \times I^T \mid \mathcal{N}$ has countable support or $\mathcal{N} = 1\}$. Clearly (T, δ) is an IFTS. Every sequence $\{t_n\}$ of points of T converges intuitionistic fuzzily uniquely if it converges. (In fact here every sequence does not converge to any point. For, let $t \in T$. Consider $R \in \delta$ such that*

$$\mathcal{M}_R(z) = \begin{cases} 1 & \text{if } z \neq t_n, t \text{ or } z = y; \\ \frac{1}{4} & \text{if } z = t_n; \\ \frac{3}{4} & \text{if } z = t. \end{cases}$$

Clearly $\mathcal{M}_R(t) > \frac{1}{2}$, *But* $\mathcal{M}_R(t_n) < \frac{1}{2}$, *for all* $n \in Z_+$ *and so* t_n *doesn't converge to* t *fuzzily).*
But it is not NIFHS. For let $x \neq y \in T$. *Suppose there exist* $R, S \in \delta$ *such that* $\mathcal{M}_R(x) > \frac{1}{2}$, $\mathcal{M}_S(y) > \frac{1}{2}$, *and* $\mathcal{N}_R(t) + \mathcal{N}_S(t) \geq 1$, *for every* $t \in T$. *Since* $R, S \in \delta$, $\mathcal{N}_R$ *and* $\mathcal{N}_S$ *have countable support* $\{x_n\}_{n\in Z^+}$ *and* $\{y_m\}_{m\in Z^+}$ *respectively. As* T *is uncountable, there exist* $t \in T - \{x_n, y_m\}_{n,m\in Z^+}$ *such that* $\mathcal{N}_R(t) = 0$, $\mathcal{N}_S(t) = 0$, *which contradicts the fact that* $\mathcal{N}_R(t) + \mathcal{N}_S(t) \geq 1$, *for every* $t \in T$.

Theorem 3.9. *Let* $f : (T, \delta) \to (Y, \sigma)$ *be a bijective IF open function. Then* (Y, σ) *is NIFHS if* (T, δ) *is NIFHS.*

Proof. Let y_1, y_2 in Y be two distinct points. Since f is bijective, there exist unique distinct points t_1, t_2 in T such that $f(t_1) = y_1$, $f(t_2) = y_2$. Since $t_1 \neq t_2$ and (T, δ) is a NIFHS, there exist $R, S \in \delta$ such that $\mathcal{M}_R(t_1) > \frac{1}{2}$, $\mathcal{M}_S(t_2) > \frac{1}{2}$, and $\mathcal{N}_R(t) + \mathcal{N}_S(t) \geq 1$, for every $t \in T$. Since f is IF open, $f(R)$, $f(S) \in \sigma$. Clearly $\mathcal{M}_{f(R)}(y_1) = \mathcal{M}_R(t_1) > \frac{1}{2}$ and $\mathcal{M}_{f(S)}(y_2) = \mathcal{M}_S(t_2) > \frac{1}{2}$. Now we claim that $f(R)$ and $f(S)$ are disjoint. That is to prove $\mathcal{N}_{f(R)}(z) + \mathcal{N}_{f(S)}(z) \geq 1$, for every $z \in Y$. Suppose $\mathcal{N}_{f(R)}(z) + \mathcal{N}_{f(S)}(z) < 1$, for some $z \in Y$, by hypothesis there exist unique $t \in T$ such that $f(t) = z$. Hence $\mathcal{N}_R(t) + \mathcal{N}_S(t) < 1$, a contradiction. Hence (Y, σ) is a NIFHS. □

Remark 3.10. *The requirement that* f *is IF open can not be dropped in the above theorem.*

Example 3.11. *Let* $T = \{p, q\}$, $Y = \{r, s\}$. *Let* $\delta = \{(0, 1), (1, 0), (\mathcal{M}_{p^x q^0}, \mathcal{N}_{p^{x^*} q^1}), (\mathcal{M}_{p^0 q^y}, \mathcal{N}_{p^1 q^{y^*}}), (\mathcal{M}_{p^x q^y}, \mathcal{N}_{p^{x^*} q^{y^*}})\}$, *and* $\sigma = \{(0, 1), (1, 0), (\mathcal{M}_{r^x s^0}, \mathcal{N}_{r^{x^*} s^1})\}$, *where* $x, y > \frac{1}{2}$, $x + x^* \leq 1$. *Let* $f : T \to Y$ *be a map defined by* $f(p) = r$, $f(q) = s$. *Clearly* f *is one-one and onto but* f *is not fuzzy open. Clearly* (T, δ) *is NIFHS, but* (Y, σ) *is not a NIFHS.*

Remark 3.12. *The requirement that* f *is onto can not be dropped in the above theorem.*

Example 3.13. *Let* $T = \{p, q\}$, $Y = \{r, s, t\}$. *Let* $\delta = \{(0, 1), (1, 0), (\mathcal{M}_{p^x q^0}, \mathcal{N}_{p^{x^*} q^1}), (\mathcal{M}_{p^0 q^y}, \mathcal{N}_{p^1 q^{y^*}}), (\mathcal{M}_{p^x q^y}, \mathcal{N}_{p^{x^*} q^{y^*}})\}$ *and* $\sigma = \{(0, 1), (1, 0), (\mathcal{M}_{r^x s^0 t^0}, \mathcal{N}_{r^{x^*} s^1 t^1}), (\mathcal{M}_{r^0 s^y t^0}, \mathcal{N}_{r^1 s^{y^*} t^1}), (\mathcal{M}_{r^x s^y t^0}, \mathcal{N}_{r^{x^*} s^{y^*} t^1}), (\mathcal{M}_{r^x s^1 t^0}, \mathcal{N}_{r^{x^*} s^0 t^1}), (\mathcal{M}_{r^1 s^y t^0}, \mathcal{N}_{r^0 s^{y^*} t^1}), (\mathcal{M}_{r^1 s^1 t^0}, \mathcal{N}_{r^0 s^0 t^1})\}$ *where* $x, y > \frac{1}{2}$, $x + x^* \leq 1$. *Let* $f : T \to Y$ *be a map defined by* $f(p) = r$, $f(q) = s$.

Now $f(\delta)=\{(0,1),(\mathcal{M}_{r^x s^0 t^0},\mathcal{N}_{r^{x^*} s^1 t^1}),(\mathcal{M}_{r^0 s^y t^0},\mathcal{N}_{r^1 s^{y^*} t^1}),(\mathcal{M}_{r^x s^y t^0},\mathcal{N}_{r^{x^*} s^{y^*} t^1}),$ $(\mathcal{M}_{r^1 s^1 t^0},\mathcal{N}_{r^0 s^0 t^1})\}$. *Clearly* f *is one-one and open. But* f *is not onto. Clearly* (T,δ) *is NIFHS, but* (Y,σ) *is not a NIFHS.*

Remark 3.14. *The requirement that* f *is one-one can not be dropped.*

Example 3.15. *Let* $T=\{t_n \mid n\in Z_+\}$, $Y=\{x,y\}$. *Let* δ *be generated by* $\{0_\sim,1_\sim,R_i \mid i\in Z_+\}$ *where* $R_i\in I^T\times I^T$ *given by*

$$\mathcal{M}_{R_i}(t_j)=\begin{cases}\frac{1}{2}(1+\frac{1}{i}), & \text{if } j=i;\\ \frac{1}{2}(1-\frac{1}{j}), & \text{if } j\neq i;\end{cases}$$

$$\mathcal{N}_{R_i}(t_j)=\begin{cases}\frac{1}{2}(1-\frac{1}{i}), & \text{if } j=i;\\ \frac{1}{2}(1+\frac{1}{j}), & \text{if } j\neq i;\end{cases}$$

Let σ *be generated by* $\{0_\sim,1_\sim,S_i \mid i\in Z_+\}$ *where* $S_i\in I^Y\times I^Y$ *given by*

$$\begin{array}{llll}\mathcal{M}_{S_{2i}}(x) &= \frac{1}{2}(1+\frac{1}{2i}) & \mathcal{N}_{S_{2i}}(x) &= \frac{1}{2}(1-\frac{1}{2i})\\ \mathcal{M}_{S_{2i}}(y) &= \frac{1}{2} & \mathcal{N}_{S_{2i}}(y) &= \frac{1}{2}\\ \mathcal{M}_{S_{2i-1}}(x) &= \frac{1}{2} & \mathcal{N}_{S_{2i-1}}(x) &= \frac{1}{2}\\ \mathcal{M}_{S_{2i-1}}(y) &= \frac{1}{2}(1+\frac{1}{2i-1}) & \mathcal{N}_{S_{2i-1}}(y) &= \frac{1}{2}(1-\frac{1}{2i-1})\end{array}$$

Define the map $f:T\to Y$ *by* $f(t_{2i})=x$ *and* $f(t_{2i-1})=y$. *Clearly* f *is onto but it is not one-one.*
Now $f(\delta)$ *is generated by* $\{0_\sim,1_\sim,f(R_i)\}$ *where*

$$\begin{array}{llll}\mathcal{M}_{f(R_{2i})}(x) &= \frac{1}{2}(1+\frac{1}{2i}) & \mathcal{N}_{f(R_{2i})}(x) &= \frac{1}{2}(1-\frac{1}{2i}).\\ \mathcal{M}_{f(R_{2i})}(y) &= \frac{1}{2}. & \mathcal{N}_{f(R_{2i})}(y) &= \frac{1}{2}.\\ \mathcal{M}_{f(R_{2i-1})}(x) &= \frac{1}{2}. & \mathcal{N}_{f(R_{2i-1})}(x) &= \frac{1}{2}.\\ \mathcal{M}_{f(R_{2i-1})}(y) &= \frac{1}{2}(1+\frac{1}{2i-1}). & \mathcal{N}_{f(R_{2i-1})}(y) &= \frac{1}{2}(1-\frac{1}{2i-1}).\end{array}$$

Clearly $f(\delta)\subseteq\sigma$. *Hence* f *is IF open. Now* (T,δ) *is a NIFHS. For* $t_i\neq t_j$ *there exist* $R_i,R_j\in\delta$ *such that* $\mathcal{M}_{R_i}(t_i)>\frac{1}{2}$, $\mathcal{M}_{R_j}(t_j)>\frac{1}{2}$ *and* $\mathcal{N}_{R_i}(t)+\mathcal{N}_{R_j}(t)\geq 1$, *for all* $t\in T$. *But* (Y,σ) *is not a NIFHS.*

Theorem 3.16. *Let* $f:(T,\delta)\to(Y,\sigma)$ *be a injective IF continuous map. Then* (T,δ) *is NIFHS if* (Y,σ) *is NIFHS.*

Proof. To prove (T,δ) is NIFHS, take two distinct points t_1, t_2 in T. Since f is injective, there exist unique distinct points t_1, t_2 in T such that $f(t_1) = y_1, f(t_2) = y_2$ and $y_1 \neq y_2$. Since $y_1 \neq y_2$ and (Y,σ) is NIFHS, there exist IFOSs $R \neq 0$,$S \neq 0$ in σ such that $\mathcal{M}_R(y_1) > \frac{1}{2}$, $\mathcal{M}_S(y_2) > \frac{1}{2}$, and $\mathcal{N}_R(z) + \mathcal{N}_S(z) \geq 1$, for every $z \in Y$.
Since f is IF continuous, $f^{-1}(R), f^{-1}(S) \in \delta$.
Then, $\mathcal{M}_{f^{-1}(R)}(t_1) = \mathcal{M}_R(f(t_1)) = \mathcal{M}_R(y_1) > \frac{1}{2}$. Similarly, $\mathcal{M}_{f^{-1}(S)}(t_2) > \frac{1}{2}$.
Now, we claim that $f^{-1}(R)$ and $f^{-1}(S)$ are disjoint.
Suppose,

$$\begin{aligned} &\mathcal{N}_{f^{-1}(R)}(t) + \mathcal{N}_{f^{-1}(S)}(t) < 1, \text{ for some, } t \in T. \\ &\Rightarrow \mathcal{N}_R(f(t)) + \mathcal{N}_S(f(t)) < 1 \\ &\Rightarrow \mathcal{N}_R(z) + \mathcal{N}_S(z) < 1, (\text{since, } f \text{ is injective.}) \end{aligned}$$

which is a contradiction. Hence, (T,δ) is NIFHS. □

Note. Let $f : (T,\delta) \to (Y,\sigma)$ be a bijective IF continuous map. Then (Y,σ) need not be NIFHS if (T,δ) is NIFHS.

Example 3.17. *Let $T = \{p,q\}$, $Y = \{r,s\}$. Let $\delta = \{(0,1),(1,0), (\mathcal{M}_{p^x q^0}, \mathcal{N}_{p^{x^*} q^1}), (\mathcal{M}_{p^0 q^y}, \mathcal{N}_{p^1 q^{y^*}}), (\mathcal{M}_{p^x q^y}, \mathcal{N}_{p^{x^*} q^{y^*}})\}$, and $\sigma = \{(0,1),(1,0), (\mathcal{M}_{r^x s^0}, \mathcal{N}_{r^{x^*} s^1})\}$, where $x, y > \frac{1}{2}$, $x + x^* \leq 1$. Define a map $f : T \to Y$ by $f(p) = r$, $f(q) = s$. So $f^{-1}(\sigma) = \{(0,1),(1,0),(\mathcal{M}_{p^x q^0}, \mathcal{N}_{p^{x^*} q^1})\} \subseteq \delta$. Clearly f is bijective IF continuous map and (T,δ) is NIFHS but (Y,σ) is not a NIFHS.*

Theorem 3.18. *Let $f : (T,\delta) \to (Y,\sigma)$ be a bijective IF closed map. Then (Y,σ) is NIFHS if (T,δ) is NIFHS.*

Proof. To prove (Y,σ) is NIFHS, take two distinct points $y_1, y_2 \in Y$. Since f is bijective, there exist unique $t_1 \neq t_2$ in T such that $f(t_1) = y_1, f(t_2) = y_2$. Since $t_1 \neq t_2$ and (T,δ) is NIFHS, there exist IFOSs $R \neq 0$,$S \neq 0$ in δ such that $\mathcal{M}_R(t_1) > \frac{1}{2}$, $\mathcal{M}_S(t_2) > \frac{1}{2}$, and $\mathcal{N}_R(z) + \mathcal{N}_S(z) \geq 1$, for every $z \in T$. Since f is IF closed, $f(R^c)^c$, $f(S^c)^c \in \sigma$. Now $f(R^c)^c = \langle y, \inf_{t \in f^{-1}(y)} \mathcal{M}_R(t), \sup_{t \in f^{-1}(y)} \mathcal{N}_R(t) \rangle$. Since f is bijective, $\inf_{t \in f^{-1}(y_1)} \mathcal{M}_R(t) = \mathcal{M}_R(t_1) > \frac{1}{2}$.

Similarly $f(S^c)^c = \langle y, \inf_{t \in f^{-1}(y)} \mathcal{M}_S(t), \sup_{t \in f^{-1}(y)} \mathcal{N}_S(t)\rangle$. Since f is bijective, $\inf_{t \in f^{-1}(y_2)} \mathcal{M}_S(t) = \mathcal{M}_S(t_2) > \frac{1}{2}$. Therefore $\mathcal{M}_{f(R^c)^c}(y_1) > \frac{1}{2}$ and $\mathcal{M}_{f(S^c)^c}(y_2) > \frac{1}{2}$. Clearly $\mathcal{N}_{f(A^c)^c}(z) + \mathcal{N}_{f(B^c)^c}(z) \geq 1$, for all $z \in Y$. Hence (Y, σ) is NIFHS. □

Note. Nearly intuitionistic fuzzy Housdorffness is a topological property.

Theorem 3.19. *Let (T, δ) be IFTS. If (T, δ) is NIFHS (as per definition 3.3), then (T, δ_1) is a FHTS (as per definition 2.12), where $\delta_1 = \{\mathcal{M}_G \mid G \in \delta\}$.*

Proof. Suppose (T, δ) is NIFHS, then for every pair of elements $x \neq y$ of T, there exist non-zero disjoint IFOSs R and S of δ such that $\mathcal{M}_R(x) > \frac{1}{2}$, $\mathcal{M}_S(y) > \frac{1}{2}$.
That is, we have

$$\begin{aligned}
&\mathcal{N}_R(t) + \mathcal{N}_S(t) && \geq 1, \text{for every } t \in T \\
&\Rightarrow 2 - (\mathcal{N}_R(t) + \mathcal{N}_S(t)) && \leq 1 \\
&\Rightarrow (1 - (\mathcal{N}_R(t)) + (1 - (\mathcal{N}_S(t))) && \leq 1 \\
&\Rightarrow \mathcal{M}_R(t) + \mathcal{M}_S(t) && \leq 1
\end{aligned}$$

Hence, there exist non-zero disjoint fuzzy open sets $\mathcal{M}_R$ and $\mathcal{M}_S$ of δ_1 such that $\mathcal{M}_R(x) > \frac{1}{2}$, $\mathcal{M}_S(y) > \frac{1}{2}$. Therefore, (T, δ_1) is FHTS (as per definition 2.12) □

4. Intuitionistic fuzzy Hausdorff spaces

Definition 4.1. (i) *Let (T, δ) be an IFTS. A singleton $\{x\} \subseteq T$ is said to be IF closed if there exist an IF closed set C with $\mathcal{M}_c(x) > \frac{1}{2}$ and $\mathcal{N}_c(z) = 1$, for all $z \in T$ where $z \neq x$.*
(ii) *Let (T, δ) be an IFTS. A singleton $\{x\} \subseteq T$ is said to be IF closed if there exist an IF open set O with $\mathcal{M}_O(x) < \frac{1}{2}$ and $\mathcal{M}_O(z) = 1$, for all $z \in T$ where $z \neq x$.*

Example 4.2. *Take $T = \{a, b\}$, $\delta = \{(0,1), (1,0), \langle(\mathcal{M}_{a^{.6}b^0}, \mathcal{N}_{a^{.4}b^1}), (\mathcal{M}_{a^0b^{.6}}, \mathcal{N}_{a^1b^{.3}}), (\mathcal{M}_{a^{.3}b^1}, \mathcal{N}_{a^{.4}b^0}), (\mathcal{M}_{a^1b^{.2}}, \mathcal{N}_{a^0b^{.4}})\rangle\}$. Clearly (T, δ) is a NIFHS. Here singletons are IF closed as per (ii) but not by (i).*

Example 4.3. *Take $T = \{a, b\}$, $\delta = \{(0,1), (1,0), \langle(\mathcal{M}_{a^{1-x}b^1}, \mathcal{N}_{a^xb^0}), (\mathcal{M}_{a^1b^{1-y}}, \mathcal{N}_{a^0b^y}), (\mathcal{M}_{a^xb^0}, \mathcal{N}_{a^{1-x}b^1}), (\mathcal{M}_{a^0b^y}, \mathcal{N}_{a^1b^{1-y}})\rangle\}$ where $x, y > \frac{1}{2}$. Clearly (T, δ) is NIFHS. Also note that every singletons are IF closed by above two definitions.*

Note. The above examples shows that the second definition is the generalised one. That is singleton $\{x\}$ is IF closed as per $(i) \Rightarrow (ii)$ but not the converse.

Note. The following results based on the generalised definition.

Note. In NIFHS, singletons need not be IF closed. This can be proved by the following example.

Example 4.4. *Take* $T = \{a, b\}$, $\delta = \{(0,1), (1,0), \langle(\mathcal{M}_{a^x b^0}, \mathcal{N}_{a^{1-x} b^1}),$ $(\mathcal{M}_{a^0 b^y}, \mathcal{N}_{a^1 b^{1-y}})\rangle\}$, *where* $x, y > \frac{1}{2}$. *Clearly* (T, δ) *is NIFHS. But singletons need not be IF closed.*

Remark 4.5. *There exist NIFHS in which every singleton is IF closed. This can be seen in the example 4.3.*

Definition 4.6. *An IFTS* (T, δ) *is said to be a IFHS if* (T, δ) *is NIFHS and every singletons are IF closed.*

Theorem 4.7. *Let* (T, δ) *be an IFHS. Then a subspace of IFHS is IFHS.*

Proof. Let (T, δ) be an IFHS. Let $Y \subseteq T$. To prove that $(Y, (\delta \mid Y))$ is IFHS, by theorem 3.4, we have $(Y, (\delta \mid Y))$ is NIFHS. It is enough to prove that singletons of Y are IF closed. Let $y \in Y \subseteq T$. Since (T, δ) is IFHS, the singleton y is IF closed in T. Hence there exist an IFOS R such that $\mathcal{M}_R(y) < \frac{1}{2}$ and $\mathcal{M}_R(t) = 1$, for all $t \in T$ where $t \neq y$. Therefore we have

$$\mathcal{M}_R(t) = \begin{cases} 1 & \text{if } t \neq y; \\ < \frac{1}{2} & \text{if } t = y. \end{cases} \textit{ and}$$

R $\in \delta$. Therefore $(R \mid Y) \in (\delta \mid Y)$.
Now

$$\mathcal{M}_{(R|Y)}(t) = \begin{cases} 1 & \text{if } t \neq y; \\ < \frac{1}{2} & \text{if } t = y. \end{cases} \textit{ Hence}$$

(R$\mid Y$) is an IFOS in $(\delta \mid Y)$ in which $\{y\}$ is IF closed. Therefore $(Y, (\delta \mid Y))$ is IFHS. □

Theorem 4.8. *Let* $(T_i, \delta_i), i = 1, 2, ..., n$ *be IFHS. Then finite product of IFHS is IFHS.*

Proof. Let $(T_i, \delta_i), i = 1, 2, ..., n$ be IFHS. By theorem (3.5), $(T = \Pi T_i, \delta = \Pi \delta_i)$ is NIFHS. Now, we prove that singletons are IF closed in PIFHS. Let $t = (t_1, t_2, ..., t_n) \in T$ be arbitrary. Since $t_i \in T_i$, and (T_i, δ_i) is IFHS, there exist IFOS $A_i \in \delta_i$ such that $\mathcal{M}_{A_i}(t_i) < \frac{1}{2}$ and $\mathcal{M}_{A_i}(y_i) = 1$, for all $y_i \neq t_i \in T_i$.
We know that the projection, $\theta_i : \Pi T_i \to T_i$ is IF continuous in PIFT. As $A_i \in \delta_i$, for every i, $\{\theta_i^{-1}(A_i)/A_i \in \delta_i, i = 1, 2, ..., n\}$ is the collection of sub-base. Now

$$\mathcal{M}_{\theta_i^{-1}(A_i)}(z) = \begin{cases} 1 & \text{if } z_i \neq t_i; \\ < \frac{1}{2} & \text{if } z_i = t_i. \end{cases}$$

Therefore

$$\bigvee_{i=1}^{n} \mathcal{M}_{\theta_i^{-1}(A_i)}(z) = \begin{cases} 1 & \text{if } z \neq t \text{ and } z \in T; \\ < \frac{1}{2} & \text{if } z = t. \end{cases}$$

Since finite union open set is open set, $\bigvee_{i=1}^{n} \mathcal{M}_{\theta_i^{-1}(A_i)}$ is our required IFOS. Hence (T, δ) is IFHS. □

Remark 4.9. *Arbitrary product of IFHS need not be IFHS as can be seen in the following example.*

Example 4.10. *Let* $T_i = \{x_i, y_i\}$ *and* $\delta_i = \{0, 1, A_{ni}/n = 1, 2, 3, 4.$ *and* $i = 1, 2, \cdots\}$
where
$A_{1i} = \left(\mathcal{M}_{x_i}^{\frac{1}{2}(1-\frac{1}{i})} y_i^1, \mathcal{N}_{x_i}^{\frac{1}{2}(\frac{1}{2}+\frac{1}{i})} y_i^0\right)$, $A_{2i} = \left(\mathcal{M}_{x_i}^1 y_i^0, \mathcal{N}_{x_i}^0 y_i^1\right)$,
$A_{3i} = \left(\mathcal{M}_{x_i}^{\frac{1}{2}(1-\frac{1}{i})} y_i^0, \mathcal{N}_{x_i}^{\frac{1}{2}(\frac{1}{2}+\frac{1}{i})} y_i^1\right)$, $A_{4i} = \left(\mathcal{M}_{x_i}^0 y_i^1, \mathcal{N}_{x_i}^1 y_i^0\right)$.

Clearly (T_i, δ_i) *is NIFHS in which singletons are IF closed. Therefore* (T_i, δ_i) *is IFHS.*

Let $x \in T = \prod T_i$ *and* $\delta = \prod \delta_i = \left\langle \{0, 1, \theta_i^{-1}(A_{ni})/n = 1, 2, 3, 4.\right.$ *and* $\left. i = 1, 2, \cdots\} \right\rangle$
where

$$\mathcal{M}_{\theta_i^{-1}(A_{1i})}(x) = \begin{cases} \frac{1}{2}(1 - \frac{1}{i}), & \text{if } \theta_i(x) = x_i; \\ 1, & \text{if } \theta_i(x) = y_i; \end{cases}$$

$$\mathcal{M}_{\theta_i^{-1}(A_{2i})}(x) = \begin{cases} 1, & \text{if } \theta_i(x) = x_i; \\ 0, & \text{if } \theta_i(x) = y_i; \end{cases}$$

$$\mathcal{M}_{\theta_i^{-1}(A_{3i})}(x) = \begin{cases} \frac{1}{2}(1-\frac{1}{i}), & \text{if } \theta_i(x) = x_i; \\ 0, & \text{if } \theta_i(x) = y_i; \end{cases}$$

$$\mathcal{M}_{\theta_i^{-1}(A_{4i})}(x) = \begin{cases} 0, & \text{if } \theta_i(x) = x_i; \\ 1, & \text{if } \theta_i(x) = y_i; \end{cases}$$

Suppose for $x \in T$, $\{x\}$ is closed then there should exit some IFOS $A \in (\prod \delta_\alpha)$ such that

$$\mathcal{M}_A(z) = \begin{cases} 1 & \text{if } z \neq x; \\ < \frac{1}{2} & \text{if } z = x. \end{cases}$$

Since $A \in (\prod \delta_\alpha)$, A can be written as arbitrary union of basic elements. That is $A = \bigvee A_\alpha$,
$A_\alpha \in B \subset (\prod \delta_\alpha)$ Now

$$\bigvee \mathcal{M}_{A_\alpha}(z) = \begin{cases} 1 & \text{if } z \neq x; \\ < \frac{1}{2} & \text{if } z = x. \end{cases}$$

which implies $\bigvee \mathcal{M}_{A_\alpha}(x) < \frac{1}{2}$ and $sup_\alpha \mathcal{M}_{A_\alpha}(y) = 1$, for all $y \neq x$.
For given $\varepsilon > 0$ there should exist A_α such that $\mathcal{M}_{A_\alpha}(x) < \frac{1}{2}$ and $\mathcal{M}_{A_\alpha}(y) > 1 - \varepsilon$, for all $y \neq x$.
Also A_α is the finite intersection of subbasis elements. That is $A_\alpha = \bigwedge \theta_i^{-1}(A_{ni})$, for some finite numbers of i.
Let $C = \{\theta_i^{-1}(A_{ni}) / \bigwedge \mathcal{M}_{\theta_i^{-1}(A_{ni})}(x) < \frac{1}{2}$ and $\mathcal{M}_{\theta_i^{-1}(A_{ni})}(y) > 1 - \varepsilon$, for all $y \neq x\}$.
Here $\theta_i^{-1}(A_{2i}) \notin C$, for all i, since $\mathcal{M}_{\theta_i^{-1}(A_{2i})}(y) = 0$ when $\theta_i(x) = y_i$.
Similarly, $\theta_i^{-1}(A_{3i}) \notin C$, for all i.
For $i < j$,

$$(\mathcal{M}_{\theta_i^{-1}(A_{1i})} \bigwedge \mathcal{M}_{\theta_j^{-1}(A_{1j})})(x) = \begin{cases} \frac{1}{2}(1-\frac{1}{i}), & \text{if } \theta_i(x) = x_i, \theta_j(x) = x_j; \\ \frac{1}{2}(1-\frac{1}{i}), & \text{if } \theta_i(x) = x_i, \theta_j(x) = y_j; \\ \frac{1}{2}(1-\frac{1}{i}), & \text{if } \theta_i(x) = y_i, \theta_j(x) = x_j; \\ 1, & \text{if } \theta_i(x) = y_i, \theta_j(x) = y_j; \end{cases}$$

Here $(\mathcal{M}_{\theta_i^{-1}(A_{1i})} \bigwedge \mathcal{M}_{\theta_j^{-1}(A_{1j})})(x) < \frac{1}{2}$, *for all* $y \neq x$ *with* $\theta_i(x) = x_i$,$\theta_j(x) = y_j$ *and also for* $\theta_i(x) = y_i$,$\theta_j(x) = x_j$. *Therefore* $\theta_i^{-1}(A_{1i}) \notin C$, *for all* i.

$$(\mathcal{M}_{\theta_i^{-1}(A_{1i})} \bigwedge \mathcal{M}_{\theta_j^{-1}(A_{4j})})(x) = \begin{cases} 0, & \text{if } \theta_i(x) = x_i, \theta_j(x) = x_j; \\ \frac{1}{2}(1-\frac{1}{i}), & \text{if } \theta_i(x) = x_i, \theta_j(x) = y_j; \\ 0, & \text{if } \theta_i(x) = y_i, \theta_j(x) = x_j; \\ 1, & \text{if } \theta_i(x) = y_i, \theta_j(x) = y_j; \end{cases}$$

Here $(\mathcal{M}_{\theta_i^{-1}(A_{1i})} \bigwedge \mathcal{M}_{\theta_j^{-1}(A_{4j})})(x) < \frac{1}{2}$, *for all* $y \neq x$ *with* $\theta_i(x) = x_i$, $\theta_j(x) = y_j$ *and also for* $\theta_i(x) = y_i$, $\theta_j(x) = x_j$. *Therefore* $\theta_i^{-1}(A_{1i}) \bigwedge \theta_j^{-1}(A_{4j}) \notin C$, *for all* i, j.
For $i < j$,

$$(\mathcal{M}_{\theta_i^{-1}(A_{4i})} \bigwedge \mathcal{M}_{\theta_j^{-1}(A_{4j})})(x) = \begin{cases} 0, & \text{if } \theta_i(x) = x_i, \theta_j(x) = x_j; \\ 0, & \text{if } \theta_i(x) = x_i, \theta_j(x) = y_j; \\ 0, & \text{if } \theta_i(x) = y_i, \theta_j(x) = x_j; \\ 1, & \text{if } \theta_i(x) = y_i, \theta_j(x) = y_j; \end{cases}$$

Here $(\mathcal{M}_{\theta_i^{-1}(A_{4i})} \bigwedge \mathcal{M}_{\theta_j^{-1}(A_{4j})})(x) < \frac{1}{2}$, *for all* $y \neq x$ *with* $\theta_i(x) = x_i$,$\theta_j(x) = y_j$ *and also for* $\theta_i(x) = y_i$,$\theta_j(x) = x_j$. *Therefore* $\theta_i^{-1}(A_{4i}) \notin C$, *for all* i. *So* C *is empty. That is there is no* A_α *satisfying the requirement that* $\mathcal{M}_{A_\alpha}(x) < \frac{1}{2}$ *and* $\mathcal{M}_{A_\alpha}(y) > 1-\varepsilon, \quad \forall y \neq x$, *which is a contradiction. Therfore* $\{x\}$ *is not closed. Hence* (T, δ) *is not IFHS.*

Note. If we change our definition as "singleton $\{x\} \subseteq T$ is said to be IF closed, if there exist an IFOS O with $\mathcal{M}_O(x) \leq \frac{1}{2}$ and $\mathcal{M}_O(z) = 1$, for all $z \in T$ where $z \neq x$"; then arbitrary product of IFHS is IFHS.

5. Comparative studies

The following study shows that the class of all IFTS introduced in this paper is finer than the classes in [5], [11] and also coarser than the classes in [13] and [8]. For convenience we denote the notations and definitions in [5], [11], [13], [8] and 3.3 as H_a, H_b, H_c, H_d and H_e respectively.

Theorem 5.1. *The following implications are true:*
$H_a \Rightarrow H_b \Rightarrow H_e \Rightarrow H_c \Rightarrow H_d$.

Proof. $H_a \Rightarrow H_b$
Consider a IFTS (T, δ) in which H_a holds. Therefore for every distinct IFPs $p^x_{(\alpha,\beta)}$, $q^y_{(\alpha,\beta)}$ in T, there exist IFOSs M and N such that $\mathcal{M}_M(x) = 1, \mathcal{N}_M(x) = 0, \mathcal{M}_N(y) = 1, \mathcal{N}_N(y) = 0$ and $M \cap N = 0_\sim$. Clearly M and N are ε - neighbourhood of p and q respectively and $\mathcal{M}_M \cap \mathcal{M}_N = 0, \mathcal{N}_M \cup \mathcal{N}_N = 1$. If $\mathcal{M}_M \cap \mathcal{M}_N = 0$ then either $\mathcal{M}_M(x)$ or $\mathcal{M}_N(x) = 0$. If $\mathcal{M}_M(x) = 0$ then $\mathcal{M}_M(x) \leq 1 - \mathcal{M}_N(x) = \mathcal{M}'_N(x)$. If $\mathcal{M}_M(x) \neq 0$ then $\mathcal{M}_N(x) = 0 \Rightarrow \mathcal{M}_M(x) \leq 1 - \mathcal{M}_N(x) = \mathcal{M}'_N(x)$. Similarly $\mathcal{N}_M \cup \mathcal{N}_N = 1 \Rightarrow \mathcal{N}_M \geq \mathcal{N}'_N$ Hence H_b holds.
$H_b \Rightarrow H_e$
Consider a IFTS (T, δ) in which H_b holds. Take two distinct points $x, y \in T$. Consider the IFPs $p^x_{(1,0)}$, $q^y_{(1,0)}$ in T. Since (T, δ) satisfy H_b, there exist ε - neighbourhood M and N of p and q respectively such that $\mathcal{M}_M \leq \mathcal{M}'_N$ and $\mathcal{N}_M \geq \mathcal{N}'_N$. Then there exist two open sets G_1 and $G_2 \in \delta$ such that$p^x_{(1,0)} \in G_1 \subseteq M$ and $q^y_{(1,0)} \in G_2 \subseteq N$. Clearly $\mathcal{M}_{G_1}(x) = 1 > \frac{1}{2}$, $\mathcal{M}_{G_2}(y) = 1 > \frac{1}{2}$. Now we have to prove that $\mathcal{N}_{G_1}(z) + \mathcal{N}_{G_2}(z) \geq 1$, for all $z \in T$. Since $G_1 \subseteq M, G_2 \subseteq N$ we have $\mathcal{N}_{G_1} \geq \mathcal{N}_M$ and $\mathcal{N}_{G_2} \geq \mathcal{N}_N$. Now $\mathcal{N}_{G_1}(z) + \mathcal{N}_{G_2}(z) \geq \mathcal{N}_M(z) + \mathcal{N}_N(z) \geq 1$, since $\mathcal{N}_M \geq \mathcal{N}'_N$. Hence H_e holds.
$H_e \Rightarrow H_c$
Now consider a IFTS (T, δ) in which H_e holds. To prove H_c holds, consider two distinct points $x, y \in T$. Since (T, δ) holds H_e, there exist non zero disjoint IFOSs A and B of δ such that $\mathcal{M}_A(x) > \frac{1}{2}$, $\mathcal{M}_B(y) > \frac{1}{2}$. That is, there exist IFOSs $A \neq 0$, $B \neq 0$ in δ such that $\mathcal{M}_A(x) > \frac{1}{2}$, $\mathcal{M}_B(y) > \frac{1}{2}$, and $\mathcal{N}_A(t) + \mathcal{N}_B(t) \geq 1$, for every $t \in T$. It is to prove that $\mathcal{N}_A(t) \geq \mathcal{M}_B(t)$ and $\mathcal{N}_B(t) \geq \mathcal{M}_A(t)$.
By definition, we have

$$\mathcal{N}_A(t) + \mathcal{N}_B(t) \geq 1, \text{ for all } t \in T.$$
$$\Rightarrow \mathcal{N}_A(t) \geq 1 - \mathcal{N}_B(t) \geq \mathcal{M}_B(t).$$
$$\Rightarrow \mathcal{N}_A(t) \geq \mathcal{M}_B(t).$$

Similarly $\mathcal{N}_B(t) \geq \mathcal{M}_A(t)$. Hence H_c holds.
$H_c \Rightarrow H_d$

Consider a IFTS (T, δ) in which H_c holds. Then for every pair of distinct points $x, y \in T$ there exist disjoint IFOSs A and B of δ such that $\mathcal{M}_A(x) > \frac{1}{2}$, $\mathcal{M}_B(y) > \frac{1}{2}$. This will follow immediately such that there exist two open sets A and B such that $\mathcal{M}_A(x) > 0, \mathcal{N}_A(x) < 1, \mathcal{M}_A(y) < 1, \mathcal{N}_A(y) > 0$, and $\mathcal{M}_B(y) > 0, \mathcal{N}_B(y) < 1, \mathcal{M}_B(x) < 1, \mathcal{N}_B(x) > 0$. Now we have to prove the condition $(\mathcal{M}_A \cap \mathcal{M}_B) \subset (\mathcal{N}_A \cup \mathcal{N}_B)$. We have $\mathcal{N}_A(t) \geq \mathcal{M}_B(t)$ and $\mathcal{N}_B(t) \geq \mathcal{M}_A(t)$ for every $t \in T$. This will imply $\mathcal{N}_A(t) \cup \mathcal{N}_B(t) \supseteq \mathcal{M}_B(t) \cup \mathcal{M}_A(t) \supseteq \mathcal{M}_B(t) \cap \mathcal{M}_A(t)$. Hence H_d holds. □

Remark 5.2. *The following examples show that the converse of above implications does not hold.*

Example 5.3. $H_d \nRightarrow H_c$
Let $T = \{e, f\}$, $\delta = \{(0,1), (1,0), (\mathcal{M}_{e^{.2}f^{.3}}, \mathcal{N}_{e^{.6}f^{.7}}), (\mathcal{M}_{e^{.2}f^{.4}}, \mathcal{N}_{e^{.7}f^{.3}}),$
$(\mathcal{M}_{e^{.2}f^{.3}}, \mathcal{N}_{e^{.7}f^{.7}}),$
$(\mathcal{M}_{e^{.2}f^{.4}}, \mathcal{N}_{e^{.6}f^{.3}})\}$. *Clearly* (T, δ) *is IFTS in which* H_d *holds. But* H_c *does not holds.*

Example 5.4. $H_c \nRightarrow H_e$
Let $T = \{e, f\}$, $\delta = \{(0,1), (1,0), (\mathcal{M}_{e^{.6}f^{.3}}, \mathcal{N}_{e^{.3}f^{.6}}), (\mathcal{M}_{e^{.2}f^{.6}}, \mathcal{N}_{e^{.6}f^{.4}}),$
$(\mathcal{M}_{e^{.6}f^{.6}}, \mathcal{N}_{e^{.3}f^{.4}}), (\mathcal{M}_{e^{.2}f^{.3}}, \mathcal{N}_{e^{.6}f^{.6}})\}$. *Clearly* (T, δ) *is IFTS in which* H_c *holds. But* H_e *does not holds.*

Example 5.5. $H_e \nRightarrow H_b$
Let $T = \{e, f\}$, $\delta = \{(0,1), (1,0), (\mathcal{M}_{e^{.6}f^{.2}}, \mathcal{N}_{e^{.3}f^{.8}}), (\mathcal{M}_{e^{.2}f^{.6}}, \mathcal{N}_{e^{.8}f^{.3}}),$
$(\mathcal{M}_{e^{.6}f^{.6}}, \mathcal{N}_{e^{.3}f^{.3}}), (\mathcal{M}_{e^{.2}f^{.2}}, \mathcal{N}_{e^{.8}f^{.8}})\}$. *Clearly* (T, δ) *is IFTS in which* H_e *holds. But* H_b *does not holds since take two IFPs* $P^e_{(.7,.1)}, q^f_{(.7,.1)}$ *there does not exists* ε *- neighbourhood* M *and* N *of* p *and* q *respectively.*

Example 5.6. $H_b \nRightarrow H_a$
Let $T = \{e, f\}$, $\delta = \{(0,1), (1,0), (\mathcal{M}_{e^{.6}f^{0}}, \mathcal{N}_{e^{.3}f^{1}}), (\mathcal{M}_{e^{0}f^{.6}}, \mathcal{N}_{e^{1}f^{.3}}),$
$(\mathcal{M}_{e^{.6}f^{.6}}, \mathcal{N}_{e^{.3}f^{.3}}), \}$.
Clearly (T, δ) *is IFTS in which* H_b *holds. But* H_a *does not holds.*

6. Hausdorffness in image processing

A raster image is a $2D$ array of numbers representing pixel intensities. In image processing points are analogous to pixels. A feature vector is a vector of numbers where each number describes a feature value of a point. The feature vector which describes a point can be conveniently modelled as a fuzzy set. Apart from the existing spatially distinct points, the feature vector promotes descriptively distinct points, descriptive remote(near)sets, descriptive proximity space. A descriptive Hausdorff space [12] is defined in the context of a descriptive distinct neighbourhoods and descriptive proximity space. Every raster image is a descriptive Hausdorff space and it is found in paintings and any digital image. Visual patterns can be recognised by searching for pairs of descriptively near neighbourhoods in descriptive Hausdorff space.

7. Significance

Separation axiom especially Hausdorffness is very much important in the field of topology. Many interesting results we got because of this axiom. Such an important result is 'In IFHS, every convergent sequence has a unique limit point' which we proved in this paper. Moreover Hausdorffness has significant role in fuzzy digital topological space. The study of digital images naturally leads to recognition of different types of sets embedded in image. The outcome of this approach is discovery of spatially near sets and descriptively near sets of picture elements. Sets of picture elements are spatially near provided sets have picture elements in common. Sets of picture elements are descriptively near provided sets contain picture elements that resemble each other. Nearness in this case based on the perception of closeness of pixel feature values. For further study we can use our Hausdorff condition to distinguish sets embedded in fuzzy digital images.

8. Conclusion

This paper attributes a new notion of IFHS and its properties have been studied. The class of all IFTS introduced in this paper is finer than the classes in [5], [11] and also coarser than the classes in [13] and [8].

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