The connected and forcing connected restrained monophonic numbers of a graph

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Abstract
For a connected graph $G = (V, E)$ of order at least two, a restrained monophonic set $S$ of a graph $G$ is a monophonic set such that either $S = V$ or the subgraph induced by $V - S$ has no isolated vertices. The minimum cardinality of a restrained monophonic set of $G$ is the restrained monophonic number of $G$ and is denoted by $m_r(G)$. A connected restrained monophonic set $S$ of $G$ is a restrained monophonic set such that the subgraph $G[S]$ induced by $S$ is connected. The minimum cardinality of a connected restrained monophonic set of $G$ is the connected restrained monophonic number of $G$ and is denoted by $m_{cr}(G)$. We determine bounds for it and find the same for some special classes of graphs. It is shown that, if $a, b$ and $p$ are positive integers such that $3 \leq a \leq b \leq p$, $p - 1 \neq a$, $p - 1 \neq b$, then there exists a connected graph $G$ of order $p$ with $m_r(G) = a$ and $m_{cr}(G) = b$. Also, another parameter forcing connected restrained monophonic number $f_{crm}(G)$ of a graph $G$ is introduced and several interesting results and realization theorems are proved.

Key Words: monophonic set, restrained monophonic set, connected restrained monophonic set, forcing connected restrained monophonic set.

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1. Introduction

By a graph $G = (V, E)$ we mean a finite undirected connected graph without loops or multiple edges. The order and size of $G$ are denoted by $p$ and $q$, respectively. For basic graph theoretic terminology we refer to Harary [8]. A block of a graph is a maximal non-separable subgraph. An end-block of $G$ is a block containing exactly one cut-vertex of $G$. For vertices $u$ and $v$ in a connected graph $G$, the distance $d(u, v)$ is the length of a shortest $u - v$ path in $G$. An $u - v$ path of length $d(u, v)$ is called an $u - v$ geodesic. It is known that $d$ is a metric on the vertex set $V$ of $G$. The neighborhood of a vertex $v$ is the set $N(v)$ consisting of all vertices $u$ which are adjacent with $v$. The closed neighborhood of a vertex $v$ is the set $N[v] = N(v) \cup \{v\}$. A vertex $v$ is an extreme vertex if the subgraph induced by its neighbors is complete. The closed interval $I[x, y]$ consists of all vertices lying on some $x - y$ geodesic of $G$, while for $S \subseteq V, I[S] = \bigcup_{x,y \in S} I[x, y]$. A set $S$ of vertices of $G$ is a geodetic set if $I[S] = V$, and the minimum cardinality of a geodetic set is the geodetic number $g(G)$. A geodetic set of cardinality $g(G)$ is called a g-set. The geodetic number of a graph was introduced in [3, 4, 9] and further studied in [5, 6]. A geodetic set $S$ of a graph $G$ is a restrained geodetic set if the subgraph $G[V - S]$ has no isolated vertex. The minimum cardinality of a restrained geodetic set of $G$ is the restrained geodetic number. The restrained geodetic number of a graph was introduced and studied in [1].

A chord of a path $P$ is an edge joining two non-adjacent vertices of $P$. A path $P$ is called a monophonic path if it is a chordless path. A set $S$ of vertices of $G$ is a monophonic set of $G$ if each vertex $v$ of $G$ lies on an $x - y$ monophonic path for some $x$ and $y$ in $S$. The minimum cardinality of a monophonic set of $G$ is the monophonic number of $G$ and is denoted by $m(G)$. The monophonic number of a graph and its related parameters was studied and discussed in [2, 10, 13, 17, 19]. A connected monophonic set of $G$ is a monophonic set $S$ such that the subgraph $G[S]$ induced by $S$ is connected. The minimum cardinality of a connected monophonic set of $G$ is the connected monophonic number of $G$ and is denoted by $m_c(G)$. The connected monophonic number of a graph was introduced and studied in [7, 20]. A restrained monophonic set $S$ of a graph $G$ is a monophonic set such that either $S = V$ or the subgraph induced by $V - S$ has no isolated vertices. The minimum cardinality of a restrained monophonic set of $G$ is the restrained monophonic number of $G$ and is denoted by $m_r(G)$. The restrained monophonic number of a graph was introduced in [14] and
further studied in [11, 12, 15, 16, 18]. These concepts have interesting applications in Channel Assignment Problem in FM radio technologies. The monophonic matrix is used to discuss different aspects of certain molecular graphs associated to the molecules arising in special situations of molecular problems in theoretical chemistry.

The following theorems will be used in the sequel.

**Theorem 1.1.** [13] Each extreme vertex of a connected graph \( G \) belongs to every monophonic set of \( G \).

**Theorem 1.2.** [20] Every cut-vertex of a connected graph \( G \) belongs to every connected monophonic set of \( G \).

**Theorem 1.3.** [14] For the complete graph \( K_p(p \geq 2) \), \( m_r(K_p) = p \).

**Theorem 1.4.** [14] Let \( G \) be a connected graph with a cut-vertex \( v \) and let \( S \) be a restrained monophonic set of \( G \). Then every component of \( G - v \) contains an element of \( S \).

**Theorem 1.5.** [14] If \( T \) is a tree of order \( p \) with \( k \) end-vertices and \( p-k \geq 2 \), then \( m_r(T) = k \).

**Theorem 1.6.** [14] For the complete bipartite graph \( G = K_{m,n}(2 \leq m \leq n) \), \( m_r(G) = \left\{ \begin{array}{ll} n+2 & \text{if } 2 = m \leq n \\ 4 & \text{if } 3 \leq m \leq n. \end{array} \right. \)

Throughout this paper \( G \) denotes a connected graph with at least two vertices.

### 2. Connected Restrained Monophonic Number

**Definition 2.1.** A connected restrained monophonic set \( S \) of \( G \) is a restrained monophonic set such that the subgraph \( G[S] \) induced by \( S \) is connected. The minimum cardinality of a connected restrained monophonic set of \( G \) is the connected restrained monophonic number of \( G \) and is denoted by \( m_{cr}(G) \). A connected restrained monophonic set of cardinality \( m_{cr}(G) \) is called a \( m_{cr} \)-set of \( G \).
Example 2.2. For the graph $G$ given in Figure 2.1, $S_1 = \{u, w\}$ and $S_2 = \{y, w\}$ are the two minimum restrained monophonic sets of $G$ and so $m_r(G) = 2$. The subgraphs induced by both $S_1$ and $S_2$ are not connected. It is clear that $S_3 = \{u, v, w\}$ and $S_4 = \{x, y, w\}$ are the two minimum connected restrained monophonic sets of $G$ and so $m_{cr}(G) = 3$. Thus the restrained monophonic number and the connected restrained monophonic number of a graph $G$ are different.

Since every extreme vertex of a connected graph $G$ belongs to every restrained monophonic set of $G$, we have the following observation.

Observation 2.3. Each extreme vertex of a connected graph $G$ belongs to every connected restrained monophonic set of $G$.

From the Observation 2.3, it is clear that for the complete graph $K_p$ ($p \geq 2$), $m_{cr}(K_p) = p$. The next observation follows from Theorem 1.4.

Observation 2.4. Let $G$ be a connected graph with cut-vertices and let $S$ be a connected restrained monophonic set of $G$. If $v$ is a cut-vertex of $G$, then every component of $G - v$ contains an element of $S$.

Theorem 2.5. Every cut-vertex of a connected graph $G$ belongs to every connected restrained monophonic set of $G$. 
Proof. Let \( v \) be any cut-vertex of \( G \) and let \( G_1, G_2, \ldots, G_r (r \geq 2) \) be the components of \( G - v \). Let \( S \) be any connected restrained monophonic set of \( G \). Then by Observation 2.4, \( S \) contains at least one element from each \( G_i (1 \leq i \leq r) \). Since \( G[S] \) is connected and \( v \) is a cut-vertex, it follows that \( v \in S \). \( \square \)

Observation 2.6. For any connected graph \( G \) of order \( p \) with \( k \) extreme vertices and \( l \) cut-vertices, \( \max\{2, k + l\} \leq m_{cr}(G) \leq p \).

For a cut-vertex \( v \) in a connected graph \( G \) and a component \( H \) of \( G - v \), the subgraph \( H \) and the vertex \( v \) together with all edges joining \( v \) and \( V(H) \) is called a branch of \( G \) at \( v \). Since every end-block \( B \) is a branch of \( G \) at some cut-vertex, it follows from Observation 2.4 that every minimum connected restrained monophonic set of \( G \) contains at least one vertex from \( B \) that is not a cut-vertex. Thus the following corollaries are consequences of Observation 2.4 and Theorem 2.5.

Corollary 2.7. If \( G \) is a connected graph with \( k \geq 2 \) end-blocks, then \( m_{cr}(G) \geq k + 1 \).

Corollary 2.8. If \( k \) is the maximum number of blocks to which a vertex in a graph \( G \) belongs, then \( m_{cr}(G) \geq k + 1 \).

Corollary 2.9. For any non-trivial tree \( T \) of order \( p \), \( m_{cr}(T) = p \).

Proof. This follows from Observation 2.3 and Theorem 2.5. \( \square \)

Theorem 2.10. For any cycle \( C_p (p \geq 3) \), \( m_r(C_p) = \begin{cases} 3 & \text{if } p = 3 \text{ or } p \geq 5 \\ 4 & \text{if } p = 4. \end{cases} \)

Proof. If \( p = 3 \), then \( G = C_3 \) is a complete graph, by Observation 2.3, we have \( m_{cr}(C_3) = 3 \).

If \( p = 4 \), then it is clear that neither 2-element subset nor 3-element subset of \( V(C_4) \) forms a connected restrained monophonic set of \( C_4 \) and so \( m_{cr}(C_4) = 4 \).

If \( p \geq 5 \), then it is easily observed that any three consecutive vertices of \( G = C_p \) forms a minimum connected restrained monophonic set of \( G = C_p \) and so \( m_{cr}(C_p) = 3 \). \( \square \)

In a complete bipartite graph, it is easy to observe that any minimum restrained monophonic set is also a connected restrained monophonic set. The next observation follows from Theorem 1.6.
Observation 2.11. For the complete bipartite graph \( G = K_{m,n} (2 \leq m \leq n) \), \( m_{cr}(G) = \begin{cases} n + 2 & \text{if } 2 = m \leq n \\ 4 & \text{if } 3 \leq m \leq n. \end{cases} \)

Theorem 2.12. For a connected graph \( G \) of order \( p \), \( 2 \leq m_r(G) \leq m_{cr}(G) \leq p \), \( p - 1 \neq m_r(G) \), \( p - 1 \neq m_{cr}(G) \).

Proof. Any restrained monophonic set needs at least two vertices and so \( m_r(G) \geq 2 \). Since every connected restrained monophonic set is also a restrained monophonic set, \( m_r(G) \leq m_{cr}(G) \). Also, \( V(G) \) induces a connected restrained monophonic set of \( G \). Hence, we have \( 2 \leq m_r(G) \leq m_{cr}(G) \leq p \). From the definition of the restrained monophonic number and the connected restrained monophonic number of a graph, we have \( p - 1 \neq m_r(G) \), \( p - 1 \neq m_{cr}(G) \).

\( \square \)

Figure 2.2: Graph \( G \)

Remark 2.13. The bounds for the Theorem 2.12 are sharp. If \( G = K_p \), \( m_r(G) = p \) and \( m_{cr}(G) = p \). All the inequalities in Theorem 2.12 can be strict. For the graph \( G \) given in Figure 2.2, \( S = \{v_1, v_5, v_6\} \) is the unique minimum restrained monophonic set of \( G \) so that \( m_r(G) = 3 \), and no 3-element or no 4-element subset of the vertex set is a connected restrained monophonic set of \( G \). Since \( \{v_1, v_4, v_5, v_6, v_7\} \) is a connected restrained monophonic set of \( G \), it follows that \( m_{cr}(G) = 5 \). Thus, we have \( 2 < m_r(G) < m_{cr}(G) < p \).

Observation 2.14. Let \( G \) be a connected graph of order \( p \geq 2 \). Then \( G = K_2 \) if and only if \( m_{cr}(G) = 2 \).

Theorem 2.15. For a connected graph \( G \) of order \( p \), \( 2 \leq m(G) \leq m_c(G) \leq m_{cr}(G) \leq p \), \( p - 1 \neq m_{cr}(G) \).
Proof. Any monophonic set needs at least two vertices and so \( m(G) \geq 2 \). Since every connected monophonic set of \( G \) is a monophonic set of \( G \), \( m(G) \leq m_c(G) \). Also, every connected restrained monophonic set of \( G \) is also a connected monophonic set of \( G \) and so \( m_c(G) \leq m_{cr}(G) \). Since \( V(G) \) induces a connected restrained monophonic set of \( G \), it follows that \( 2 \leq m(G) \leq m_c(G) \leq m_{cr}(G) \leq p \). From the definition of the connected restrained monophonic number of a graph, we have \( m_{cr}(G) \neq p - 1 \). \( \square \)

Remark 2.16. The bounds for the Theorem 2.15 are sharp. If \( G = K_p \), then \( m(G) = m_c(G) = p \) and \( m_{cr}(G) = p \). All the inequalities in Theorem 2.15 can be strict. For the graph \( G \) given in Figure 2.3, \( S_1 = \{v_1, v_3, v_5\} \) is the unique minimum monophonic set of \( G \) so that \( m(G) = 3 \) and \( S_2 = \{v_1, v_2, v_3, v_5\} \) is a connected monophonic set of \( G \) and so \( m_c(G) = 4 \). It is easily verified that \( S_2 = \{v_1, v_2, v_3, v_5\} \) is not a connected restrained monophonic set of \( G \) since the subgraph induced by \( V - S_2 \) has an isolated vertex \( v_4 \). Hence \( S_2 \cup \{v_4\} \) is a connected restrained monophonic set of \( G \). It follows that \( m_{cr}(G) = 5 \). Thus, we have \( 2 < m(G) < m_c(G) < m_{cr}(G) < p \).

![Graph G](image)

**Figure 2.3: Graph G**

**Theorem 2.17.** Let \( G \) be a connected graph of order \( p \) with every vertex of \( G \) either a cut-vertex or an extreme vertex. Then \( m_{cr}(G) = p \).

Proof. Let \( G \) be a connected graph with every vertex of \( G \) either a cut-vertex or an extreme vertex. Then the result follows from Observation 2.3 and Theorem 2.5. \( \square \)

Remark 2.18. The Converse of the Theorem 2.17 need not be true. For the cycle \( C_4 \), the vertex set is the unique minimum connected restrained
monophonic set of $C_4$. However any vertex of $C_4$ is neither a cut-vertex nor an extreme vertex.

Thus there are a number of classes of graphs $G$ (complete and non-complete) of order $p$ with $m_{cr}(G) = p$. This leads to the following open problem.

**Problem 2.19.** Characterize the class of graphs $G$ of order $p$ for which $m_{cr}(G) = p$.

In view Theorem 2.12, we have the following realization theorem.

**Theorem 2.20.** If $a, b$ and $p$ are positive integers such that $3 \leq a \leq b \leq p, p - 1 \neq a, p - 1 \neq b$, then there exists a connected graph $G$ of order $p$ with $m_{r}(G) = a$ and $m_{cr}(G) = b$.

**Proof.** We prove this theorem by considering four cases.

**Case 1.** $3 \leq a = b = p$. Let $G$ be a complete graph of order $p$. Then by Theorem 1.3 and Observation 2.3, $G$ has the desired properties.

![Figure 2.4: Graph $G$](image)

**Case 2.** $3 \leq a = b < p$. Let $P_3 : u_1, u_2, u_3$ be a path of order 3. Let $G$ be the graph obtained from $P_3$ by adding $p-3$ new vertices $v_1, v_2, \ldots, v_{a-3}, w_1, w_2, \ldots,$
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$w_{p-a}$ and joining each $v_i (1 \leq i \leq a - 3)$ with $u_2$ and joining each $w_i (1 \leq i \leq p - a)$ with $u_1$ and $u_3$; and joining each $w_j (1 \leq j \leq p - a - 1)$ to $w_k (j + 1 \leq k \leq p - a)$, thereby producing the graph $G$ of order $p$. The graph $G$ is shown in Figure 2.4. Let $S = \{v_1, v_2, \ldots, v_{a-3}\}$ be the set of all extreme vertices of $G$ and $u_2$ is the cut-vertex of $G$. By Theorem 1.1 and Observation 2.3, every restrained and connected restrained monophonic set of $G$ contain $S$. It is easily verified that $S$ is not a restrained monophonic set of $G$. Also, for any $x, y \in V(G) - S$, $S \cup \{x, y\}$ is not a restrained monophonic set of $G$. Let $S' = S \cup \{u_1, u_2, u_3\}$. Clearly, $S'$ is a minimum restrained monophonic and connected restrained monophonic set of $G$ and so $m_r(G) = m_{cr}(G) = a$.

**Case 3.** $3 \leq a < b = p$. Let $G$ be any tree with number of pendant vertices $a$. Then, by Theorem 1.5 and Corollary 2.9, $G$ has the desired properties.

![Graph G](image)

**Figure 2.5:** Graph $G$

**Case 4.** $3 \leq a < b < p$. Let $P_{b-a+2} : u_1, u_2, \ldots, u_{b-a+2}$ be a path of length $b - a + 1$. Add $p - b + a - 2$ new vertices $w_1, w_2, \ldots, w_{p-b}, v_1, v_2, \ldots, v_{a-2}$ to $P_{b-a+2}$ and join each $w_i (1 \leq i \leq p - b)$ to the vertices $u_1, u_2$ and $u_3$ and also join each $v_j (1 \leq j \leq a - 2)$ to $u_{b-a+1}$; and join each $w_j (1 \leq j \leq p - b - 1)$ to $w_k (j + 1 \leq k \leq p - b)$, thereby producing the graph $G$ of order $p$ which is shown in Figure 2.5. Let $S = \{u_1, u_{b-a+2}, v_1, v_2, \ldots, v_{a-2}\}$ be the set of all extreme vertices of $G$. By Theorem 1.1 and Observation 2.3, every restrained and connected restrained monophonic set of $G$ contain $S$. It is
clear that $S$ is a restrained monophonic set of $G$ and so $m_r(G) = a$. Let $T = \{u_3, u_4, \ldots, u_{b-a+1}\}$ be the set of all cut-vertices of $G$. By Observation 2.3 and Theorem 2.5, every connected restrained monophonic set of $G$ contains $S \cup T$. It is easily seen that $S \cup T$ is not a connected restrained monophonic set of $G$. Let $M = S \cup T \cup \{u_2\}$. It is clear that $M$ is a minimum connected restrained monophonic set of $G$ and so $m_{cr}(G) = b$.

We leave the following problem as an open question.

**Problem 2.21.** Characterize graphs $G$ for which $m_r(G) = m_{cr}(G)$.

**Theorem 2.22.** If $a, b$ and $c$ are positive integers such that $3 \leq a < b \leq c$, then there exists a connected graph $G$ with $m(G) = a$, $m_c(G) = b$ and $m_{cr}(G) = c$.

**Proof.** We prove this theorem by considering two cases.

**Case 1.** $3 \leq a < b = c$. For the graph $G$ given in Figure 2.5, $S = \{u_1, u_{b-a+2}, v_1, v_2, \ldots, v_{a-2}\}$ be the set of all extreme vertices of $G$. By Theorem 1.1 and Observation 2.3, every monophonic and connected restrained monophonic set of $G$ contain $S$. It is clear that $S$ is a monophonic set of $G$ and so $m(G) = a$. Let $T = \{u_3, u_4, \ldots, u_{b-a+1}\}$ be the set of all cut-vertices of $G$. By Theorems 1.1, 1.2, 2.5 and Observation 2.3, every connected monophonic and connected restrained monophonic set of $G$ contain $S \cup T$. It is clear that $S \cup T$ is not a connected monophonic and connected restrained monophonic set of $G$. Let $M = S \cup T \cup \{u_2\}$. It is clear that $M$ is a minimum connected monophonic and minimum connected restrained monophonic set of $G$ and so $m_c(G) = m_{cr}(G) = b$. 

![Figure 2.6: Graph $G$](image-url)
Case 2. $3 \leq a < b < c$. Let $P_{b-a+2} : v_1, v_2, \ldots, v_{b-a+2}$ be a path of length $b - a + 1$. Add $c - b + a - 2$ new vertices $w_1, w_2, \ldots, w_{c-b}, u_1, u_2, \ldots, u_{a-2}$ to $P_{b-a+2}$ and join $w_1, w_2, \ldots, w_{c-b}$ to both the vertices $v_1$ and $v_3$ and join $u_1, u_2, \ldots, u_{a-2}$ to the vertex $v_{b-a+1}$, thereby producing the graph $G$ which is shown in Figure 2.6. Let $S = \{v_{b-a+2}, v_1, u_1, \ldots, u_{a-2}\}$ be the set of all end-vertices of $G$. By Theorem 1.1, every monophonic set of $G$ contains $S$. Since no vertex in $V-S$ lies on a $u-v$ monophonic path for some $u,v \in S$, $S$ is not a monophonic set of $G$. Let $S_1 = S \cup \{v_1\}$. It is easily verified that $S_1$ is a monophonic set of $G$ and so $m(G) = a$. Let $T = \{v_3, v_4, \ldots, v_{b-a+1}\}$ be the set of all cut-vertices of $G$. By Theorems 1.1 and 1.2, every connected monophonic set of $G$ contains $S \cup T$. Let $S_2 = S \cup T$. It is clear that $S_2$ is not a connected monophonic and connected restrained monophonic set of $G$. Observe that $S_2 \cup \{v_1\}$ is a monophonic set of $G$ and it is not a connected monophonic set of $G$, since the induced subgraph $G[S_2]$ is not connected. Let $S_3 = S_2 \cup \{v_1, v_2\}$. It is easily seen that $S_3$ is a connected monophonic set of $G$ and so $m_c(G) = b$. By Observation 2.3 and Theorem 2.5, every connected restrained monophonic set of $G$ contains $S \cup T$. Since the subgraph induced by $V-S_3$ has the isolated vertices $v_1, v_2, \ldots, w_{c-b}$. It is easy to observe that every connected restrained monophonic set of $G$ contains all the vertices $w_1, w_2, \ldots, w_{c-b}$. Thus, $S_3 \cup \{w_1, w_2, \ldots, w_{c-b}\}$ is the unique minimum connected restrained monophonic set of $G$ and so $m_{cr}(G) = c$. 

3. Forcing Connected Restrained Monophonic Number

For the graph $G$ given in Figure 2.1, the sets $S_3 = \{u, v, w\}$ and $S_4 = \{x, y, w\}$ are the two minimum connected restrained monophonic sets of $G$ and so $m_{cr}(G) = 3$. Thus a connected graph $G$ may contain more than one minimum connected restrained monophonic sets. For each minimum connected restrained monophonic set $S$ in $G$, there is always some subset $T$ of $S$ that uniquely determines $S$ as the minimum connected restrained monophonic set containing $T$, that is, $T$ is not contained in any other minimum connected restrained monophonic set of $G$. Such sets are called “forcing connected restrained monophonic subsets” and we discuss these sets in this section.

Definition 3.1. Let $S$ be a minimum connected restrained monophonic set of $G$. A subset $T$ of a minimum connected restrained monophonic set $S$ of $G$ is called a forcing connected restrained monophonic subset for $S$ if $S$ is the unique minimum connected restrained monophonic set containing
A forcing connected restrained monophonic subset for $S$ of minimum cardinality is a minimum forcing connected restrained monophonic subset of $S$. The forcing connected restrained monophonic number of $S$, denoted by $f_{crm}(S)$, is the cardinality of a minimum forcing connected restrained monophonic subset for $S$. The forcing connected restrained monophonic number of $G$ is $f_{crm}(G) = \min \{ f_{crm}(S) \}$, where the minimum is taken over all minimum connected restrained monophonic sets $S$ in $G$.

![Figure 3.1: Graph $G$](image)

**Example 3.2.** For the graph $G$ given in Figure 2.1, $S_3 = \{u, v, w\}$ and $S_4 = \{x, y, w\}$ are the two minimum connected restrained monophonic sets of $G$. It is clear that $f_{crm}(S_3) = 1$ and $f_{crm}(S_4) = 1$ so that $f_{crm}(G) = 1$. For the graph $G$ given in Figure 3.1, $S = \{v_1, v_2, v_3, v_4\}$ is the unique minimum connected restrained monophonic set of $G$ and so $f_{crm}(G) = 0$.

The next theorem follows immediately from the definitions of the connected restrained monophonic number and the forcing connected restrained monophonic number of a graph $G$.

**Theorem 3.3.** For a connected graph $G$ of order $p$, $0 \leq f_{crm}(G) \leq m_{cr}(G) \leq p$, $p - 1 \neq m_{cr}(G)$.

The bounds in Theorem 3.3 are sharp. For the graph $G$ given in Figure 3.1, $f_{crm}(G) = 0$. By Observation 2.3, for the complete graph $K_p(p \geq 2), m_{cr}(K_p) = p$. The inequalities in Theorem 3.3 can be strict. For the graph $G$ given in Figure 2.1, $m_{cr}(G) = 3$ and $f_{crm}(G) = 1$. Thus $0 < f_{crm}(G) < m_{cr}(G) < p$.

The following theorem is an easy consequence of the definitions of the connected restrained monophonic number and forcing connected restrained
monophonic number. In fact, the theorem characterizes graphs $G$ for which the lower bound in Theorem 3.3 is attained and also graphs $G$ for which $f_{crm}(G) = 1$ and $f_{crm}(G) = m_{cr}(G)$.

**Theorem 3.4.** Let $G$ be a connected graph. Then

(i) $f_{crm}(G) = 0$ if and only if $G$ has the unique minimum connected restrained monophonic set.

(ii) $f_{crm}(G) = 1$ if and only if $G$ has at least two minimum connected restrained monophonic sets, one of which is a unique minimum connected restrained monophonic set containing one of its elements, and

(iii) $f_{crm}(G) = m_{cr}(G)$ if and only if no minimum connected restrained monophonic set of $G$ is the unique minimum connected restrained monophonic set containing any of its proper subsets.

A vertex $v$ of a connected graph $G$ is said to be a connected restrained monophonic vertex of $G$ if $v$ belongs to every minimum connected restrained monophonic set of $G$.

**Observation 3.5.** If $G$ has an unique minimum connected restrained monophonic set $S$, then every vertex in $S$ is a connected restrained monophonic vertex of $G$. Also, if $x$ is an extreme vertex or a cut-vertex of $G$, then by Observation 2.3 and Theorem 2.5, $x$ is a connected restrained monophonic vertex of $G$.

The following theorem and corollary follow immediately from the definitions of connected restrained monophonic vertex and forcing connected restrained monophonic subset of $G$.

**Theorem 3.6.** Let $G$ be a connected graph and let $\rho_{dm}$ be the set of relative complements of the minimum forcing connected restrained monophonic subsets in their respective minimum connected restrained monophonic sets in $G$. Then $\bigcap_{F \in \rho_{dm}} F$ is the set of connected restrained monophonic vertices of $G$.

**Corollary 3.7.** Let $G$ be a connected graph and let $S$ be a minimum connected restrained monophonic set of $G$. Then no connected restrained monophonic vertex of $G$ belongs to any minimum forcing connected restrained monophonic subset of $S$.

Let $S$ be any minimum connected restrained monophonic set of $G$. Then
\( m_{cr}(G) = |S|, M \subseteq S \) and \( S \) is the unique minimum connected restrained monophonic set containing \( S - M \). Thus \( f_{crm}(G) \leq |S - M| = |S| - |M| = m_{cr}(G) - |M| \).

**Theorem 3.8.** Let \( G \) be a connected graph and let \( M \) be the set of all connected restrained monophonic vertices of \( G \). Then \( f_{crm}(G) \leq m_{cr}(G) - |M| \).

**Corollary 3.9.** If \( G \) is a connected graph with \( l \) extreme vertices and \( k \) cut-vertices, then \( f_{crm}(G) \leq m_{cr}(G) - (l + k) \).

The bound in Theorem 3.8 is sharp. For the graph \( G \) given in Figure 3.1, \( m_{cr}(G) = 4 \) and \( f_{crm}(G) = 0 \). Also, \( M = \{v_1, v_2, v_3, v_4\} \) is the unique minimum connected restrained monophonic set of \( G \). By Observation 3.5, every vertex of \( M \) is a connected restrained monophonic vertex of \( G \) and so \( f_{crm}(G) = m_{cr}(G) - |M| \). Also the inequality in Theorem 3.8 can be strict. For the graph \( G \) given in Figure 2.1, \( S_3 = \{u, v, w\} \) and \( S_4 = \{x, y, w\} \) are the minimum connected restrained monophonic sets so that \( m_{cr}(G) = 3 \) and \( f_{crm}(G) = 1 \). Also, since only one vertex \( w \) of \( G \) is a connected restrained monophonic vertex of \( G \), we have \( f_{crm}(G) < m_{cr}(G) - |M| \).

**Theorem 3.10.** If \( G \) is a connected graph with \( m_{cr}(G) = 2 \), then \( f_{crm}(G) = 0 \).

**Proof.** Let \( m_{cr}(G) = 2 \). Then by Observation 2.14, \( S = V(G) \) is the unique minimum connected restrained monophonic set of \( G \). It follows from Theorem 3.4(i) that \( f_{crm}(G) = 0 \).

Now, we proceed to determine the forcing connected restrained monophonic number of certain classes of graphs. The next observation follows from Theorem 2.10.

**Observation 3.11.** For any cycle \( C_p (p \geq 3) \), \( f_{crm}(C_p) = \) \( \begin{cases} 0 & \text{if } p = 3 \text{ or } 4 \\ 3 & \text{if } p \geq 5 \end{cases} \).

**Theorem 3.12.** For any complete graph \( G = K_p (p \geq 2) \) or any non-trivial tree \( G = T \), \( f_{crm}(G) = 0 \).
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**Proof.** For $G = K_p$, it follows from Observation 2.3, that the set of all vertices of $G$ is the unique minimum connected restrained monophonic set of $G$. Now, it follows from Theorem 3.4(i) that $f_{crm}(G) = 0$. If $G$ is a non-trivial tree, then by Corollary 2.9, the set of all vertices of $G$ is the unique minimum connected restrained monophonic set of $G$ and so by Theorem 3.4(i), $f_{crm}(G) = 0$. \qed

**Theorem 3.13.** For the complete bipartite graph $G = K_{m,n}$ ($2 \leq m \leq n$),

$$f_{crm}(G) = \begin{cases} 
0 & \text{if } 2 = m \leq n \\
4 & \text{if } 3 \leq m \leq n.
\end{cases}$$

**Proof.** Let $U = \{x_1, x_2, \ldots, x_m\}$ and $W = \{y_1, y_2, \ldots, y_n\}$ be the partite sets of $G$, where $m \leq n$. We prove this theorem by considering two cases.

**Case 1.** $2 = m \leq n$. By Observation 2.11, $V(G)$ is the unique minimum connected restrained monophonic set of $G$ and so by Theorem 3.4(i), $f_{crm}(G) = 0$.

**Case 3.** $3 = m \leq n$. Then any minimum connected restrained monophonic set is got by choosing any two elements from each of $U$ and $W$, and $G$ has at least two minimum connected restrained monophonic sets. Hence $m_{cr}(G) = 4$. Clearly, no minimum connected restrained monophonic set of $G$ is the unique minimum connected restrained monophonic set containing any of its proper subsets. Then by Theorem 3.4(iii), we have $f_{crm}(G) = m_{cr}(G) = 4$. \qed
Theorem 3.14. For every pair $a, b$ of positive integers with $0 \leq a < b$ and $b > 2a + 1$, there exists a connected graph $G$ such that $f_{crm}(G) = a$ and $m_{cr}(G) = b$.

Proof. If $a = 0$, let $G = K_b$. Then by Theorem 3.12, $f_{crm}(G) = 0$ and by Observation 2.3, $m_{cr}(G) = b$. Thus we assume that $0 < a < b$. For each $i$ with $1 \leq i \leq a$, let $C_i : v_{i,1}, v_{i,2}, v_{i,3}, v_{i,1}$ be a cycle of order 3 and let $K_{1,b-2a-1}$ be a star with the cut-vertex $x$ and $V(K_{1,b-2a-1}) = \{x, u_1, u_2, \cdots, u_{b-2a-1}\}$. Let $G$ be the graph obtained from $C_i$ and $K_{1,b-2a-1}$ by joining $v_{i,1}$ and $v_{i,3}(1 \leq i \leq a)$ to the vertices $v_{j,1}$ and $v_{j,3}(1 \leq j \leq a, i \neq j)$. The graph $G$ is shown in Figure 3.2. Let $S = \{u_1, u_2, \cdots, u_{b-2a-1}, v_{1,2}, v_{2,2}, \cdots v_{a,2}, x\}$ be the set of all extreme vertices and cut-vertex of $G$. By Observation 2.3 and Theorem 2.5, every connected restrained monophonic set of $G$ contains $S$. It is easily verified that $S$ is not a connected restrained monophonic set of $G$. We observe that every minimum connected restrained monophonic set of $G$ contains exactly one vertex from $\{v_{i,1}, v_{i,3}\}$ for every $i(1 \leq i \leq a)$. Hence $m_{cr}(G) \geq b - a + a = b$. On the other hand, $S' = S \cup \{v_{1,1}, v_{2,1}, \cdots, v_{a,1}\}$ is a connected restrained monophonic set of $G$, then it follows that $m_{cr}(G) \leq b$. Thus $m_{cr}(G) = b$.

Next, we show that $f_{crm}(G) = a$. By Observation 3.5, $S$ is the set of all connected restrained monophonic vertices of $G$ and so by Theorem 3.8,
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\[ f_{crm}(G) \leq m_{cr}(G) - |S| = b - (b - a) = a. \]

Now, since \( m_{cr}(G) = b \) and every minimum connected restrained monophonic set of \( G \) contains \( S \), it is easily seen that every minimum connected restrained monophonic set \( S_1 \) of \( G \) is of the form \( S \cup \{x_1, x_2, \ldots, x_a\} \), where \( x_i \in \{v_{i,1}, v_{i,3}\} \) for every \( i \) \((1 \leq i \leq a)\).

Let \( T \) be any proper subset of \( S_1 \) with \(|T| < a\). Then there is a vertex \( x \in S_1 - S \) such that \( x \notin T \). If \( x = v_{i,1} \) \((1 \leq i \leq a)\), then \( S_2 = (S_1 - \{v_{i,1}\}) \cup \{v_{i,3}\} \) is a minimum connected restrained monophonic set of \( G \) containing \( T \). Similarly, if \( x = v_{i,3} \) \((1 \leq i \leq a)\), then \( S_3 = (S_1 - \{v_{i,3}\}) \cup \{v_{i,1}\} \) is a minimum connected restrained monophonic set of \( G \) containing \( T \). Thus \( S_1 \) is not the unique minimum connected restrained monophonic set of \( G \) containing \( T \) and so \( T \) is not a forcing connected restrained monophonic subset of \( S_1 \).

Since this is true for all minimum connected restrained monophonic sets of \( G \), it follows that \( f_{crm}(G) \geq a \) and so \( f_{crm}(G) = a \). \( \square \)

References


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