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Uniqueness of fixed points for sums of operators in ordered Banach spaces and applications

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Abstract

In this article, we are concerned by existence and uniqueness of a fixed point for the sum of two operators A and B, defined on a closed convex subset of an ordered Banach space, where the order is induced by a normal and minihedral cone. In such a structure, an absolute value function is generated by the order and this provide the ability to introduce new versions of the concepts of lipschitzian and expansive mappings. Therefore we prove that if A is expansive and B is contractive, then the sum $A + B$ has a unique fixed point.

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1. Introduction

In this paper, we are concerned by existence of a solution to the fixed point equation:

$$x = Ax + Bx, \ x \in \Omega \tag{1.1}$$

where Ω is a closed convex set (a ball of radius r in practice), of a Banach space X and $A, B : \Omega \to X$ are two continuous mapping.

The Krasnoselskii's theorem (1958) is the first and the most famous fixed-point principle solving Equation (1.1), see [20]. His result states that if

(a) A is a contraction,

(b) B is compact and

(c) for all $x, y \in \Omega$, $Ax + By \in \Omega$,

then the sum of operators $A + B$ has at least one fixed point.

The proof of this result is based on the following three facts. Under Condition (a), the inverse map of $I - A$ exists and is continuous. Condition (b) implies that the mapping $(I-A)^{-1} B$ is compact and Condition (c) leads to $(I-A)^{-1} B(\Omega) \subset \Omega$. Since Ω is closed and convex, and fixed points of $(I-A)^{-1} B$ are also fixed points of $A + B$, the author of [20] used Schauder's fixed point theorem to conclude that the sum of operators $A + B$ has at least one fixed point in Ω.

Because of the lack of compactness in infinite dimensional Banach spaces and in order to measure how a set is not compact, Kuratowsky introduced in 1930 his measure of noncompactness, see [21]. This new measure has been used by Darbo, see [11], to introduce a new version of contraction hypothesis. He calls A a k-set contraction mapping if

$$\alpha\left(A(D)\right) \leq k\alpha\left(D\right) \text{ for all } D \subset \Omega \text{ bounded},$$

where α denotes Kuratowsky's measure of noncompactness and $k \in [0, 1)$. Then he proved that any k-set contraction mapping which maps continuously Ω into itself, has at least a fixed point. Since the sum of a contraction and a completely continuous mapping turned out to be a k-set contraction, Darbo's theorem is considered an extension of Krasnoselskii's fixed point theorem. Since then, the existence of fixed points for the sum of two operators has attracted tremendous interest and many forms and improvements of Krasnoselskii's fixed point theorem, have been established in the literature by modifying the above assumptions, see, for example, [2], [7], [14], [15], [16], [24], [23] and references therein.

Recently, in [6] and [5] a new version of hypothesis of contraction is introduced. In these works, authors consider the case where X is an ordered Banach space and the order is induced by a normal and minihédral cone. It is well known that in such a framework an absolute value function $|\cdot|$ is defined. Hence the authors of [6] and [5] replaced the condition of contraction on A by

$$\tag{1.2} \begin{cases} \text{there exists a positive operator } L \text{ in } \mathcal{L}(X) \\ \text{and } k \geq 0, \text{ such that } kr(L) < 1 \text{ and} \\ |Ax - Ay| \preceq kL\left(|x-y|\right) \text{ for all } x, y \in \Omega, \end{cases}$$

where $r(L)$ is the spectral radius of L. Then they proved that any operator T mapping continuously the closed and convex set Ω into itself and satisfying Hypothesis (1.2) under hypothesis if the contraction mapping T maps continuously the closed convex set Ω into itself, has a unique fixed point. In [5], the author considered the case of Equation (1.1) in such an ordered Banach space X. Assuming that the mappings A and B satisfy Hypotheses (1.2), (b) and (c), he proved that the sum $A + B$ admits at least one fixed point in Ω.

Motivated by the work in [5], we consider in this paper Equation (1.1) in the same framework of [6] and [5]. In this contribution, we introduce a new version of expansivity, we say that $T : \Omega \subset X \to X$ is (k, L)-expansive if

$$|Tx - Ty| \succeq kL\left(|x-y|\right) \text{ for all } x, y \in \Omega,$$

where $L \in \mathcal{L}(X)$ is positive and $k > 0$. Therefore, we prove that if A is (α, L_A)-expansive, B is (β, L_B)-lipschtizian (ie B satisfies (1.2) without $\beta r(L_B) < 1$), and $A(\Omega) \subset (I - B)(\Omega)$ and if $x = Ax + By$ with $y \in \Omega$ implies $x \in \Omega$, then the sum of operators $A + B$ has a unique fixed point in Ω.

This work is ended by two applications of our main result. The first application concerns algebraic equations and the second one concerns the integral equations of Urysohn-type posed on unbounded interval.

2. Abstract background

In all this section we let $(E, \|\cdot\|)$ be a real Banach space. The standard notation $\mathcal{L}(E)$ refers to the set of all linear bounded self-mapping defined on E. For $L \in \mathcal{L}(E)$ the notations $\sigma(L)$ and $r(L)$ are commonly used and refer respectively to the spectrum of L and the spectral radius of L. We recall below their definitions,

$$\sigma(L) = \{\mu \in \mathbf{C} : (\mu I - L) \text{ is not invertible}\},$$

$$r(L) = \sup\{|\mu| : \quad \mu \in \sigma(L)\}.$$

Now let us recall some basic facts related to cones and positivity.

Definition 2.1. *A nonempty closed and convex set K of E is said to be a cone in E if, $(tK) \subset K$ for all $t \geq 0$ and $K \cap (-K) = \{0_E\}$.*

It is well known that if K is a cone in E, then K induces a partial order on the Banach space E. We write for all $x, y \in E :\ x \preceq_K y$ (or $y \succeq_K x$) if $y - x \in K$ and $x \prec_K y$ (or $y \succ_K x$) if $y - x \in K\,\{0_E\}$. Thus, vectors lying in $K\,\{0_E\}$ are said to be positive.

Definition 2.2. *Let Ω be a nonempty set in E. Then*

a) *$u \in E$ is said to be an upper bound of Ω if $v \preceq_K u$ for all $v \in \Omega$;*

b) *$u \in E$ is said to be a lower bound of Ω if $v \succeq_K u$ for all $v \in \Omega$;*

c) *$u \in E$ is said to be the least upper bound of Ω and we write $u = \sup \Omega$, if u is an upper bound of Ω and $v \preceq_K w$ for all $v \in \Omega$ implies $u \preceq_K w$;*

d) *$u \in E$ is said to be the greatest lower bound of Ω and we write $u = \inf \Omega$, if u is a lower bound of Ω and $v \succeq_K w$ for all $v \in \Omega$ implies $u \succeq_K w$.*

Definition 2.3. *Let K be a cone in E. The cone K is said to be*

a) *normal, if there is a positive constant n_K such that for all $u, v \in E$, $0_E \preceq_K u \preceq_K v$ implies $\| \leq n_K \|v\|$,*

b) *minihedral if $\sup(x, y)$ exists for all $x, y \in E$.*

Remark 2.4. *Notice that if a cone K is minihedral then $\inf(x, y)$ exists for all $x, y \in E$. Moreover, we have $\inf(x, y) = -\sup(-x, -y)$.*

Remark 2.5. *It is well known that if K is a minihedral cone inducing the order $\preceq_K$ on E, then $(E, \preceq_K)$ is a Riesz space or a Banach lattice in the sence given in [22]..*

Definition 2.6. *Let K be a minihedral cone in E inducing the order $\preceq_K$ on E. For $x \in E$, we define the positive part, the negative part and the absolute value of the vector x respectively by*

$$x^+ = \sup(x, 0), \qquad x^- = \sup(-x, 0) \qquad \text{and} \qquad |x| = x^+ + x^-.$$

Proposition 2.7. *([22]) Let K be a minihedral cone in E inducing the order $\preceq_K$ on E. Then the absolute value define then a self-mapping on E and it has the following properties:*

i) $|x| \succeq_K 0_E$ *for all* $x \in E$,

ii) $|x| = 0_E \Rightarrow x = 0_E$,

iii) $|tx| = |t|\,|x|$ *for all* $x \in E$ *and* $t \in \mathbf{R}$,

iv) $|x + y| \preceq_K |x| + |y|$ *for all* $x, y \in E$,

v) $||x| - |y|| \preceq_K |x - y|$ *for all* $x, y \in E$.

Proposition 2.8. *([5])Let K be a minihedral cone in E, then the following assertions are equivalents.*

i) *The mapping* $|\cdot| : E \to K$ *is continuous.*

ii) *The mapping* $|\cdot| : E \to K$ *is continuous at* 0_E.

iii) *There exists* $\eta > 0$ *such that* $\||u|\| \leq \eta \|u\|$ *for all* $u \in E$.

iv) $\sup_{\|u\|=1} \||u|\| < \infty$.

Definition 2.9. *Let K be a cone in E, a mapping $L \in \mathcal{L}(E)$ is said to be positive, if $L(K) \subset K$.*

Throughout the notation $\mathcal{L}_K(E)$ will refer to the subset of $\mathcal{L}(E)$ of positive operators.

For detailled presentations on cones and positivity we refer the reader to [13] and [17]. The reader will observe that the definition of the minihedrality given here is that of [13]. In [17], a cone C is said to be minihedral if $\sup(x, y)$ exists for all pair $(x, y) \in E^2$ having an upper bounded. To ensure the existence of $\sup(x, y)$ for all $x, y \in E$ when such is the definition of the minihedrality, one may assume that the cone C is generating (i.e. $E = K - K$). Indeed, for all $x, y \in E$ there exist $x_1, x_2, y_1, y_2 \in K$ such that $x = x_1 - x_2$ and $y = y_1 - y_2$. Therefore, we have $x \preceq_K x_1 + y_1$ and $y \preceq_K x_1 + y_1$.

3. Main results

In all this section, we let $(E, \|\cdot\|)$ be a real Banach space and we let K be a normal and minihedral cone in E, inducing the order $\preceq$ and the absolute function $|\cdot|$ on E. All the notations of this section are that introduced in Section 2 and we suppose throughout that the absolute value function

$$|\cdot| : E \to K \text{ is continuous.} \tag{3.1}$$

We introduce below two new concepts.

Definition 3.1. *Let $k \geq 0$ and $L \in \mathcal{L}_K(E)$. A mapping $T : \Omega \subset E \to E$ is said (k, L)-lipschitzian if,*

$$|Tu - Tv| \preceq kL(|u - v|), \text{ for all } u, v \in \Omega.$$

Definition 3.2. *Let $k > 0$ and $L \in \mathcal{L}_K(E)$. A mapping $T : \Omega \subset E \to E$ is said (k, L)-expansive if,*

$$|Tu - Tv| \succeq kL(|u - v|), \text{ for all } u, v \in \Omega.$$

We will use in the proof of the main result of this work, the following theorem which is a cosequence of Corollary 2 and Remark 4 in [6]. It provides a variant of Banach contraction principle in ordered Banach spaces.

Theorem 3.3. *Let Ω is nonempty closed convex subset in E and $T : \Omega \to \Omega$ (k, L)-lipschitzian mapping, where $k \geq 0$ and $L \in \mathcal{L}_K(E)$. If the absolute function $|\cdot|$ is continuous on E and $kr(L) < 1$, then T admits a unique fixed point.*

Let now Ω be a nonempty closed convex subset in E and $T : E \to E$ and $S : \Omega \to E$ be two mappings such that

$$S(\Omega) \subset (I-T)(\Omega)\,, \tag{3.2}$$

$$v = Tv + Su,\ u \in \Omega \text{ implies } v \in \Omega, \tag{3.3}$$

The main result of this work consists in the following theorem.

Theorem 3.4. *Assume that Hypotheses (3.1)-(3.3) hold true, the mapping T is (β, L_T)-expansive and S is (α, L_S)-lipschitzian, where, $\alpha \geq 0,\ \beta > 0$ and $L_S, L_T \in \mathcal{L}_K(E)$ are such that*

$$\begin{cases} \beta^{-1} \notin \sigma(L_T)\,,\ (\beta L_T - I)^{-1} \in \mathcal{L}_K(E) \text{ and} \\ \alpha r((\beta L_T - I)^{-1} L_S) < 1. \end{cases} \tag{3.4}$$

Then the sum of operators, $T + S$ has a unique fixed point in Ω.

Proof. For all $u, v \in E$ we have

$$\begin{aligned} |(I-T)(u) - (I-T)(v)| &= |(T(u) - T(v)) - (u-v)| && (3.5)\\ &\succeq |T(u) - T(v)| - |u-v| && (3.6)\\ &\succeq (\beta L_T - I)(|u-v|)\,. && (3.7) \end{aligned}$$

This shows that the mapping $(I-T)$ is one to one and the inverse-mapping of $(I-T) : E \to (I-T)(E)$ exists. Moreover, since from Hypothesis (3.4), $(\beta L_T - I)^{-1}$ is positive, the estimate (3.5) leads to

$$\left|(I-T)^{-1}(x) - (I-T)^{-1}(y)\right| \preceq (\beta L_T - I)^{-1}(|x-y|) \quad \text{for all } x, y \in (I-T)(\Omega)\,.$$

Therefore, Hypothesis (3.1) and the continuity of the mappings $(\beta L_T - I)^{-1}$ lead to that of the mapping $(I-T)^{-1}$.

At this stage, since by Hypothesis (3.2), $S(\Omega) \subset (I-T)(\Omega)$ the mapping $(I-T)^{-1} S : \Omega \to E$ is well defined. Moreover, if $v = (I-T)^{-1} S(u)$ with $u \in \Omega$, then Hypothesis (3.3) leads to $v = Tv + Su \in \Omega$. Therefore we have proved that $(I-T)^{-1} S(\Omega) \subset \Omega$.

Since $x \in \Omega$ is a fixed point of $(I-T)^{-1} S$ if and only if x is a fixed point of $T + S$, Theorem 3.4 will be proved once we prove that $(I-T)^{-1} S$ has a unique fixed in Ω. Thus, let us prove that the mapping $(I-T)^{-1} S$ has a unique fixed point in Ω.

Since S is (α, L_S)-lipschitzian and T is (β, L_T)-expansive for all $u, v \in \Omega$ we have

$$\begin{aligned}\left|(I-T)^{-1} S(u)-(I-T)^{-1} S(v)\right| &\preceq (\beta L_T - I)^{-1}(|S(u)-S(v)|)\\ &\preceq \alpha(\beta L_T - I)^{-1} L_S(|u-v|). \qquad (3.8)\end{aligned}$$

Since the continuity of the absolute value function implies that there is $\eta > 0$ such that $\||u|\| \leq \eta \|u\|$ for all $u \in E$ (see iii) in Proposition 2.8), and taking in consideration the normality of the cone K, for all $u, v \in \Omega$ we have from (3.8)

$$\begin{aligned}\left\|(I-T)^{-1} S(u)-(I-T)^{-1} S(v)\right\| &\leq \left\|\left|(I-T)^{-1} S(u)-(I-T)^{-1} S(v)\right|\right\|\\ &\leq n_K \left\|(\beta L_T - I)^{-1} L_S\right\| \||u-v|\|\\ &\leq n_K \eta \left\|(\beta L_T - I)^{-1} L_S\right\| \|u-v\|.\end{aligned}$$

This shows that the mapping $(I-T)^{-1} S$ is continuous. At the end, since $(\beta L_T - I)^{-1} L_S \in \mathcal{L}_K(E)$ and $\alpha r\left((\beta L_T - I)^{-1} L_S\right) < 1$, we obtain from Theorem 3.3 that $(I-T)^{-1} S$ has a unique fixed point in Ω. The proof is complete. □

Since the condition $\alpha r((\beta L_T - I)^{-1} L_S) < 1$ in Hypothesis (3.4) is difficult to verify when the Banach space E is of infinite dimension, we present below particular situations of Theorem 3.4 where the verification of this condition is less difficult.

Corollary 3.5. *Assume that Hypotheses (3.1)-(3.3) hold true, the mapping T is (β, I)-expansive and S is (α, L)-lipschitzian, where, $\alpha \geq 0,\ \beta > 0$ are such that*

$$\beta > 1 \text{ and } \alpha(\beta-1)^{-1} r(L) < 1. \qquad (3.9)$$

Then the sum of operators, $T + S$ has a unique fixed point in Ω.

Corollary 3.6. *Assume that Hypotheses (3.1)-(3.3) hold true, the mapping T is (β, I)-expansive and S is (α, I)-lipschitzian, where , $\alpha \geq 0,\ \beta > 0$ are such that*

$$\beta > 1 \text{ and } \alpha(\beta-1)^{-1} < 1. \qquad (3.10)$$

Then the sum of operators, $T + S$ has a unique fixed point in Ω.

The following corollary consider the particular case of Theorem 3.4 where $L_S = L_T$.

Corollary 3.7. *Assume that Hypotheses (3.1)-(3.3) hold true, the mapping T is (β, L)-expansive and S is (α, L)-lipschitzian, where , $\alpha \geq 0,\ \beta > 0$ are such that*

$$(3.11)\quad (\beta L - I)^{-1} \in \mathcal{L}_K(E) \text{ and } (\beta - \alpha) \inf \{|\lambda| : \lambda \in \sigma(L)\} > 1.$$

Then the sum of operators, $T + S$ has a unique fixed point in Ω.

Proof. Set $\lambda_* = \inf \{|\lambda| : \lambda \in \sigma(L)\}$. It follows from Hypothesis (3.12) that $\beta^{-1} \notin \sigma(L)$ and for all $\lambda \in \sigma(L)$, $\beta |\lambda| \geq \beta\lambda_* \geq (\beta - \alpha)\lambda_* > 1$. Let $\epsilon > 0$ small enough such that $(\beta - \epsilon)\lambda_* > 1$ and set $r_\epsilon = (\beta - \epsilon)^{-1} > \beta^{-1}$. Therefore there exists $r > 0$ such that

$$\sigma(L) \subset C(0, r_\epsilon, r) = \{z \in \mathbf{C} : r_\epsilon < |z| < r\}$$

and the function $f(z) = \frac{\alpha z}{1 - \beta z}$ is holomorphic in $C(0, r_\epsilon, r)$. By the spectral mapping theorem, (see [25], p. 227), we have $\sigma\left(\alpha (I - \beta L)^{-1} L\right) = f(\sigma(L))$ and

$$r\left(\alpha (I - \beta L)^{-1} L\right) = \sup\left\{|f(\lambda)| = \left|\frac{\alpha\lambda}{1 - \beta\lambda}\right| : \lambda \in \sigma(L)\right\}.$$

Since for all $\lambda \in \sigma(L)$,

$$|f(\lambda)| = \left|\frac{\alpha\lambda}{\beta\lambda - 1}\right| = \frac{\alpha|\lambda|}{|\beta\lambda - 1|} \leq \frac{\alpha|\lambda|}{\beta|\lambda| - 1}$$

and the real function $f(x) = \frac{\alpha x}{\beta x - 1}$ is decreasing, we obtain from (3.12),

$$r\left(\alpha (\beta L - I)^{-1} L\right) \leq \frac{\alpha\lambda_*}{\beta\lambda_* - 1} < 1.$$

Therefore, Hypothesis (3.4) is satisfied and Theorem 3.4 guarantees the existence and the uniqueness of a fixed point for the sum $T + S$. $\square$

4. Applications

4.1. Fixed point equation in $\mathbf{R}^2$

Let $a, b, \alpha, \beta, \theta, \eta, \xi$ and ε be real numbers with $a, b, \alpha, \beta, \theta, \eta$ positive and consider the functions $T, S : \mathbf{R}^2 \to \mathbf{R}^2$ defined for all $(x, y) \in \mathbf{R}^2$ by:

$$T(x, y) = (ax^3 + \alpha x + \xi, by^5 + \beta y + \varepsilon) \text{ and } S(x, y) = (\theta \sin x, \eta \cos y).$$

We are concerned here by the existence and uniqueness in $\mathbf{R}^2$ of the solution to the fixed point equation associated to the sum of operators $T + S$,

$$(x, y) = T(x, y) + S(x, y).$$

Theorem 4.1. *Assume that*

$$\left(1 - \max\left(\frac{\theta}{\alpha}, \frac{\eta}{\beta}\right)\right) \min(\alpha, \beta) > 1,$$

then the sum of operators $T + S$ admits a unique fixed point.

Proof. Let K be the cone of $\mathbf{R}^2$ defined by

$$K = \left\{(x, y) \in \mathbf{R}^2 : x \geq 0 \text{ and } y \geq 0.\right\}$$

Clearly, the cone K is normal and minihedral and generates on $\mathbf{R}^2$ the absolute value function $|\cdot|$, where $|(x, y)| = (|x|, |y|)$ for all $(x, y) \in \mathbf{R}^2$.

Straightforward computations lead to

$$|T(x, y) - T(u, v)| \succeq L\left(|(x, y) - (u, v)|\right)$$

and

$$|S(x, y) - S(u, v)| \preceq \max\left(\frac{\theta}{\alpha}, \frac{\eta}{\beta}\right) L\left(|(x, y) - (u, v)|\right)$$

for all $(x, y), (u, v) \in \mathbf{R}^2$, where $L(x, y) = (\alpha x, \beta y)$.

Since α and β are positive real numbers, we have that $L \in \mathcal{L}_K\left(\mathbf{R}^2\right)$. Moreover, since the assumption of Theorem 4.1 implies that $\alpha, \beta \in (1, +\infty)$ and for all $(x, y) \in \mathbf{R}^2$, we have $(L - I)^{-1}(x, y) = \left(\frac{x}{\alpha - 1}, \frac{y}{\beta - 1}\right)$, we obtain that $(L - I)^{-1} \in \mathcal{L}_K\left(\mathbf{R}^2\right)$.

At the end, all the conditions of Corollary 3.7 are satisfied since $\sigma(L) = \{\alpha, \beta\}$ and $\left(1 - \max\left(\frac{\theta}{\alpha}, \frac{\eta}{\beta}\right)\right) \min(\alpha, \beta) > 1$. Therefore the sum of operators $T + S$ admits a unique fixed point. $\square$

4.2. Uniqueness of the positive solution for Urysohn-type integral equation

The theory of integral equations is an important branch of nonlinear Analysis. Integral equations originate from numerous areas of science, such as biology, transport theory, kinetic theory of gases and so on; see for instance, [1], [8], [9] and [10]. Urysohn-type integral equations is one of the most classes of nonlinear integral equations describing real word problems; see [3], [4], [12], [18], [19] and references therein.

We are concerned in this section by uniqueness of the solution in $BC(\mathbf{R})$ to the integral equation of Urysohn-type,

$$u(t) = f(t, u(t)) + \int_{\mathbf{R}} G(t, s) g(s, u(s)) ds, \tag{4.1}$$

where $BC(\mathbf{R})$ denotes the set of all continuous bounded real functions defined on $\mathbf{R}$ and $G, f, g : \mathbf{R}^2 \to \mathbf{R}$ are continuous functions such that $G(t, s) \geq 0$ and $g(t, s) \geq 0$ for all $(t, s) \in \mathbf{R}^2$. Throughout this section we assume that there exist continuous functions $\phi, \gamma : \mathbf{R} \to \mathbf{R}^+$, $h : \mathbf{R}^2 \to \mathbf{R}^+$ and nonnegative constants α and β such that

$$\begin{cases} \int_{\mathbf{R}} \gamma(s) \phi(s) \, ds < \infty, \ G(t,s) \leq \gamma(s) \ \text{ for all } t, s \in \mathbf{R} \text{ and} \\ G(t,s) \phi(s) > 0 \text{ for all } t, s \in [\xi, \eta] \subset \mathbf{R}, \end{cases} \tag{4.2}$$

$$\begin{cases} \text{for all } R > 0 \text{ there exists } M_R > 0 \text{ such that} \\ f(s,u) \leq M_R \text{ for all } s \in \mathbf{R} \text{ and } u \in [-R, R], \end{cases} \tag{4.3}$$

$$u - f(t,u) = v, \ v \geq 0 \ \Rightarrow u = h(t,v), \tag{4.4}$$

$$\begin{cases} \int_{\mathbf{R}} \gamma(s) \, g(s,0) ds < \infty, \\ |f(s,u) - f(s,v)| \geq \alpha \, |u - v| \ \text{ and} \\ |g(s,u) - g(s,v)| \leq \beta \phi(s) \, |u - v| \ \text{ for all } s \in \mathbf{R} \text{ and } u, v \in \mathbf{R}^+. \end{cases} \tag{4.5}$$

Let K denote the cone of nonnegative functions in $BC(\mathbf{R})$. It is well known that the cone K is normal and it generates a continuous absolute value function $|\cdot|$, where for $u \in BC(\mathbf{R})$ $|u|(t) = \max(-u(t), u(t))$.

Lemma 4.2. *Assume that Hypothesis (4.2) holds and let for $u \in BC(\mathbf{R})$,*

$$Lu(x) = \int_{\mathbf{R}} G(x,s)\phi(s)\, u(s)ds.$$

Then L is a linear operator belonging to $\mathcal{L}_K(BC(\mathbf{R}))$ having a positive spectral radius.

Proof. For any $u \in BC(\mathbf{R})$ Hypothesis (4.2) leads to the following estimates:

$$|Lu(t)| \le \int_{\mathbf{R}} G(t,s)\phi(s)\,|u(s)|\,ds \le \left(\int_{\mathbf{R}} \gamma(s)\phi(s)ds\right)\|u\|_{\infty} < \infty, \text{ for all } t \in \mathbf{R} \tag{4.6}$$

and

$$|Lu(t_1) - Lu(t_2)| \le \left(\int_{-\infty}^{+\infty} |G(t_1,s) - G(t_2,s)|\,\phi(s)ds\right)\|u\|_{\infty} \text{ for all } t_1, t_2 \in \mathbf{R}. \tag{4.7}$$

Estimate (4.6) shows that $Lu(t)$ is defined for all $t \in \mathbf{R}$. If $(t_n) \subset \mathbf{R}$ is a sequence converging to t_*, Estimates (4.6) and (4.7) lead to

$$|Lu(t_n) - Lu(t_*)| \le |Lu(t_n)| + |Lu(t_*)| \le 2\left(\int_{\mathbf{R}} \gamma(s)\phi(s)ds\right)\|u\|_{\infty} \text{ for all } n \ge 1$$

and

$$|Lu(t_n) - Lu(t_*)| \le \left(\int_{-\infty}^{+\infty} |G(t_n,s) - G(t_*,s)|\,\phi(s)ds\right)\|u\|_{\infty} \text{ for all } n \ge 1.$$

Since G is continuous, we conclude by Lebesgue's dominated convergence theorem that $Lu \in C(\mathbf{R})$. Noticing that (4.6) provide a uniform bound, we conclude that $Lu \in BC(\mathbf{R})$ and the linear operator L is bounded. Clearly the positivity of the function G makes of L an operator in $\mathcal{L}_K(BC(\mathbf{R}))$.

At the end, let us prove that $r(L) > 0$. Let $u_0 : \mathbf{R} \to \mathbf{R}^+$ be the function defined by

$$u_0(t) = \begin{cases} 0, \text{ if } t \in (-\infty, \xi] \\ \frac{4}{\eta-\xi}(t-\xi), \text{ if } t \in \left[\xi, \frac{3\xi+\eta}{4}\right] \\ 1, \text{ if } t \in \left[\frac{3\xi+\eta}{4}, \frac{\xi+3\eta}{4}\right] \\ \frac{4}{\eta-\xi}(\eta-t), \text{ if } t \in \left[\frac{\xi+3\eta}{4}, \eta\right] \\ 0, \text{ if } t \in [\eta, +\infty), \end{cases}$$

$G_0 = \min\{G_2(t,s)\phi(s) : t, s \in [\xi, \eta]\}$,

$\theta_0 = \int_\xi^\eta u_0(s)\, ds = \frac{3(\eta-\xi)G_0}{4}$.

Therefore, we have

$$\begin{array}{ll} Lu_0(t) & \geq 0 = \theta_0 u_0(t) \text{ for } t \in (-\infty, \xi] \cup [\eta, +\infty) \text{ and} \\ Lu_0(t) & \geq \int_\xi^\eta G(t,s)\phi(s)\, u_0(x)\, ds \geq \theta_0 \geq \theta_0 u_0(t) \text{ for } t \in [\xi, \eta], \end{array}$$

leading to $Lu_0 \geq \theta_0 u_0$ and $r(L) \geq \theta_0 > 0$. The proof is complete. □

The following theorem is the main result of this section.

Theorem 4.3. *Assume that Hypotheses (4.2)-(4.5) hold. If $\alpha > 1$ and $(\alpha-1)^{-1}\beta r(L) < 1$, then the integral equation (4.1) admit a unique solution lying in the cone K.*

Proof. Let $\begin{array}{l} Tu(t) = f(t, u(t)) \text{ for all } u \in BC(\mathbf{R}) \text{ and} \\ Su(t) = \int_{\mathbf{R}} G(t,s)g(s,u(s))ds \text{ for all } u \in K. \end{array}$

Clearly, for any $u \in BC(\mathbf{R})$, the continuity of the function f and Hypothesis (4.3) lead to $Tu \in BC(\mathbf{R})$ as well as to the continuity of the operator T. Now for any $u \in K$, we have

$$\text{(4.8)} \quad \begin{array}{ll} Su & (t \leq \beta \int_{\mathbf{R}} G(t,s)\phi(s)\, u(s)ds + \int_{\mathbf{R}} G(t,s)g(s,0)ds \\ & leq \int_{\mathbf{R}} \gamma(s)\phi(s)ds + \int_{\mathbf{R}} \gamma(s)g(s,0)ds < \infty, \text{ for all } t \in \mathbf{R}, \end{array}$$

$$\text{(4.9)} \quad \begin{array}{ll} | & Su(t_1) - Su(t_2) \leq \|u\| \int_{\mathbf{R}} |G(t_1,s) - G(t_2,s)|\, \phi(s)ds \\ & + \int_{\mathbf{R}} |G(t_1,s) - G(t_2,s)|\, g(s,0)ds \text{ for all } t_1, t_2 \in \mathbf{R}. \end{array}$$

Arguing as in the proof of Lemma 4.2, we obtain from Estimates (4.8) and (4.9) that for any $u \in K$, $Su \in K$.

Now for $u, v \in K$ we obtain from Hypothesis (4.5),

$$|Tu(t) - Tv(t)| \geq \alpha\, |u(t) - v(t)|$$

and

$$\begin{aligned} |Su(t) - Sv(t)| &\leq \int_{\mathbf{R}} G(t,s)\phi(s) \left|g(s,u(s)) - g(s,v(s))\right| ds \\ &\leq \beta \int_{\mathbf{R}} G(t,s)\phi(s) \left|u(s) - v(s)\right| ds \\ &= \beta L \left|u - v\right|(t). \end{aligned}$$

From all the above calculations, we see that all conditions of Corollary 3.5 are satisfied, hence the Urysohn integral equation (4.1) admits a unique positive solution. □

References

[1] R. P. Agarwal, D. O'Regan and P. J. Y. Wong, *Positive solutions of differential, difference and Integral equations.* Dordrecht: Kluwer Academic Publishers, 1999. doi: 10.1007/978-94-015-9171-3

[2] R. P. Agarwal, D. O'Regan, and M.-A. Taoudi, "Browder-Krasnoselskii-type fixed point theorems in Banach spaces", *Fixed Point Theory and Applications*, vol. 2010, no. 1, p. 243716, 2010. doi: 10.1155/2010/243716

[3] H. K. Awad and M. A. Darwish, "On Erdélyi-Kober cubic fractional integral equation of Urysohn-Volterra type", *Differential Equations and control processes*, vol. 1, 2019. [On line]. Available: https://bit.ly/3oyznno

[4] J. Banas and L. Olszowy, "On solutions of a quadratic Urysohn integral equation on an unbouded interval", *Dynamic Systems and Applications*, vol. 17, no. 2 pp. 255-270, 2008. [On line]. Available: https://bit.ly/3BrMRJD

[5] A. Benmezaï, "Krasnoselskii-type fixed point theorem in ordered Banach spaces and application to integral equations", *Advances in Pure and Applied Mathematics*, vol. 13, no. 1, pp. 50-61, 2022. doi: 10.21494/ISTE.OP.2021.0759

[6] A. Benmezaïand N. Benkaci-Ali, "Krein-Rutman operators and a variant of Banach contraction principle in ordered Banach spaces", *Bulletin mathématique de la Société des Sciences Mathématiques de Roumanie*, vol. 64, no. 3, pp. 255-280, 2021. [On line]. Available: https://bit.ly/3BmCBC3

[7] M. Berzig, S. Chandok and M. S. Khan, "Generalized Krasnoselskii fixed point theorem involving auxiliary functions in bimetric spaces and application to two-point boundary value problem", *Applied Mathematics and Computation*, vol. 248, pp. 323-327, 2014. doi: 10.1016/j.amc.2014.09.096

[8] L. W. Busbridge, *The Mathematics of Radiative Transfer.* Cambridge: Cambridge University, 1960.

[9] B. Cahlon and B. Eskin, "Existence theorems for an integral equation of the Chandrasekhar H-equation with perturbation", *Journal of Mathematical Analysis and Applications,* vol. 83, no. 1, pp. 159-171, 1981. doi: 10.1016/0022-247x(81)90254-7

[10] K. M. Case and P. F. Zweigel, *Linear Transport Theory*. Reading, MA: Addison-Wesley, 1967.

[11] G. Darbo, "Punti uniti in trasformazioni a codomio non compatto", *Rendiconti del Seminario Matematico della Università di Padova,* vol. 24, pp. 84-92, 1955.

[12] M. A. Darwish, "On a perturbed functional integral equation of Urysohn type", *Applied Mathematics and Computation*, vol. 218, no. 17, pp. 8800–8805, 2012. doi: 10.1016/j.amc.2012.02.037

[13] K. Deimling, *Nonlinear Functional Analysis*. Springer-Verlag, 1985.

[14] B. C. Dhage, "Some variants of two basic hybrid fixed point theorems of Krasnoselskii with applications", *Nonlinear Studies*, vol. 25, pp. 559-573, 2018.

[15] J. Garcia-Falset, K. Latrach, E. Moreno-Gálvez, and M.-A. Taoudi, "Schaefer–Krasnoselskii fixed point theorems using a usual measure of weak noncompactness", *Journal of Differential Equations,* vol. 252, no. 5, pp. 3436–3452, 2012. doi: 10.1016/j.jde.2011.11.012

[16] J. Garcia-Falset and K. Latrach, "Krasnoselskii-type fixed-point theorems for weakly sequentially continuous mappings," *Bulletin of the London Mathematical Society,* vol. 44, no. 1, pp. 25–38, 2011. doi: 10.1112/blms/bdr035

[17] D. Guo and V. Lakshmikantaham, *Nonlinear Problems in Abstract Cones.* San Diego: Academic Press, 1988.

[18] S. Hu, M. Khavanin, and W. A. N. Zhuang, "Integral equations arising in the kinetic theory of gases", *Applicable Analysis*, vol. 34, no. 3-4, pp. 261-266, 1989. doi: 10.1080/00036818908839899

[19] B. Ílhan and Í. Ozdemir, ˝Existence and asymptotic behavior of solutions for some nonlinear integral equations on an unbounded interval", *Electronic Journal of Differential Equations*, vol. 2016, no. 271, pp. 1-15, 2016. [On line]. Available: https://bit.ly/3owipX0

[20] M. A. Krasnosel'skii, "Some problems of nonlinear analysis", *American Mathematical Society Translations, Series 2,* vol. 2, no. 10, pp. 345-409, 1958.

[21] K. Kuratowski, “Sur les espaces completes”, *Fundamenta Mathematicae,* vol. 15, pp. 209-301, 1930. [On line]. Available: https://bit.ly/3zaewMo

[22] W. A. J. Luxemburg and A. C. Zaanen, *Riesz spaces I.* North-Holland publishing company, 1971.

[23] M. A. Taoudi, “Krasnosel’skii type fixed point theorems under weak topology features”, *Nonlinear Analysis: Theory, Methods and Applications,* vol. 72, pp. 478-482, 2010. doi: 10.1016/j.na.2009.06.086

[24] T. Xiang and S. G. Georgiev, “Noncompact-type Krasnoselskii fixed-point theorems and their applications”, *Mathematical Methods in the Applied Sciences,* 2015. doi: 10.1002/mma.3525

[25] K. Yoshida, *Functional Analysis*. Springer-Verlag, 1980.

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