Proyecciones Journal of Mathematics
Vol. 43, N$^o$ 1, pp. 103-117, February 2024.
Universidad Católica del Norte
Antofagasta - Chile

10.22199/issn.0717-6279-5401

# Extreme outer connected geodesic graphs

*K. Ganesamoorthy*
*Coimbatore Institute of Technology, India*
*and*
*D. Jayanthi*
*Coimbatore Institute of Technology, India*
*Received : April 2022. Accepted : April 2023*

**Abstract**

*For a connected graph $G$ of order at least two, a set $S$ of vertices in a graph $G$ is said to be an outer connected geodetic set if $S$ is a geodetic set of $G$ and either $S = V$ or the subgraph induced by $V - S$ is connected. The minimum cardinality of an outer connected geodetic set of $G$ is the outer connected geodetic number of $G$ and is denoted by $g_{oc}(G)$. The number of extreme vertices in $G$ is its extreme order $ex(G)$. A graph $G$ is said to be an extreme outer connected geodesic graph if $g_{oc}(G) = ex(G)$. It is shown that for every pair $a, b$ of integers with $0 \leq a \leq b$ and $b \geq 2$, there exists a connected graph $G$ with $ex(G) = a$ and $g_{oc}(G) = b$. Also, it is shown that for positive integers $r, d$ and $k \geq 2$ with $r < d \leq 2r$, there exists an extreme outer connected geodesic graph $G$ of radius $r$, diameter $d$ and outer connected geodetic number $k$.*

**Key Words:** *Outer connected geodetic set; outer connected geodetic number; extreme order; extreme outer connected geodesic graph.*

**AMS Subject Classification:** *05C12.*

## 1. Introduction

By a graph $G$ we mean a simple finite undirected connected graph with *vertex set* $V(G) = V$ and *edge set* $E(G) = E$. The *order* and *size* of $G$ are denoted by $p = |V|$ and $q = |E|$ respectively. For basic graph theoretic terminology we refer to Harary [1, 12]. The *distance* $d(x, y)$ between two vertices $x$ and $y$ in $G$ is the length of a shortest $x - y$ path in $G$. A $x - y$ path of length $d(x, y)$ is called $x - y$ *geodesic*. For any vertex $u$ of $G$, the *eccentricity* of $u$ is defined as $e(u) = max\{d(u, v) : v \in V(G)\}$. The *radius* $rad(G)$ of $G$ is the minimum eccentricity among the vertices of $G$ and *diameter* $diam(G)$ of $G$ is the maximum eccentricity among the vertices of $G$. The *degree* of a vertex $x$ in graph $G$ is the number of edges incident with $x$. A vertex $v$ of $G$ is called an *endvertex* of $G$ if its degree is 1. The *neighborhood* of a vertex $v$ is the set $N(v)$ consisting of all vertices $u$ which are adjacent with $v$. A vertex $v$ is an *extreme vertex* if the subgraph induced by its neighbors is complete. The number of extreme vertices in $G$ is its *extreme order* $ex(G)$. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs, then the sum $G_1 + G_2$ is a graph $G = (V, E)$, where $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$ together with all vertices in $V_1$ is adjacent to all the vertices in $V_2$. In this paper, $m_iK_j$ denotes $m_i$-copies of the complete graph $K_j$.

The *closed interval* $I[x, y]$ consists of all vertices lying on some $x - y$ geodesic of $G$, while for $S \subseteq V$, $I[S] = \bigcup_{x,y \in S} I[x, y]$. A set $S$ of vertices of $G$ is a *geodetic set* if $I[S] = V$, and the minimum cardinality of a geodetic set of $G$ is the *geodetic number* $g(G)$ of $G$. The geodetic number of a graph and its variants have been studied by several authors in [2, 3, 4, 5, 6, 13, 14, 16, 17]. These concepts have many applications in location theory and convexity theory. There are interesting applications of these concepts to the problem of designing the route for a shuttle and communication network design. A set $S$ of vertices in a graph $G$ is said to be an *outer connected geodetic set* if $S$ is a geodetic set of $G$ and either $S = V$ or the subgraph induced by $V - S$ is connected. The minimum cardinality of an outer connected geodetic set of $G$ is the *outer connected geodetic number* of $G$ and is denoted by $g_{oc}(G)$. The outer connected geodetic number of a graph was introduced in [7] and further studied in [8, 9, 10, 11]. This concept can be mainly used in fault-tolerance in communication networks [7].

The following theorems will be used in the sequel.

**Theorem 1.1.** [6] Each extreme vertex of a connected graph $G$ belongs to every geodetic set of $G$.

**Theorem 1.2.** [3] If $G$ is a non-trivial connected graph of order $p$ and diameter $diam(G)$, then $g(G) \leq p - diam(G) + 1$.

**Theorem 1.3.** [7] Each extreme vertex of a connected graph $G$ belongs to every outer connected geodetic set of $G$.

**Theorem 1.4.** [7] For the complete graph $K_p (p \geq 2)$, $g_{oc}(K_p) = p$.

**Theorem 1.5.** [7] If $T$ is a tree with $k$ endvertices, then $g_{oc}(T) = k$.

Throughout this paper $G$ denotes a connected graph with at least two vertices.

## 2. Main Results

**Definition 2.1.** A graph $G$ is said to be an *extreme outer connected geodesic graph* if $g_{oc}(G) = ex(G)$.

**Example 2.2.** For the graph $G_1$ given in Figure 2.1 of order 6, $u_1$ and $u_4$ are the only two extreme vertices and so $ex(G_1) = 2$. It is clear that $S = \{u_1, u_4\}$ is the unique minimum outer connected geodetic set of $G_1$ so that $g_{oc}(G_1) = 2 = ex(G_1)$. Hence the graph $G_1$ is an extreme outer connected geodesic graph. The graph $G_2$ given in Figure 2.1 has only one extreme vertex $v_1$ and so $ex(G_2) = 1$. It is clear that $S_1 = \{v_1, v_4\}$ is the unique minimum outer connected geodetic set of $G_2$, so that $g_{oc}(G_2) = 2 \neq ex(G_2)$. Therefore $G_2$ is not an extreme outer connected geodesic graph. The graph $G_3$ given in Figure 2.1 contains no extreme vertices and so it is not an extreme outer connected geodesic graph.

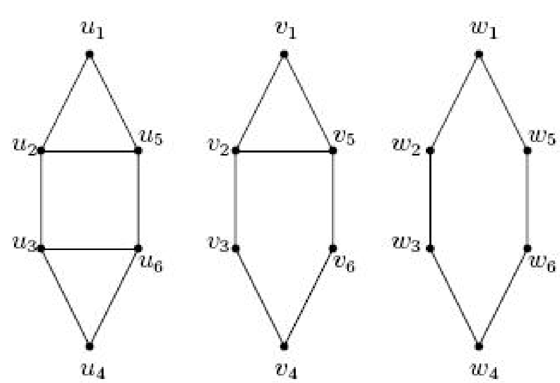


Figure 2.1: Graphs $G_1, G_2, G_3$

**Remark 2.3.** *For any non-trivial tree $T$ with $k$ endvertices, $ex(T) = k$ and by Theorem 1.5, $g_{oc}(T) = k = ex(T)$. Thus any non-trivial tree is an extreme outer connected geodesic graph. For the complete graph $K_p(p \geq 2)$, $ex(K_p) = p$ and by Theorem 1.4, $g_{oc}(K_p) = p = ex(K_p)$. It follows that $K_p$ is an extreme outer connected geodesic graph.*

**Observation 2.4.** *Any graph $G$ with no extreme vertices is not an extreme outer connected geodesic graph.*

**Remark 2.5.** *Any cycle $C_n(n \geq 4)$ and the complete bipartite graph $K_{m,n}(2 \leq m \leq n)$ contains no extreme vertices. Hence any cycle $C_n(n \geq 4)$ and the complete bipartite graph $K_{m,n}(2 \leq m \leq n)$ are not extreme outer connected geodesic graphs.*

**Theorem 2.6.** *For any connected graph $G$ of order $p\,(p \geq 2)$, $0 \leq ex(G) \leq g(G) \leq g_{oc}(G) \leq p$.*

**Proof.** Any graph $G$ may or may not contain extreme vertices and so $ex(G) \geq 0$. By Theorem 1.1, every geodetic set of $G$ contains all the extreme vertices of $G$ and so $g(G) \geq ex(G)$. Since every outer connected geodetic set of $G$ is a geodetic set of $G$, $g(G) \leq g_{oc}(G)$. Also, $V(G)$ induces an outer connected geodetic set of $G$. It follows that $g_{oc}(G) \leq p$. Hence, we have $0 \leq ex(G) \leq g(G) \leq g_{oc}(G) \leq p$. □

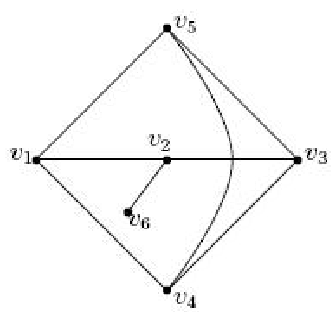


Figure 2.2: Graph $G$

**Remark 2.7.** The bounds in Theorem 2.6 are sharp. For any cycle $C_n(n \geq 4)$, $ex(G) = 0$ and for the complete graph $K_p(p \geq 2), g_{oc}(K_p) = p$. Also, all the inequalities in Theorem 2.6 can be strict. For the graph $G$ given in Figure 2.2 of order 6, $v_6$ is the only one extreme vertex of $G$ and so $ex(G) = 1$. It is clear that no 2-element subset of $V(G)$ is a geodetic set of $G$. It is easily verified that $S = \{v_1, v_3, v_6\}$ is a geodetic set of $G$ and so

$g(G) = 3$. Since the subgraph induced by $V - S$ is not connected, $S$ is not an outer connected geodetic set of $G$. It is clear that no 2-element subset or 3-element subset of $V(G)$ is an outer connected geodetic set of $G$. Since $S_1 = \{v_1, v_2, v_3, v_6\}$ is an outer connected geodetic set of $G$, $g_{oc}(G) = 4$. Thus, we have $0 < ex(G) < g(G) < g_{oc}(G) < p$.

**Theorem 2.8.** If $G = K_2 + \bigcup m_i K_j$, where each $m_i$ is a positive integer such that $\sum m_i \geq 2$ and $j \geq 1$, then $G$ is an extreme outer connected geodesic graph with $g_{oc}(G) = p - 2$.

**Proof.** Let $V(K_2) = \{x, y\}$. Since every vertex of $G$ is an extreme vertex except the vertices $x$ and $y$, $ex(G) = p - 2$. It is clear that the set $S$ of all extreme vertices of $G$ is a minimum geodetic set of $G$ and the subgraph induced by $V - S$ is connected. Hence $S$ is the unique minimum outer connected geodetic set of $G$ and so $g_{oc}(G) = p - 2 = ex(G)$. Thus $G$ is an extreme outer connected geodesic graph with $g_{oc}(G) = p - 2$. □

**Remark 2.9.** *The converse of Theorem 2.8 need not be true. For the path $P_4 : v_1, v_2, v_3, v_4$ of order 4, $S = \{v_1, v_4\}$ is the set of all extreme vertices of $P_4$ and so $ex(P_4) = 2$. It is clear that $S$ is the unique minimum outer connected geodetic set of $P_4$ and so $g_{oc}(P_4) = 2 = p - 2 = ex(P_4)$. Thus $G$ is an extreme outer connected geodesic graph, and it is not in the form $G = K_2 + \bigcup m_i K_j$.*

**Theorem 2.10.** *If $G$ is a non-trivial connected graph of order $p$ and diameter $diam(G)$, then $ex(G) \leq p - diam(G) + 1$.*

**Proof.** It follows from Theorems 1.2 and 2.6. □

**Remark 2.11.** *The bound in Theorem 2.10 is sharp. For the complete graph $K_p (p \geq 2)$, $ex(G) = p$ and $diam(K_p) = 1$ so that $ex(G) = p - diam(K_p) + 1$. Also, all the inequality in Theorem 2.10 can be strict. For the graph $G$ given in Figure 2.2 of order 6, $v_6$ is the only one extreme vertex of $G$ and so $ex(G) = 1$. It is easy to verify that $2 \leq e(x) \leq 3$ for any vertex $x$ in $G$, $e(v_6) = 3$. Then $diam(G) = 3$. Since $ex(G) = 1 < p - diam(G) + 1 = 4$, we have $ex(G) < p - diam(G) + 1$.*

**Theorem 2.12.** *For every pair $k, p$ of integers with $2 \leq k \leq p$, there exists an extreme outer connected geodesic graph $G$ of order $p$ with outer connected geodetic number $k$ and $ex(G) = k$.*

**Proof.** For $k = p$, it follows from the Remark 2.3 by taking $G = K_p$. For $2 \leq k \leq p-1$, the tree $T$ given in Figure 2.3 has $p$ vertices and it follows from the Remark 2.3 that $g_{oc}(T) = k = ex(T)$. As the graph $T$ is a tree, it is minimal with respect to edges. □

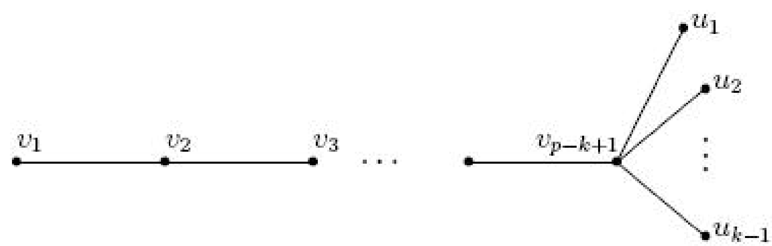


Figure 2.3: Tree $T$

**Theorem 2.13.** *For every pair $a$, $b$ of integers with $0 \leq a \leq b$ and $b \geq 2$, there exists a connected graph $G$ with $ex(G) = a$ and $g_{oc}(G) = b$.*

**Proof.** We prove this theorem by considering two cases.
**Case 1.** $a = 0$ and $b \geq 2$. Let $P_3 : x, y, z$ be a path of order 3. The graph $G$ in Figure 2.4 is obtained from $P_3$ by adding $b$ new vertices $u_1, v_1, v_2, ..., v_{b-1}$ and joining each $v_i (2 \leq i \leq b-1)$ to the vertices $x$ and $z$; and also joining the vertices $u_1, v_1$ to the vertices $x$, $y$, $z$. Clearly, no vertex of $G$ is an extreme vertex and so $ex(G) = 0$. It is easy to observe that any subset $S \subseteq V(G)$ with cardinality $|S| \leq b-1$ is not an outer connected geodetic set of $G$. Let $S' = \{u_1, v_1, v_2, ..., v_{b-1}\}$. Since $S'$ is a geodetic set of $G$ and the subgraph induced by $V - S'$ is connected, $S'$ is an outer connected geodetic set of $G$. It follows that $g_{oc}(G) = |S'| = b$.

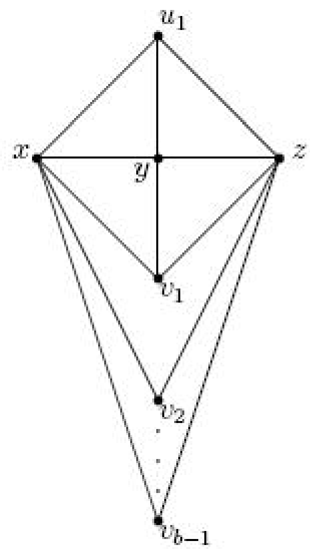


Figure 2.4: Graph $G$

**Case 2.** $a \geq 1$ and $b \geq 2$. If $a = b$, then by Remark 2.3 that the complete graph $G = K_a$ has the desired properties. If $a < b$, then we construct the required graph $G$ as follows: let $P_3 : x, y, z$ be a path of order 3 and let $G$ be the graph obtained from $P_3$ by adding $b$ new vertices $v_1, v_2, \ldots, v_{b-a}, u_1, u_2, \ldots, u_a$ and joining each $u_i (1 \leq i \leq a)$ to the vertex $y$ of $P_3$; and also joining each $v_i (1 \leq i \leq b-a)$ to both the vertices $x, z$ of $P_3$. The graph $G$ is shown in Figure 2.5. Since $S = \{u_1, u_2, \ldots, u_a\}$ is the set of all extreme vertices, $ex(G) = a$. By Theorem 1.3, every outer connected geodetic set of $G$ contains $S$. It is clear that $S$ is not an outer connected geodetic set of $G$. It is easy to observe that every minimum outer connected geodetic set of $G$ contains $\{v_1, v_2, \ldots, v_{b-a}\}$. Clearly, $S \cup \{v_1, v_2, \ldots, v_{b-a}\}$ is a minimum outer connected geodetic set of $G$ and so $g_{oc}(G) = b$. $\square$

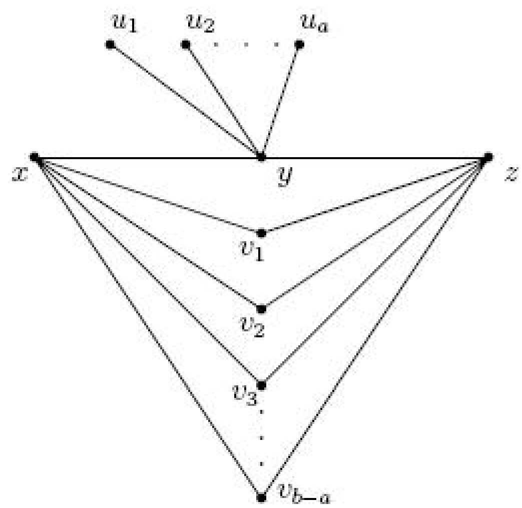


Figure 2.5: Graph $G$

For every connected graph $G$, $rad(G) \leq diam(G) \leq 2rad(G)$. Ostrand [15] showed that every two positive integers $a$ and $b$ with $a \leq b \leq 2a$ are realizable as the radius and diameter respectively, of some connected graph. Now, Ostrand's theorem can be extended so that an extreme outer connected geodesic graph can also be prescribed.

**Theorem 2.14.** *For any three positive integers $r, d$ and $k \geq 2$ with $r < d \leq 2r$, there exists an extreme outer connected geodesic graph $G$ such that $rad(G) = r$, $diam(G) = d$ and $g_{oc}(G) = k$.*

**Proof.** If $r = 1$, then $d = 2$. By Theorem 1.5 and Remark 2.3, the star $K_{1,k}$ has the desired property.

Now, let $r \geq 2$ and $r < d \leq 2r$. Let $C_{2r} : u_1, u_2, \ldots, u_{2r}, u_1$ be a cycle of order $2r$ and let $P_{d-r+1} : v_0, v_1, \ldots, v_{d-r}$ be a path of length $d - r$. Let $H$ be the graph obtained from $C_{2r}$ and $P_{d-r+1}$ by identifying the vertex $v_0$ of $P_{d-r+1}$ and the vertex $u_1$ of $C_{2r}$; and also joining the vertex $u_{r+2}$ to the vertex $u_r$. The graph $G$ in Figure 2.6 is obtained from $H$ by adding $k - 2$ new vertices $w_1, w_2, \ldots, w_{k-2}$ and joining each $w_i (1 \leq i \leq k - 2)$ to the vertex $v_{d-r-1}$. It is easy to verify that $r \leq e(x) \leq d$ for any vertex $x$ in $G$, $e(u_1) = r$ and $e(v_{d-r}) = d$. Then $rad(G) = r$ and $diam(G) = d$. Since $S = \{u_{r+1}, w_1, w_2, \ldots, w_{k-2}, v_{d-r}\}$ is the set of all extreme vertices of $G$, $ex(G) = k$. By Theorem 1.3, every outer connected geodetic set of $G$ contains $S$. It is clear that $S$ is the unique minimum outer connected geodetic set of $G$ and so $g_{oc}(G) = k = ex(G)$. Thus $G$ is an extreme outer connected geodesic graph such that $rad(G) = r$, $diam(G) = d$ and $g_{oc}(G) = k$. □

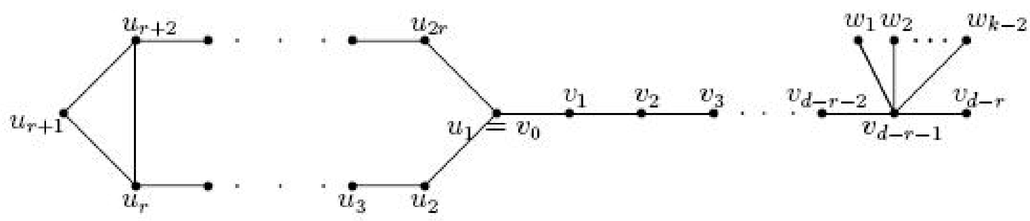


Figure 2.6: Graph $G$

**Problem 2.15.** *For positive integers $r$, $d$ and $k \geq 2$ with $r = d \leq 2r$, does there exist an extreme outer connected geodesic graph $G$ with $rad(G) = r$, $diam(G) = d$ and $g_{oc}(G) = k$?*

**Theorem 2.16.** *For each triple $p, d$ and $k$ of positive integers with $k \geq 2$, $d \geq 2$ and $p - d - k + 1 \geq 0$, there exists an extreme outer connected geodesic graph $G$ of order $p$ such that $diam(G) = d$ and $g_{oc}(G) = k$.*

**Proof.** Let $P_{d+1} : u_1, u_2, \ldots, u_{d+1}$ be a path of length $d$. Add $p - d - 1$ new vertices $v_1, v_2, \ldots, v_{k-2}, w_1, w_2, \ldots, w_{p-d-k+1}$ to $P_{d+1}$ and join each $w_i (1 \leq i \leq p-d-k+1)$ to the vertices $u_1$, $u_2$ and $u_3$; and join each $v_j (1 \leq j \leq k-2)$ to the vertex $u_2$; and also join each vertex $w_i (1 \leq i \leq p-d-k)$ to the vertex $w_j (i+1 \leq j \leq p-d-k+1)$. The graph $G$ of order $p$ with diameter $d$ is shown in Figure 2.7. Since $S = \{v_1, v_2, \ldots, v_{k-2}, u_1, u_{d+1}\}$ is the set of all extreme vertices of $G$, $ex(G) = k$. By Theorem 1.3, every outer connected geodetic set of $G$ contains $S$. It is clear that $S$ is the unique minimum outer connected geodetic set of $G$ and so $g_{oc}(G) = k = ex(G)$. Thus $G$ is an extreme outer connected geodesic graph of order $p$ with $diam(G) = d$ and $g_{oc}(G) = k$. □

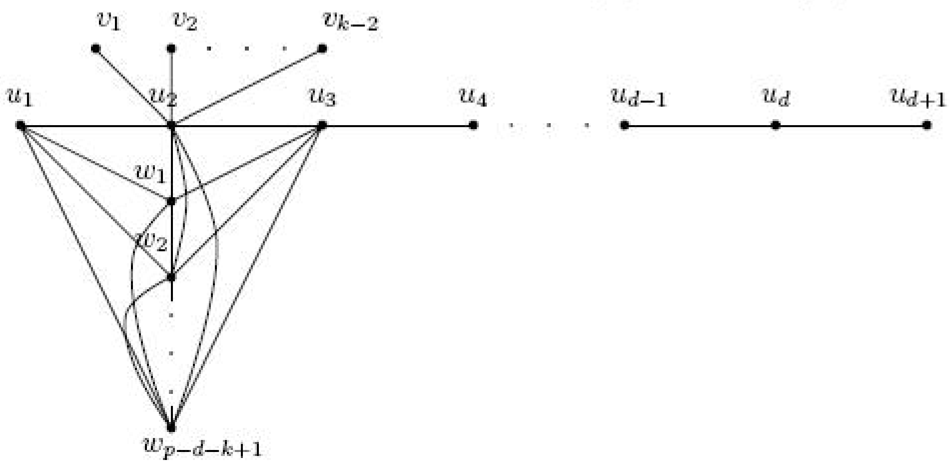


Figure 2.7: Graph $G$

In the following theorem we construct a non-extreme outer connected geodesic graph $G$ of order $p$ such that $diam(G) = d$ and $g_{oc}(G) = k$.

**Theorem 2.17.** *For each triple $p$, $d$ and $k$ of positive integers with $k \geq 2$, $d \geq 2$ and $p - d - k + 1 \geq 0$, there exists a non-extreme outer connected geodesic graph $G$ of order $p$ such that $diam(G) = d$ and $g_{oc}(G) = k$.*

**Proof.** Let $P_{d+1} : u_1, u_2, \ldots, u_{d+1}$ be a path of length $d$. Add $p - d - 1$ new vertices $v_1, v_2, \ldots, v_{k-2}, w_1, w_2, \ldots, w_{p-d-k+1}$ to $P_{d+1}$ and join each $w_i (1 \leq i \leq p - d - k + 1)$ to the vertices $u_1$, $u_2$ and $u_3$; and also join each $v_j (1 \leq j \leq k - 2)$ to the vertex $u_2$. The graph $G$ of order $p$ with diameter $d$ is shown in Figure 2.8. If $d = 2$, then $S_1 = \{v_1, v_2, \ldots, v_{k-2}\}$ is the set of all extreme vertices of $G$, $ex(G) = k - 2$. By Theorem 1.3, every outer connected geodetic set of $G$ contains $S_1$. It is clear that neither $S_1$ nor $S_1 \cup \{x\}$, where $x \notin S_1$, is an outer connected geodetic set of $G$. Since $S_2 = S_1 \cup \{u_1, u_3\}$ is a minimum geodetic set of $G$ and the subgraph induced by $V - S_2$ is connected, $S_2$ is an outer connected geodetic set of $G$ and so $g_{oc}(G) = k$. If $d \geq 3$, then $S_3 = \{v_1, v_2, \ldots, v_{k-2}, u_{d+1}\}$ is the set of all extreme vertices of $G$, $ex(G) = k - 1$. By Theorem 1.3, every outer connected geodetic set of $G$ contains $S_3$. It is clear that $S_3$ is not an outer connected geodetic set of $G$. It is easily verified that $S_3 \bigcup \{u_1\}$ is the unique minimum outer connected geodetic set of $G$ and so $g_{oc}(G) = k$.

Since $g_{oc}(G) = k \neq ex(G)$, $G$ is a non-extreme outer connected geodesic graph of order $p$ with $diam(G) = d$ and $g_{oc}(G) = k$. □

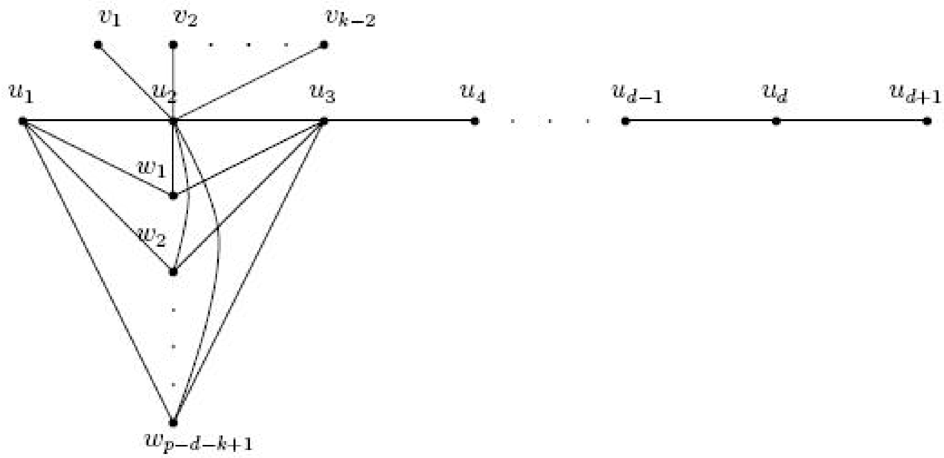


Figure 2.8: Graph $G$

Next, we analyse how the extreme outer connected geodesic graphs are affected by the addition of a pendant edge.

**Theorem 2.18.** *If $G^{'}$ is a graph obtained by adding $l$ pendant edges to an extreme outer connected geodesic graph $G$, then $ex(G) \leq ex(G^{'}) \leq ex(G) + l$ and $G'$ is an extreme outer connected geodesic graph.*

**Proof.** Let $G^{'}$ be the graph obtained from an extreme outer connected geodesic graph $G$ by adding $l$ pendant edges $u_iv_i(1 \leq i \leq l)$, where each $u_i(1 \leq i \leq l)$ is a vertex of $G$ and each $v_i(1 \leq i \leq l)$ is not a vertex of $G$. Let $S$ be a minimum outer connected geodetic set of $G$. Since $G$ is an extreme outer connected geodesic graph, $S$ is the unique minimum outer connected geodetic set of $G$ and $S$ is the set of all extreme vertices of $G$. Then it is clear that $S \cup \{v_1, v_2, \ldots, v_l\}$ is an outer connected geodetic set of $G^{'}$. Now, we claim that $ex(G^{'}) \leq ex(G)+l$ and $G'$ is an extreme outer connected geodesic graph. If each $u_i(1 \leq i \leq l)$ is an extreme vertex of $G$ then each $v_i(1 \leq i \leq l)$ is an extreme vertex of $G^{'}$ and each $u_i(1 \leq i \leq l)$ is not an extreme vertex of $G^{'}$. It is clear that $S' = (S - \{u_1, u_2, \ldots, u_l\}) \cup \{v_1, v_2, \ldots, v_l\}$ is the set of all extreme vertices of $G^{'}$ and so $ex(G^{'}) = |S'|$. Hence, we have $ex(G^{'}) = ex(G)$. If each $u_i(1 \leq i \leq l)$ is not an extreme vertex of $G$ then each $v_i(1 \leq i \leq l)$ is an extreme vertex of $G^{'}$. It is clear that $S' = S \cup \{v_1, v_2, \ldots, v_l\}$ is the set of all extreme vertices of $G^{'}$ and so $ex(G^{'}) = |S'|$.

Hence, we have $ex(G^{'}) = ex(G) + l$. Without loss of generality, if each $u_i(1 \leq i \leq k, k < l)$ is an extreme vertex of $G$ and each $u_j(k+1 \leq j \leq l)$ is not an extreme vertex of $G$, then $\{u_1, u_2, \ldots, u_k\} \subseteq S$. It is clear that $S' = (S - \{u_1, u_2, \ldots, u_k\} \cup \{v_1, v_2, \ldots, v_k\}) \cup \{v_{k+1}, v_{k+2}, \ldots, v_l\}$ is the set of all extreme vertices of $G^{'}$ and so $ex(G') = |S'|$. Hence, we have $ex(G^{'}) < ex(G) + l$. Note that in all the above cases, it is easily verified that $S'$ is the unique minimum outer connected geodetic set of $G^{'}$, $g_{oc}(G^{'}) = |S'| = ex(G^{'})$. Thus $G^{'}$ is an extreme outer connected geodesic graph.

Next, we show that $ex(G) \leq ex(G^{'})$. Suppose that $ex(G) > ex(G^{'})$. Let $S_1$ be a minimum outer connected geodetic set of $G^{'}$. Since $G^{'}$ is an extreme outer connected geodesic graph, $S_1$ is the unique minimum outer connected geodetic set of $G^{'}$ and $S_1$ is the set of all extreme vertices of $G^{'}$. Then with $|S_1| = ex(G^{'}) < ex(G)$. Since each $v_i(1 \leq i \leq l)$ is an extreme vertex of $G^{'}$, it follows from Theorem 1.3 that $\{v_1, v_2, ..., v_l\} \subseteq S_1$. Let $S_2 = (S_1 - \{v_1, v_2, \ldots, v_l\}) \cup \{u_1, u_2, \ldots, u_l\}$. Then $S_2$ is a subset of $V(G)$ and $|S_2| = |S_1| < ex(G)$. Now, we show that $S_2$ is an outer connected geodetic set of $G$. Let $w \in V(G) - S_2$. Since $S_1$ is an outer connected geodetic set of $G^{'}$, $w$ lies on an $x - y$ geodesic $P$ in $G^{'}$ for some vertices $x, y \in S_1$. If neither $x$ nor $y$ is $v_i(1 \leq i \leq l)$, then $x, y \in S_2$. If exactly one of $x, y$ is $v_i(1 \leq i \leq l)$, say $x = v_i$, then $w$ lies on the $u_i - y$ geodesic path in $G$ obtained from $P$ by removing $v_i$. If both $x, y \in \{v_1, v_2, \ldots, v_l\}$, then let $x = v_i$ and $y = v_j$ where $i \neq j$. Hence $w$ lies on the $u_i - u_j$ geodesic in $G$ obtained from $P$ by removing $v_i$ and $v_j$. Thus $S_2$ is a geodetic set of $G$. By Theorem 1.1, every geodetic set of $G$ contains all the extreme vertices of $G$, $ex(G) \leq |S_2|$. Also, since $G$ is an extreme outer connected geodesic graph, $ex(G) = g_{oc}(G)$. Hence $ex(G) = g_{oc}(G) \leq |S_2| < ex(G)$, which is a contradiction. □

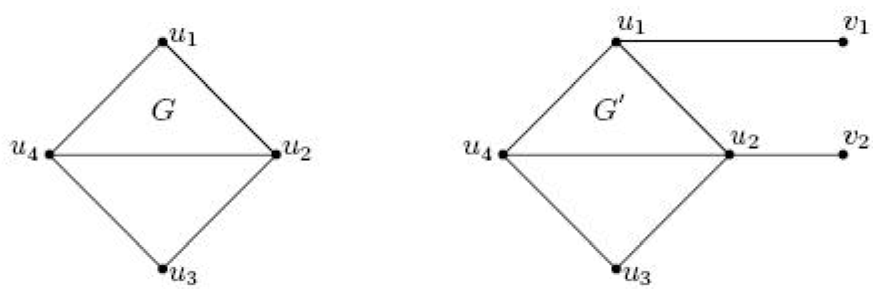


Figure 2.9: Graphs $G$ and $G'$

**Remark 2.19.** *The bounds in Theorem 2.18 are sharp. Consider a tree $T$ with number of endvertices $k \geq 2$. Let $S = \{v_1, v_2, \ldots, v_k\}$ be the set of all endvertices of $T$. Then by Theorem 1.5, $g_{oc}(T) = k = ex(T)$ and hence $T$ is an extreme outer connected geodesic graph. If we add a pendant edge to an endvertex of $T$, then we obtain another tree $T^{'}$ with $k$ endvertices. Then by Theorem 1.5, $g_{oc}(T^{'}) = k = ex(T^{'})$. Hence $ex(T) = ex(T^{'})$. On the otherhand, if we add $l$ pendant edges to a cutvertex of $T$, then we obtain another tree $T^{'}$ with $k + l$ endvertices. Then by Theorem 1.5, $ex(T^{'}) = k+l = ex(T)+l$. In both cases, $T^{'}$ is an extreme outer connected geodesic graph. Also, all the inequalities in Theorem 2.18 can be strict. For the graph $G$ given in Figure 2.9, it is clear that $S = \{u_1, u_3\}$ is the set of all extreme vertices of $G$ and so $ex(G) = 2$. Since $S$ is the unique minimum outer connected geodetic set of $G$, $g_{oc}(G) = 2 = ex(G)$. Thus $G$ is an extreme outer connected geodesic graph. The graph $G^{'}$ given in Figure 2.9 is obtained from the graph $G$ in Figure 2.9 by adding $l = 2$ pendant edges $u_iv_i (1 \leq i \leq 2)$. Since $S_1 = \{v_1, v_2, u_3\}$ is the set of all extreme vertices of $G^{'}$, $ex(G^{'}) = 3$. It is easy to see that $S_1$ is the unique minimum outer connected geodetic set of $G^{'}$ and so $g_{oc}(G^{'}) = 3 = ex(G^{'})$. Thus $G^{'}$ is an extreme outer connected geodesic graph. Hence we have $ex(G) < ex(G^{'}) < ex(G) + l$.*

**Theorem 2.20.** *For each triple $a$, $b$ and $l$ of integers with $2 \leq a \leq b$, $1 \leq l \leq b$, and $a + l - b \geq 0$, there exists a connected graph $G$ with $g_{oc}(G) = a$ and $g_{oc}(G^{'}) = b$, where $G^{'}$ is an extreme outer connected geodesic graph obtained by adding $l$ pendant edges to an extreme outer connected geodesic graph $G$.*

**Proof.** Let $G$ be a tree with number of endvertices $a$. Let $G^{'}$ be a graph obtained by adding $b - a$ pendant edges to a cutvertex of $G$ and also adding $a + l - b$ pendant edges each with different endvertices of $G$. Then $G^{'}$ is another tree with $b$ endvertices. By Theorem 1.5, $g_{oc}(G) = a$ and $g_{oc}(G^{'}) = b$. □

## References

[1] F. Buckley and F. Harary, *Distance in Graphs*, Addison-Wesley, Redwood City, CA, 1990.

[2] F. Buckley and F. Harary, L.v. Quintas, Extremal results on the geodetic number of a graph, *Scientia* **A2**, pp. 17-26, 1998.

[3] G. Chartrand, F. Harary and P. Zhang, On the geodetic number of a graph, *Networks*, Vol. 39 (1), pp. 1-6, 2002.

[4] G. Chartrand, F. Harary , H. C. Swart and P. Zhang, Geodomination in graphs, *Bulletin of the ICA*, Vol. 31, pp. 51-59, 2001.

[5] G. Chartrand, G. L.Johns, and P.Zhang, On the Detour Number and Geodetic Number of a Graph, *Ars Combin.*, Vol. 72, pp. 3-15, 2004.

[6] G. Chartrand, E. M. Palmer, P. Zhang, The geodetic number of a graph, A survey, *Congr.Numer.*, Vol. 156, pp. 37-58, 2002.

[7] K. Ganesamoorthy and D. Jayanthi, The Outer Connected Geodetic Number of a Graph, *Proc. Natl. Acad. Sci., India, Sect. A Phys. Sci.*, Vol. 91 (2), pp. 195-200, 2021.

[8] K. Ganesamoorthy and D. Jayanthi, On the Outer Connected Geodetic Number of a Graph, *Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. Mathematics,* Vol. 41 (1), pp. 78-84, 2021.

[9] K. Ganesamoorthy and D. Jayanthi, Further Results on the Outer Connected Geodetic Number of a Graph, *Publications de l'Institut Mathematique*, Vol. 108 (122), pp. 79-89, 2020.

[10] K. Ganesamoorthy and D. Jayanthi, The Upper and Forcing Connected Outer Connected Geodetic Numbers of a Graph, *Journal of Interconnection Networks*, Vol. 22, No. 01, 2142022, 2022. https://doi.org/10.1142/S0219265921420226.

[11] K. Ganesamoorthy and D. Jayanthi, More on the outer connected geodetic number of a graph, *Discrete Mathematics, Algorithms and Applications*, 2022. https://doi.org/10.1142/S1793830922501282.

[12] F. Harary, *Graph Theory*, Addision-Wesely, 1969.

[13] F. Harary, E. Loukakis, C. Tsouros, The geodetic number of a graph, *Mathl. Comput.Modeling*, Vol. 17 (11), pp. 89-95, 1993.

[14] R. Muntean and P. Zhang, On geodomination in graphs, *Congr. Numer.*, Vol. 143, pp. 161-174, 2000.

[15] P. A. Ostrand, Graphs with specified radius and diameter, *Discrete Math.*, Vol. 4, pp. 71-75, 1973.

[16] A. P. Santhakumaran, P. Titus and K. Ganesamoorthy, Extreme Restrained Geodesic Graphs, *Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. Mathematics,* Vol. 42 (1), pp. 172-178, 2022.

[17] A. P. Santhakumaran and K. Ganesamoorthy, The restrained double geodetic number of a graph, *Discrete Mathematics, Algorithms and Applications*, 2022. https://doi.org/10.1142/S1793830922501002.

**K. Ganesamoorthy**
Department of Mathematics,
Coimbatore Institute of Technology
Coimbatore-641 014,
India
e-mail: kvgm_2005@yahoo.co.in
Corresponding author

and

**D. Jayanthi**
Department of Mathematics,
Coimbatore Institute of Technology
Coimbatore-641 014,
India
e-mail: djayanthimahesh@gmail.com