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Subspace graph topological space of graphs

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Abstract

A graph topology defined on a graph G is a collection $\mathcal{T}$ of subgraphs of G which satisfies the properties such as K_0, $G \in \mathcal{T}$ and $\mathcal{T}$ is closed under arbitrary union and finite intersection. Let (X, T) be a topological space and $Y \subseteq X$ then, $T_Y = \{U \cap Y : U \in T\}$ is a topological space called a subspace topology or relative topology defined by T on Y. In this P1 we discusses the subspace or the relative graph topology defined by the graph topology $\mathcal{T}$ on a subgraph H of G. We also study the properties of subspace graph topologies, open graphs, d-closed graphs and nbd-closed graphs of subspace graph topologies.

Keywords: *Graph topology, subspace graph topology, d-closure, nbd-closure.*

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1. Introduction

Defining topologies on discrete structure has been a challenging area of research which is relatively obscure. The bare bones of this area were studied in [1]. Later, in [2], the authors proposed that the collection of all subgraphs of a graph G forms a topology (in the usual sense) of a graph under the union and intersection of graphs. Even though there are similarities between discrete and geometric objects, the differences are huge. Defining topologies for discrete structures have many limitations. For example, the concepts like continuity, compactness etc. cannot be defined in the traditional manner. As a consequence of this, the studies on graph topology have their own existence and significance.

We refer to [3, 4] for the terms and definitions in general topology and [5, 6] for the terminology in graph theory.

A *graph topological space* (see [7]) on a graph $G = (V, E)$ is a pair $(G, \mathcal{T}_G)$ where $\mathcal{T}g$ is a collection of subgraphs of G satisfying the following three axioms:

(i) $K_0, G \in \mathcal{T}_G$, where K_0 is the null graph with empty vertex set as well as edge set;

(ii) Any union of members of $\mathcal{T}_G$ is in $\mathcal{T}_G$;

(iii) Finite intersection of members of $\mathcal{T}_G$ is in $\mathcal{T}_G$.

If the context is clear, we can use the notation $\mathcal{T}$ in place of $\mathcal{T}_G$. A subgraph H is said to be *open* in a graph topological space if $H \in \mathcal{T}$.

[7] A subfamily is called a *base* of the graph topology $\mathcal{T}$ if the union of members of generates all the graphs of $\mathcal{T}$. It is the smallest collection of open subgraphs which generates $\mathcal{T}$ with the union of members.

The following theorem characterises the base of a graph topology.

[7] Let $(G, \mathcal{T})$ be a graph topological space and let $\subset \mathcal{T}$ then, is a base for the topological space if and only if,

(i) for each $v \in V(G)$, there exists $G_i \in$ such that $v \in V(G_i) \subseteq V(G)$.

(ii) for each $e \in E(G)$, there exist $G_j \in$ such that $e \in E(G_j) \subseteq E(G)$.

In the next section, we discuss the subspace graph topological space. In point set topology, a subspace or relative topology is defined as follows:

[3] Let X be a topological space with topology T . If Y is a subset of X, the collection $T_Y = \{Y \cap U : U \in T\}$ is a topology on Y, called the subspace topology or relative topology.

Now, we consider the subspace graph topological space of a graph topological space.

2. Subspace Graph Topology

For any graph topological space $(G, \mathcal{T})$, consider a subgraph H of G, the collection of graphs obtained by taking the intersection of members of $\mathcal{T}$ and H is a graph topology. The formal definition is as follows:

Let $(G, \mathcal{T})$ be a graph topological space and let H be a subgraph of G. Then, the collection of graphs $\mathcal{T}_H$ defined as, $\mathcal{T}_H = \{G_i \cap H : G_i \in \mathcal{T}\}$ is called *subspace graph topological space.*

In order to prove that the collection $\mathcal{T}_H = \{G_i \cap H : G_i \in \mathcal{T}\}$ is a graph topology, we first prove the following lemma which shows the distributive property of subgraphs of a graph under the graph operations union and intersection union.

Any graphs H_1, H_2 and H_3 with the graph operations union and intersection satisfies the following conditions:

$$H_1 \cup (H_2 \cap H_3) = (H_1 \cup H_2) \cap (H_1 \cup H_3) \tag{2.1}$$

$$H_1 \cap (H_2 \cup H_3) = (H_1 \cap H_2) \cup (H_1 \cap H_3) \tag{2.2}$$

Proof. First, consider the graph $H_1 \cup (H_2 \cap H_3)$ in Equation (2.1). This graph is characterised by the vertex set and edge set

$$\begin{aligned} V(H_2 \cap H_3) &= V(H_2) \cap V(H_2) \text{ and} \\ E(H_2 \cap H_3) &= E(H_2) \cap E(H_2) \end{aligned} \tag{2.3}$$

respectively. Now, for $H_1 \cup (H_2 \cap H_3)$, we have

$$\begin{aligned} V(H_1 \cup (H_2 \cap H_3)) \quad &= V(H_1) \cup \Big[V(H_2 \cap H_3)\Big] \\ &= V(H_1) \cup \Big[V(H_2) \cap V(H_3)\Big] \\ \text{Similarly, } E(H_1 \cup (H_2 \cap H_3)) \quad &= E(H_1) \cup \Big[E(H_2 \cap H_3)\Big] \\ &= E(H_1) \cup \Big[E(H_2) \cap E(H_3)\Big] \end{aligned} \tag{2.4}$$

Since $V(H_i)$ and $E(H_i)$ for $i = 1, 2, 3$, being ordinary sets they satisfies the distributive properties, we have

$$\begin{aligned} V(H_1) \cup \Big[V(H_2) \cap V(H_3)\Big] &= \Big[V(H_1) \cup V(H_2)\Big] \cap \Big[V(H_1) \cup V(H_3)\Big] \\ &= \Big[V(H_1 \cup H_2)\Big] \cap \Big[V(H_1 \cup H_3)\Big] \end{aligned} \tag{2.5}$$

Similarly,

$$\begin{aligned} E(H_1) \cup \Big[E(H_2) \cap E(H_3)\Big] &= \Big[E(H_1) \cup E(H_2)\Big] \cap \Big[E(H_1) \cup E(H_3)\Big] \\ &= \Big[E(H_1 \cup H_2)\Big] \cap \Big[E(H_1 \cup H_3)\Big] \end{aligned} \tag{2.6}$$

Therefore, by Equation (2.4), (2.5) and (2.6)

$$V(H_1 \cup (H_2 \cap H_3)) = \Big[V(H_1 \cup H_2)\Big] \cap \Big[V(H_1 \cup H_3)\Big] \tag{2.7}$$

and

$$E(H_1 \cup (H_2 \cap H_3)) = \Big[E(H_1 \cup H_2)\Big] \cap \Big[E(H_1 \cup H_3)\Big] \tag{2.8}$$

The vertex set as well as the edge set of the graphs obtained from the left hand side and right hand side of Equation (2.1) and Equation (2.2) are same and hence the graphs thus obtained are same.

Similarly, we can prove the same for Equation (2.2), where we consider intersection over union. Here, we consider three cases.
Case i: Let $(H_1 \cap H_2)$ and $(H_1 \cap H_3)$ be trivial graphs. Then, $| V(H_1 \cap H_2) | = 1$, $| V(H_1 \cap H_3) | = 1$ and their edge sets $E(H_1 \cap H_2)$ and $E(H_1 \cap H_3)$ becomes empty sets. Then,

$$\begin{aligned} \Big[V(H_1 \cap H_2)\Big] \cup \Big[V(H_1 \cap H_3)\Big] &= N_2, \text{ and} \\ \Big[E(H_1 \cap H_2)\Big] \cup \Big[E(H_1 \cap H_3)\Big] &= \emptyset, \end{aligned} \tag{2.9}$$

where N_2 is the null graph with two vertices and $\emptyset$ is the empty set. Considering the left hand side of Equation (2.2), $H_2 \cup H_3$ is the graph with vertices and edges of both H_2 and H_3. Since H_1 and H_2 has only one vertex in common, the edge set remains empty and also for H_1 and H_3, $| V(H_1 \cap H_3) | = 1$ and $| E(H_1 \cap H_3) | = \emptyset$. Then, $H_1 \cap (H_2 \cup H_3)$ is the null graph with two vertices one which is common in H_1 and H_2 and other common in H_1 and H_3 and the edge set remains empty. Hence, in this case the graph obtained is the null graph with two vertices that is, N_2. Therefore, we can conclude that, $H_1 \cap (H_2 \cup H_3) = (H_1 \cap H_2) \cup (H_1 \cap H_3)$.
Case ii: In this case we assume that one among $(H_1 \cap H_2)$ and $(H_1 \cap H_3)$ is trivial and other one is non-trivial. Here we need to consider three subcases.

Subcase i: Let $H_1 \cap H_2$ be a trivial graph with $v \in V(H_1 \cap H_2)$ and let $(H_1 \cap H_3)$ be a non trivial graph containing v. Then, the union of $H_1 \cap H_2$ and $H_1 \cap H_3$ will be the graph $H_1 \cap H_3$. Now, consider $H_1 \cap (H_2 \cup H_3)$. Here, $H_2 \cup H_3$ is a subgraph containing all the vertices and edges of both H_2 and H_3. Now, taking the intersection of this graph with H_1 we have, the vertices and edges that are common in H_1 and H_3 as $v \in H_1 \cap H_2 \cap H_3$ and v is the only vertex which is common H_1 and H_2. Therefore, $H_1 \cap (H_2 \cup H_3)$ will be the graph $H_1 \cap H_3$.

Subcase ii: Suppose $| V(H_1 \cap H_2) |= 1$ with $v \in V(H_1 \cap H_2)$ and $(H_1 \cap H_3)$ is a non trivial graph that does not contain v. Then, $(H_1 \cap H_2)$ and $(H_1 \cap H_3)$ are disjoint graphs and their union will be a disconnected graph. Consider $H_1 \cap (H_2 \cup H_3)$, here the graph $H_2 \cup H_3$ is either connected or disconnected. Since the vertex v does not belongs to $H_1 \cap H_3$, it will be an isolated vertex in the graph $H_1 \cap (H_2 \cup H_3)$ also since H_1 and H_2 share nothing other than the vertex v, the graph $H_1 \cap (H_2 \cup H_3)$ will have all the vertices and edges that are common in H_1 and H_3 along with the isolated vertex v. Hence, in this case $H_1 \cap (H_2 \cup H_3) = (H_1 \cap H_2) \cup (H_1 \cap H_3)$.

Subcase iii: Assume the case when $| V(H_1 \cap H_2) |= 1$ with $v \in V(H_1 \cap H_2)$ and $(H_1 \cap H_3)$ is the empty graph K_0. In this case, the only common factors in H_1 and $H_2 \cup H_3$ is the vertex v and hence $H_1 \cap (H_2 \cup H_3) = (H_1 \cap H_2)$.

Case iii: Both $(H_1 \cap H_2)$ and $(H_1 \cap H_3)$ are non-trivial graphs. The proof is analogous to the proof of part (i). □

The result can be extended or generalised to any graphs as follows:

Let $H_1, H_2, H_3, \ldots, H_j, \ldots$ be any graphs. Then, for a fixed integer $i \in \{1, 2, \ldots, \infty\}$,

$$H_i \bigcap \left(\bigcup_{j=1}^{\infty} H_j \right) = \bigcup_{j=1}^{\infty} (H_i \cap H_j)$$

Now, we shall prove that the collection $\mathcal{T}_H = \{G_i \cap H : G_i \in \mathcal{T}\}$ is a graph topology on the subgraph H of G.

Consider a graph topological space $(G, \mathcal{T})$. Let H be a subgraph of G and $\mathcal{T}_H = \{G_i \cap H : G_i \in \mathcal{T}\}$. Then, $\mathcal{T}_H$ is a graph topological space on H.

Proof. We need to prove that $\mathcal{T}_H = \{G_i \cap H : G_i \in \mathcal{T}\}$ is a graph topology, where H is a subgraph of G and $(G, \mathcal{T}_G)$ a topological space on

G. That is, we need to prove that the collection $\mathcal{T}_H = \{G_i \cap H : G_i \in \mathcal{T}\}$ satisfies the three axioms of graph topologies.

Let $G_1, G_2, \ldots, G_n$ be open graphs in $\mathcal{T}$. Since $\mathcal{T}$ is a graph topology on G, the null graph K_0 with empty vertex and edge set and the graph G are open in $\mathcal{T}$. W e shall verify the three axioms of graph topologies.

(i) First we check whether K_0 and H is in $\mathcal{T}_H$. By the definition of $\mathcal{T}_H$, the members of $\mathcal{T}_H$ are the graphs obtained by the intersection of members of $\mathcal{T}$ and H. Since G and K_0 are in $\mathcal{T}$, we have $K_0 \cap H = K_0 \in \mathcal{T}_H$ and $G \cap H = H \in \mathcal{T}_H$. Hence, K_0 and H are open in $\mathcal{T}_H$.

(ii) To prove that $\mathcal{T}_H$ is closed under any intersection. We have $\mathcal{T} = \{G, K_0, G_1, G_2, G_3, \ldots, G_n\}$, the graph topology on G. Consider the collection, $\mathcal{T}_H = \{G \cap H, K_0 \cap H, G_1 \cap H, G_2 \cap H, G_3 \cap H, \ldots, G_n \cap H\}$. Since $\mathcal{T}$ is a graph topology, it is closed under any intersection. That is for any open graphs G_i and G_j in $\mathcal{T}$ their intersection $G_i \cap G_j$ is open in $\mathcal{T}$. Now consider the graphs $(G_i \cap H)$ and $(G_j \cap H)$ in $\mathcal{T}_H$ then,

$$\begin{aligned}(G_i \cap H) \cap (G_j \cap H) &= G_i \cap H \cap G_j \cap H \\ &= (G_i \cap G_j) \cap H \in \mathcal{T}_H.\end{aligned}$$

Therefore, $\{G_i \cap G_j\} \cap H \in \mathcal{T}_H$. This can be extended to any intersection of graphs. Hence, we can say that $\mathcal{T}_H$ is closed under intersection.

(iii) Now, we need to show that $\mathcal{T}_H$ is closed under union. For this, let us consider two open subgraphs G_i and G_j of G. Then, $(G_i \cap H)$ and $(G_j \cap H)$ belongs to $\mathcal{T}_H$ by Definition 2.1. Now, we have to prove that $(G_i \cap H) \cup (G_j \cap H) \in \mathcal{T}_H$. By Lemma 2.2, we have

$$(G_i \cap H) \cup (G_j \cap H) = (G_i \cup G_j) \cap H \in \mathcal{T}_H.$$

By Theorem 2.3, we can say that any union of graphs are closed in $\mathcal{T}_H$.

Therefore, the collection $\mathcal{T}_H = \{G_i \cap H : G_i \in \mathcal{T}\}$ satisfies the three axioms of graph topology and hence $(H, \mathcal{T}_H)$ is a graph topological space and this graph topological space is called subspace graph topological space. □

The following theorem gives the relation between basis of the graph topological space $(G, \mathcal{T})$ and the subspace graph topological space of $(G, \mathcal{T})$.

Let $= \{\Gamma_i, i \in I\}$be a basis for the graph topological space $(G, \mathcal{T})$ and H be a subgraph of G with subspace graph topology $\mathcal{T}_H = \{G_i \cap H : G_i \in \mathcal{T}\}$. Then, $= \{\Gamma_i \cap H, \Gamma_i \in\}$ is a basis for the $\mathcal{T}_H$.

Proof. Let $(G, \mathcal{T})$ be a graph topological space on a graph G and $(H, \mathcal{T}_H)$ be a subspace graph topological space of a subgraph H of G. Let $= \{\Gamma_1, \Gamma_2, \ldots\}$ be a basis of $\mathcal{T}$. We need to show that $= \{\Gamma_i \cap H, \Gamma_i \in\}$ for $\mathcal{T}_H$. By Theorem 1.2, it is enough to prove that for every vertex and edge of H, there exists a graph in containing them. Let v and e be an arbitrary vertex and edge of the graph H and let H_i be an open subgraph of H containing them(either both in H_i or a vertex or an edge in H_i). Then, $v, e \in H_i = G_i \cap H \subseteq G \cap H$ where $G_i \in \mathcal{T}$ and $G_i \cap H \in \mathcal{T}_H$. Since $v, e \in G_i$ and since G_i is an open subgraph of $\mathcal{T}$ by Theorem 1.2, there exist a graph $\Gamma_i \subset$ containing v and e. It is clear that v and e belongs to $\Gamma_i \cap H \in$. Hence, for every vertex and edge there exist a graph in and hence is a base for the graph topology $\mathcal{T}_H$. □

The subgraphs which are open in the subspace graph topological space need not be always open in the graph topological space. The following proposition discusses the condition when the subgraph H in a subgraph topological space $(H, \mathcal{T}_H)$ is an open subgraph.

Let $(H, \mathcal{T}_H)$ be a subspace graph topological space of $(G, \mathcal{T})$. If S is open subgraph in $\mathcal{T}_H$ and H is an open subgraph of $\mathcal{T}$, then S is an open subgraph of $\mathcal{T}$.

Proof. Given that H is a subgraph of G and $(H, \mathcal{T}_H)$ a subspace graph topological space of $(G, \mathcal{T})$. Let S be an open subgraph of H which implies that $S \in \mathcal{T}_H$ and let H be an open subgraph of $\mathcal{T}$, that is $H \in \mathcal{T}$. We need to show that S is an open subgraph of G. Since S is an open subgraph of H, $S = G_i \cap H$ for some $G_i \in \mathcal{T}$. Since G_i and H are open, by the third axiom of graph topology, their intersection is closed in $\mathcal{T}$. That is, $S = G_i \cap H \in \mathcal{T}$. Hence, S is open in G. □

Let $(G, \mathcal{T})$ be a graph topological space and $(H, \mathcal{T}_H)$ be the subspace graph topological space of $(G, \mathcal{T})$. Let S be a subgraph of H. Then, the subspace graph topology on S inherited from the graph topology $(H, \mathcal{T}_H)$ is same as the subspace graph topology on S inherited from the graph topology $(G, \mathcal{T})$.

Proof. Let $(G, \mathcal{T})$ be a graph topological space and $(H, \mathcal{T}_H)$ be a subspace graph topological space of G. Let S be a subgraph of H. Consider the subspace graph topological space of S inherited from H that is, $\mathcal{T}s = \{H_i \cap S : H_i \in \mathcal{T}_H\}$. Since $H_i \in \mathcal{T}_H$, by Definition 2.1, we have $H_i = G_i \cap H$ for some $i \in I$. Therefore,

$$\begin{aligned} \mathcal{T}s &= \{(G_i \cap H) \cap S : G_i \in \mathcal{T}\} \\ &= \{G_i \cap (H \cap S) : G_i \in \mathcal{T}\} \\ &= \{G_i \cap S : G_i \in \mathcal{T}\} \end{aligned}$$

Hence, we can say that the subspace topology on S obtained from the subspace graph topological space on H is same as the subspace graph topology on S inherited from the graph topology on G. □

Let H_i be a subgraph of $H \subseteq G$ which is open in the graph topological space $(G, \mathcal{T})$ then, H_i is open in the subspace graph topology $(H, \mathcal{T}_H)$.

Proof. Let H_i be a subgraph of H. Given that H_i is open in $\mathcal{T}$ then, $H_i \cap H$ is open in the subspace graph topological space $(H, \mathcal{T}_H)$ by Definition 2.1. Since $H_i \subseteq H$, the intersection, $H_i \cap H = H_i$, which implies that $H_i \in \mathcal{T}_H$. □

3. Closed Graphs in Subspace Graph Topology

The notions of two types of graph complements, called the decomposition-complement and the neighbourhood complement of graphs have been introduced in [7]. The decomposition complement of a graph is defined as follows:

[7] Let G be a graph and let $H = (V_H, E_H)$ be a subgraph of the graph $G = (V, E)$. The complement of the subgraph H with respect to the graph G is the graph $H^* = (V^*, E^*)$ where $E^* = E - E_H$ and the vertex set of H^* is the set of all vertices incident to the edges in $E(H^*)$ is called the *decomposition-complement* of H. A subgraph H in a topological space is *decomposition-closed* or *d-closed* if its decomposition-complement H^* is open in the topological space.

In point set topology, a set is closed if its complement is open. But, in graph topology, a graph is either d-closed or is neighbourhood closed in a topological space if the decomposition complement or the neighbourhood complement is open. The concepts were discussed in detailed in [7]. In this section, we discuss the d-closed graphs and neighbourhood closed graphs in subspace graph topology.

The following theorem gives the edge set ofd-closed graphs in the subspace graphtopological space.

Let $(G, \mathcal{T})$ be a graph topological space and $(H, \mathcal{T}_H)$ be a subspace graph topological space then a graph J is d-closed in the subspace topological space then, $E(J) = E(H) - E(G_i \cap H)$ where $G_i \in \mathcal{T}$.

Proof. Let $(H, \mathcal{T}_H)$ be a subspace graph topological space of $(G, \mathcal{T})$. Let J be a d-closed graph in the subspace topological space then, by Definition 3.1, the decomposition complement of J is an open subgraph in the subspace graph topology $\mathcal{T}_H$. Let $\overline{J}$ denote the decomposition complement of the subgraph J. Since $\overline{J}$ is open in $\mathcal{T}_H$, $\overline{J} = G_i \cap H$, by Definition 2.1. Then, by Definition 2.1 and 3.1 we have,

$$\begin{aligned} E^*(\overline{J}) &= E(H) - E(J) \\ E(G_i \cap H) &= E(H) - E(J) \\ E(J) &= E(H) - E(G_i \cap H) \end{aligned}$$

When $E(G_i \cap H)$ is a non empty set with cardinality less than the cardinality of $E(H)$, $E(J)$ will be a proper subset of $E(H)$ and the graph induced will be a proper subgraph of H. □

Now, we show that the graph and the empty graph in the subspace graph topological space is d-closed.

The graph H and K_0 are d-closed in the subspace graph topological space $(H, \mathcal{T}_H)$ of the graph topological space $(G, \mathcal{T})$.

Proof. By Theorem 3.2, for any d-closed graph J of subspace topological space, $E(J) = E(H) - E(G_i \cap H)$ where $G_i \in \mathcal{T}$. Suppose $G_i \cap H = H$ then, $E^*(\overline{J}) = E(H) - E(H) = \emptyset$. Then,the subgraph induced by empty edge set becomes K_0. Hence, J will be K_0.

Now, suppose that $E(G_i \cap H) = \emptyset$ then, $E^*(\overline{J}) = E(H) - \emptyset = E(H)$ and the subgraph induced by the edge set $E(J)$ will be H(if H is simple). Hence, H is closed.

If H is a subgraph with isolated vertices, then the graph induced by the edge set $E(H)$ will be a subgraph of H. □

Let $(G, \mathcal{T})$ be a graph topological space and $(H, \mathcal{T}_H)$ be a subspace graph topological space. Let G_i be a d-closed subgraph in $\mathcal{T}$, then $G_i \cap H$ is a d-closed subgraph in the subspace graph topological space $(H, \mathcal{T}_H)$.

Proof. Let $(G, \mathcal{T})$ be a graph topological space and G_i be a d-closed graph in the subspace graph topological space. Then, by Definition 3.1, the decomposition complement of G_i is open in $\mathcal{T}$ and

$$E^*(G_i) = E(G) - E(G_i)$$

The subgraph induced the edge set $E^*(G)$ is open in $\mathcal{T}$.

$$\begin{aligned}
E^*(G_i) &= E(G) - E(G_i) \\
E^*(G_i) \cap E(H) &= \Big(E(G) - E(G_i)\Big) \cap E(H) \\
&= \Big(E(G) \cap E(H)\Big) - \Big(E(G_i) \cap E(H)\Big) \\
&= E(H) - \Big(E(G_i) \cap E(H)\Big)
\end{aligned}$$

Since the H is d-closed $E(H) = E^*(H)$ and $E(G_i) \cap E(H) = E(G_i \cap H)$, we have

$$\begin{aligned}
E^*(G_i) \cap E^*(H) &= E(H) - \Big(E(G_i \cap H)\Big) \\
E^*(G_i \cap H) &= E(H) - \Big(E(G_i \cap H)\Big)
\end{aligned}$$

By Definition 2.1, $G_i^* \cap H \in \mathcal{T}_H$ and hence $G_i \cap H$ is closed in $\mathcal{T}_H$. □

4. Conclusion

Analogous to the similar concepts in point set topology, the notions of subspace graph topology and related graph topological space have been introduced and studied in this P1. The topic is promising for further investigation as topological structures of graphs and their analyses are relatively new and have enough within themselves. Some of the research problems we have identified in this area during our study are the following:

1. Other topological properties such as countability, relationship between topological connectedness and usual connectedness of graphs and compactness can be studied

2. Possibility of defining a subgraph distance metric and topology with the same can be explored.

3. The properties of n-closed graphs of subspace graph topology can be studied. The relation between the n-closed graphs of subspace graph topology and the ambient graph topological space can be verified.

Some significant works in a similar area can be found in [8, 9]. All these facts highlights the wide scope for further investigation in the area concerned.

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Declarations

The authors declare that they don't have any competing interests regarding the publication of the article. All authors have significantly contributed to the article.

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