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On ultra algebras

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Abstract

In this paper any superalgebra is generalized to a new form, namely ultra algebra. We study some algebraic structure results for ultra algebras and we have defined some maps on ultra algebras to know ultra derivation and Jordan ultra derivation. Moreover on certain assumptions we have proved that Jordan ultra derivation is an ultra derivation.

Keywords: *Ultra algebra, Alternative ultra algebra, Jordan ultra algebra, Lie ultra algebra.*

Subjclass [2010]: *17A01, 17A30, 17A36.*

Introduction

A new kind of graded Lie algebra, namely $Z_{2,2}$ graded Lie algebra have been introduced by Weimin and Sicong [10]. In their paper the usual Z_2 graded Lie algebra have been generalized to this new class of graded Lie algebra. They have shown that there exist close connections between the $Z_{2,2}$ graded Lie algebra and parastatistics and so they have used $Z_{2,2}$ graded Lie algebra in order to study and analyse various symmetries and supersymmetries of the paraparticle systems. Our work has been mainly influenced by the paper [10] where we have taken the liberty of using the definition of $Z_{2,2}$ graded Lie algebra from Weimin and Sicong with the purpose to provide the definition of general $Z_{2,2}$ graded algebras, that we will name *ultra algebras*, as well as introduce new kinds of $Z_{2,2}$ graded nonassociative algebras. Differently of the interest of those authors, our goal here is to introduce, to study some algebraic structures and also the behavior of a certain map on this new kind of algebras.

1. Ultra algebras

Let vector space A over a field $\mathbf{F}$ be a direct sum of four subspaces A_{ij} $(i, j = 0, 1)$,

$$A = A_{00} \oplus A_{10} \oplus A_{01} \oplus A_{11}.$$

For any two elements in A, we define a product rule, written as $\cdot$, with the following properties :

(i) Closure: For all $a, b \in A$ we have $a \cdot b \in A$.

(ii) Bilinearity: For all $a, b, c \in A$ and $\alpha, \beta \in \mathbf{F}$ we have

$$(\alpha a + \beta b) \cdot c = \alpha a \cdot c + \beta b \cdot c,$$

$$c \cdot (\alpha a + \beta b) = \alpha c \cdot a + \beta c \cdot b,$$

(iii) Grading: For all $a \in A_{ij}, b \in A_{mn}$, $(i, j, m, n = 0, 1)$, we have $a \cdot b \in A_{(i+m)mod2,(j+n)mod2}$.

A linear space satisfying the above conditions is called a $Z_{2,2}$ *graded algebra* that henceforth we will name it *ultra algebra*. It is worth pointing out that an ultra algebra is actually an $Z_{2,2}$-color algebra when the particular antisymmetric bi-character is used, i.e., $\epsilon : Z_{2,2} \times Z_{2,2} \to \mathbf{F}^{\times}$, where $\epsilon((i, j), (m, n)) = (-1)^{(im+jn)mod2}$. It is not difficult to see that A_{00} is an subalgebra of A and $A_{10} \oplus A_{01} \oplus A_{11}$ is an A_{00}-sub-bimodule of A. Let us point out the following remark regarding to degree of some $a \in A_{ij}$. Let $a \in A_{ij}$ a degree $\bar{a} = (i, j)$ satisfies

$$\bar{a}\bar{b} = (i, j)(m, n) = (im + jn) mod 2$$

$$\bar{a} + \bar{b} = (i, j) + (m, n) = (i + m, j + n) mod 2.$$

The elements of each A_{ij}, $i, j \in \{0, 1\}$ are said to be *homogeneous.* Note that ultra algebra is a generalization of the usual superalgebra in the following some aspects: their product rules are the same, but the difference between them is that, for super case we have a direct sum of two subspaces, the grading is one-dimensional, and for ultra case, we have a direct sum of four subspaces, the grading is two-dimensional, the degree of elements in superalgebra is only a number and in an ultra algebra we have a two-dimensional vector.

It is well known that if A and B are superalgebras then there exists a Z_2 natural graduation $A \otimes B$, where $(A \otimes B)_0 = A_0 \otimes B_0 + A_1 \otimes B_1$ and $(A \otimes B)_1 = A_1 \otimes B_0 + A_0 \otimes B_1$. Let us consider $A = A_{00} + A_{01} + A_{10} + A_{11}$ and $B = B_{00} + B_{01} + B_{10} + B_{11}$ a couple of ultra algebras then we have an ultra $Z_{2,2}$ natural graduation $A \otimes B$, where $(A \otimes B)_{00} = A_{00} \otimes B_{00} + A_{01} \otimes B_{01} + A_{10} \otimes B_{10} + A_{11} \otimes B_{11}$, $(A \otimes B)_{01} = A_{00} \otimes B_{01} + A_{10} \otimes B_{11} + A_{01} \otimes B_{00} + A_{11} \otimes B_{10}$, $(A \otimes B)_{10} = A_{00} \otimes B_{10} + A_{01} \otimes B_{11} + A_{10} \otimes B_{00} + A_{11} \otimes B_{01}$ and $(A \otimes B)_{11} = A_{00} \otimes B_{11} + A_{01} \otimes B_{10} + A_{10} \otimes B_{01} + A_{11} \otimes B_{00}$. We denote by $A \widetilde{\otimes} B$ the graduated tensor product of the ultra algebras A and B, that is the tensorial product of spaces A and B with the multiplication $(a \widetilde{\otimes} b)(x \widetilde{\otimes} y) = (-1)^{\bar{b}\bar{x}}(ax \widetilde{\otimes} by)$, where $a \in A$, $x \in A_{ij}$, $b \in B_{mn}$ and $y \in B$. Let $A = A_{00} \oplus A_{10} \oplus A_{01} \oplus A_{11}$ be an ultra algebra, as we have just seen, $G \widetilde{\otimes} A$ has an ultra algebra structure. We call the *Grassmann envelope* of the ultra algebra A, and we will denote by $G(A)$, the subalgebra $(G \widetilde{\otimes} A)_{00}$, that is, $G(A) = G_{00} \widetilde{\otimes} A_{00} \oplus G_{01} \widetilde{\otimes} A_{01} \oplus G_{10} \widetilde{\otimes} A_{10} \oplus G_{11} \widetilde{\otimes} A_{11}$. An ideal I of the ultra algebra A is called *ultra graded ideal* if $I = I \cap A_{00} + I \cap A_{01} + I \cap A_{10} + I \cap A_{11}$. An ultra algebra A is called *simple ultra algebra* if it contains no proper ultra graded ideals, and $A^2 \neq 0$. We say that A is a *prime ultra algebra* if the product of any two nonzero ultra graded ideals in A is nonzero. Let $\bar{a} = (i, j)$ with $i, j \in \{0, 1\}$. We say that a $\mathbf{F}$-linear map $D : A \to A$ is a *ultra derivation of degree* $\bar{a}$ if it satisfies $D(A_{\bar{b}}) \subseteq A_{\bar{a}+\bar{b}}$ (index modulo 2) where $\bar{b} = (s, t)$ and

$$D(xy) = D(x)y + (-1)^{\bar{a}\bar{x}} xD(y) \quad \text{for all} \quad x, y \in A_{00} \cup A_{01} \cup A_{10} \cup A_{11}.$$

An *ultra derivation* is the sum of an ultra derivation of degree $(0,0)$, an ultra derivation of degree $(0,1)$, an ultra derivation of degree $(1,0)$ and an ultra derivation of degree $(1,1)$.

Consider again $\bar{a} = (i,j)$ with $i,j \in \{0,1\}$. We say that a $\mathbf{F}$-linear map $d_u : A \to A$ is a *inner ultra derivation of degree* $\bar{a}$ if it satisfies $d_u(A_{\bar{b}}) \subseteq A_{\bar{a}+\bar{b}}$ (index modulo 2) where $\bar{b} = (s,t)$ and

$$d_u(x) = xu - (-1)^{\bar{u}\bar{x}}ux \quad \text{for all} \quad x, u \in A_{00} \cup A_{01} \cup A_{10} \cup A_{11}.$$

An *inner ultra derivation* is the sum of the inner ultra derivations of degrees $(0,0)$, $(0,1)$, $(1,0)$ and $(1,1)$. Introducing a new product by

$$x \circ_U y = xy + (-1)^{\bar{x}\bar{y}}yx, \qquad x, y \in A_{00} \cup A_{01} \cup A_{10} \cup A_{11},$$

we will say that a $\mathbf{F}$-linear map $D : A \to A$ is a *Jordan ultra derivation of degree* $\bar{a}$ if $D(A_{\bar{b}}) \subseteq A_{\bar{b}+\bar{a}}$ and

$$D(x \circ_U y) = D(x) \circ_U y + (-1)^{\bar{a}\bar{x}} x \circ_U D(y) \quad \text{for all} \quad x, y \in A_{00} \cup A_{01} \cup A_{10} \cup A_{11}.$$

We define a *Jordan ultra derivation* as the sum of Jordan ultra derivations of degrees $(0,0)$, $(0,1)$, $(1,0)$ and $(1,1)$. Clearly every ultra derivation is also a Jordan ultra derivation.

Definition 1. *Let $\mathcal{M}$ be a class of algebras defined by multilinear identities, we say that an ultra algebra $A = A_{00} \oplus A_{10} \oplus A_{01} \oplus A_{11}$ is said to be a $\mathcal{M}$-ultra algebra if $G(A)$, considered as algebra, is an element of $\mathcal{M}$.*

Besides the ultra algebra let us introduce other some examples of new kinds of ultra nonassociative algebras, namely, alternative ultra algebra and Jordan ultra algebra.

Definition 2. *Let A be a ultra algebra if A satisfies:*

$$(a,b,c) = (-1)^{\bar{a}\bar{b}+1}(b,a,c) = (-1)^{\bar{b}\bar{c}+1}(a,c,b) \qquad (\textit{ultra alternative identity}),$$

where $(a,b,c) = (ab)c - a(bc)$, $a, b, c \in A_{00} \cup A_{10} \cup A_{01} \cup A_{11}$ then A is called an alternative ultra algebra.

Definition 3. *Let A be an ultra algebra if A satisfies:*

$$ab = (-1)^{\bar{a}\bar{b}}ba \qquad (\textit{ultra commutator identity})$$

and

$$\begin{aligned}
&((ab)c)d + (-1)^{\bar{b}\bar{c}+\bar{b}\bar{d}+\bar{c}\bar{d}}((ad)c)b + (-1)^{\bar{a}\bar{b}+\bar{a}\bar{c}+\bar{a}\bar{d}+\bar{c}\bar{d}}((bd)c)a \\
&= (ab)(cd) + (-1)^{\bar{b}\bar{c}}(ac)(bd) \\
+ \quad &(-1)^{\bar{d}(\bar{b}+\bar{c})}(ad)(bc) \qquad (\textit{ultra Jordan identity}),
\end{aligned}$$

where $a, b, c, d \in A_{00} \cup A_{10} \cup A_{01} \cup A_{11}$ *then* A *is called a Jordan ultra algebra.*

Definition 4 ($Z_{2,2}$ *graded Lie algebras*, **[10]).** *Let* A *be an ultra algebra if* A *satisfies:*

$$ab = -(-1)^{\bar{a}\bar{b}}ba \qquad (\textit{ultra anticommutator identity})$$

and

$$a(bc)(-1)^{\bar{a}\bar{c}} + b(ca)(-1)^{\bar{a}\bar{b}} + c(ab)(-1)^{\bar{b}\bar{c}} = 0 \qquad (\textit{ultra Jacobi identity}),$$

where $a, b, c \in A_{00} \cup A_{10} \cup A_{01} \cup A_{11}$ *then* A *is called a Lie ultra algebra.*

It is easy to see that an alternative (Jordan, Lie) ultra algebra is a generalization of the usual alternative (Jordan, Lie) superalgebra.

Our main goal is to prove under some certain assumptions that any Jordan ultra derivation of ultra algebra is an ultra derivation.

2. Basic results

It is well-known and easy to see that the primeness of an superalgebra A can be characterized by the condition that $aAb = 0$, where $a, b \in A_0 \cup A_1$, implies $a = 0$ or $b = 0$. Making some obvious modifications in the argument we get an analogous result for ultra algebras:

Lemma 1. *An ultra algebra* $A = A_{00} \oplus A_{01} \oplus A_{10} \oplus A_{11}$ *is prime if and only if for any homogeneous elements* a, b, $(a, b \in A_{00} \cup A_{10} \cup A_{01} \cup A_{11})$, $aAb = 0$ *implies* $a = 0$ *or* $b = 0$.

Lemma 2. *If* A *is an ultra algebra with unity* 1, *then* $1 \in A_{00}$.

Proof. Let $1 = a_{00} + a_{10} + a_{01} + a_{11}$, where $a_{00} \in A_{00}$, $a_{10} \in A_{10}$, $a_{01} \in A_{01}$ and $a_{11} \in A_{11}$. Multiplying on the left by a_{00}, we obtain $a_{00} = a_{00}^2 + a_{00}a_{10} + a_{00}a_{01} + a_{00}a_{11}$, whence $a_{00}a_{10} = a_{00}a_{01} = a_{00}a_{11} = 0$. Similarly we prove that $a_{10}a_{00} = a_{01}a_{00} = a_{11}a_{00} = 0$. Also multiplying on the left by a_{10}, we obtain $a_{10} = a_{10}a_{00} + a_{10}^2 + a_{10}a_{01} + a_{10}a_{11}$, whence $a_{10}a_{01} = a_{10}a_{11} = 0$ and similarly $a_{01}a_{10} = a_{11}a_{10} = 0$. Now multiplying on the left by a_{01}, we obtain $a_{01} = a_{01}a_{00} + a_{01}a_{10} + a_{01}^2 + a_{01}a_{11}$ and $a_{01}a_{10} = a_{01}a_{11} = 0$. And finally multiplying on the left by a_{11}, we obtain $a_{11} = a_{11}a_{00} + a_{11}a_{10} + a_{11}a_{01} + a_{11}^2$, whence $a_{11}a_{10} = a_{11}a_{01} = 0$. Furthermore, $1 = 1^2 = a_{00}^2 + a_{10}^2 + a_{01}^2 + a_{11}^2 \in A_{00}$. □

Lemma 3. *Let A be an unital ultra algebra over ground field $\mathbf{F}$ with $1, \frac{1}{2} \in \mathbf{F}$. If there is no zerodivisors A and satifies $ab = (-1)^{\bar{a}\bar{b}}ba$ for all $a, b \in A$ then any idempotent lies in A_{00}.*

Proof. Let $e = e_{00} + e_{01} + e_{10} + e_{11}$ be an idempotent, then

1) $e_{00}^2 + e_{11}^2 = e_{00}$;

2) $e_{01} = 0$;

3) $2e_{00}e_{10} = e_{10}$;

4) $2e_{00}e_{11} = e_{11}$.

Multiplying 3) by e_{00} we get $e_{10}e_{00} = 2e_{00}^2e_{10} = 2e_{00}e_{10} - 2e_{11}^2e_{10} = 2e_{10}e_{00} - 2e_{11}^2e_{10}$, hence $e_{10}e_{00} = 2e_{11}^2e_{10}$. Thus if $2e_{00} = 1_A$ then by 1) we get $e_{11} = \frac{1}{2} \cdot 1_A$ or $e_{11} = -\frac{1}{2} \cdot 1_A$. In both cases we have $e_{11} = 0$ and $e_{10} = 0$. □

Lemma 4. *Let A be an unital ultra algebra over a ground field of characteristic $\neq 2, 3$. Consider $e_i \in A_{00}$, $i = 1, \dots, n$, mutually orthogonal idempotents such that $\sum_i e_i = 1$. Put $A_{ij} = e_iAe_j$, then the following hold:*

1) $\bigoplus_{i,j=1}^{n} A_{ij} = A$ *is a direct sum of vector spaces;*

2) $A_{ij}A_{jk} \subseteq A_{ik}$ *for all* $i, j, k = 1, \dots, n$*;*

3) $A_{ij}A_{kl} = \{0\}$ *for* $i, j = 1, \dots, n$, $j \neq k$.

Proof. Let us prove only 2) because the others one have their proves similar.
2) We have $A_{ij} = A_{ij}^{00} + A_{ij}^{01} + A_{ij}^{10} + A_{ij}^{11}$, where $A_{ij}^{kt} = eiA_{kt}ej$, $k, t \in \{0, 1\}$. Let $x \in A_{ij}^{kt}$ and $y \in A_{ij}^{rs}$, where $k, t, r, s \in \{0, 1\}$. Consider g_{kt}^{x} and g_{rs}^{y} elements in $\langle G_{kt}\rangle$ and $\langle G_{rs}\rangle$ respectively. Pick, $g_{kt}^{x} \otimes x \in G(A)_{ij}$ and $g_{rs}^{y} \otimes y \in G(A)_{jk}$ then $(g_{kt}^{x} \otimes x)(g_{rs}^{y} \otimes y) = g_{kt}^{x} g_{rs}^{y} \otimes xy \in G(A)_{ik}$, hence $xy \in A_{ik}$. $\square$

Notation: Let A be an unital ultra algebra, such that A_{00} containing a nontrivial idempotent e_1 and $e_2 = 1_A - e_1$. When is necessary to use the two decomposition of A then we can differentiate the decomposition of the ultra algebra and its decomposition regard to the idempotents e_1, e_2 by

$$A = A^{(00)} \oplus A^{(01)} \oplus A^{(10)} \oplus A^{(11)}$$

and

$$A = A_{11} \oplus A_{12} \oplus A_{21} \oplus A_{22}$$

respectively.

3. Jordan ultra derivation of ultra algebras

It is well known that given an associative algebra A, it is possible to construct another algebra A^{+}, by defining the new product as $a \circ b = ab + ba$ where ab is the product of a and b in A. This new algebra is called a Jordan algebra. In 1957, Herstein [9] defined the Jordan derivation and he proved that any Jordan derivation of an prime algebra A over ground field of characteristic different from 2 is an ordinary derivation of A. With this picture in mind, we have proved the following

Theorem 1. *Let A be an unital ultra algebra over ground field of characteristic not 2, such that A_{00} containing a nontrivial idempotent e_1. Consider $A = A_{11} \oplus A_{12} \oplus A_{21} \oplus A_{22}$ where $A_{ij} = e_iAe_j$ regard to e_1 and $e_2 = 1_A - e_1$ and satisfying the following conditions:*

- ($\clubsuit$) *For each homogeneous a_{jj} element, if $x_{ij}a_{jj} = 0$, for all $x_{ij} \in A_{ij}$, then $a_{jj} = 0$;*
- ($\diamondsuit$) *For each homogeneous a element, if $e_ixe_jae_ixe_j = 0$, for all $x \in A$, then $e_jae_i = 0$.*

If D is a Jordan ultra derivation of A then D is an ultra derivation.

Before proceeding with the proof of the theorem let us point out the following: If we consider the map $D - f : A \to A$ where f is inner ultra derivation of A then it is easy to see that $(D - f)(e_1) = 0$. Thus, we will consider without loss of generality Jordan ultra derivation D such that $D(e_1) = 0$. Let D be Jordan ultra derivation of degree $\bar{D}$ we need to prove that $D(x_{kt}y_{rs}) = D(x_{kt})y_{rs} + (-1)^{\bar{D}\bar{x}}x_{kt}D(y_{rs})$ for every homogeneous element $x_{kt} \in A_{kt}$ and $y_{rs} \in A_{rs}$, $(k, t, r, s \in \{1, 2\})$.

Proof. First we note that $D(x_{kt}) \in A_{kt}$, $(k, t \in \{1, 2\})$. If

$$x \circ_U y = xy + (-1)^{\bar{x}\bar{y}}yx = xy + yx;$$
$$D(x \circ_U y) = D(x) \circ_U y + (-1)^{\bar{D}\bar{x}}x \circ_U D(y) = D(x) \circ_U y + x \circ_U D(y).$$

that is when $(-1)^{\bar{x}\bar{y}} = (-1)^{\bar{D}\bar{x}} = 1$ we have the same case as Lemma 2.4 of [8]. We need to study the other three cases, that is,

(1) $(-1)^{\bar{x}\bar{y}} = 1$ and $(-1)^{\bar{D}\bar{x}} = -1$;

(2) $(-1)^{\bar{x}\bar{y}} = -1$ and $(-1)^{\bar{D}\bar{x}} = 1$;

(3) $(-1)^{\bar{x}\bar{y}} = -1$ and $(-1)^{\bar{D}\bar{x}} = -1$.

(1) We get $D(xy+yx) = D(x)y+(-1)^{\overline{D(x)}\bar{y}}yD(x)-(xD(y)+(-1)^{\bar{x}\overline{D(y)}}D(y)x)$. Let $a_{ii} \in A_{ii}$, $i = 1, 2$, we have

$$2D(a_{ii}) = D(a_{ii})a_{ii}+(-1)^{\overline{D(a_{ii})}\overline{a_{ii}}}a_{ii}D(a_{ii})-(a_{ii}D(a_{ii})+(-1)^{\overline{a_{ii}}\overline{D(a_{ii})}}D(a_{ii})a_{ii}).$$

When $(-1)^{\overline{D(a_{ii})}\overline{a_{ii}}} = -1$ we have the same as Lemma 2.4 of [8]. And when $(-1)^{\overline{D(a_{ii})}\overline{a_{ii}}} = 1$ we have $D(a_{ii}) = 0 \in A_{ii}$. Now let $a_{12} \in A_{12}$ for any $b_{12} \in A_{12}$ we have

$$0 = D(a_{12}b_{12} + b_{12}a_{12}) = D(a_{12})b_{12} + (-1)^{\overline{D(a_{12})}\overline{b_{12}}}b_{12}D(a_{12})$$
$$-(a_{12}D(b_{12}) + (-1)^{\overline{a_{12}}\overline{D(b_{12})}}D(b_{12})a_{12}).$$

Note that since $D(e_i) = 0$, becomes $D(a_{12}) = D(a_{12})e_1+(-1)^{\overline{D(a_{12})}\overline{e_1}}e_1D(a_{12})$. This implies that $e_1D(a_{12})e_1 = e_2D(a_{12})e_2 = 0$. Also we have

$$0 = 2D(b_{12}^2) = D(b_{12}b_{12} + b_{12}b_{12})$$
$$= D(b_{12})b_{12} + (-1)^{\overline{D(b_{12})}\overline{b_{12}}}b_{12}D(b_{12}) - (b_{12}D(b_{12}) + (-1)^{\overline{b_{12}}\overline{D(b_{12})}}D(b_{12})b_{12}),$$

multiplying this equation by e_1 from the left and using $e_1D(b_{12})e_1 = 0$, we get $b_{12}D(b_{12}) = 0$ when $(-1)^{\overline{D(b_{12})}\overline{b_{12}}} = -1$ and $D(b_{12})b_{12} = 0$ when $(-1)^{\overline{D(b_{12})}\overline{b_{12}}} = 1$. Hence, $b_{12}D(a_{12})b_{12} = 0$ and by $(\diamondsuit)$ we get $e_2D(a_{12})e_1 = 0$. Similarly, one can verify that $D(a_{21}) \in A_{21}$.
(2) and (3) are trivial.
We need to prove that

$$D(xy) = D(x)y + (-1)^{\overline{D}\overline{x}}xD(y) \quad \text{for all} \quad x, y \in A_{00} \cup A_{01} \cup A_{10} \cup A_{11},$$

for all $\overline{D} = (i, j)$, $i, j \in 0, 1$. Since D is linear, it is sufficient to prove that

$$D(x_{kt}y_{rs}) = D(x_{kt})y_{rs} + (-1)^{\overline{D}\overline{x}}x_{kt}D(y_{rs}) \ \text{ for all } \ x_{kt}, y_{rs} \in A_{00} \cup A_{01} \cup A_{10} \cup A_{11},$$

for all $x_{kt} \in A_{kt}$ and $y_{rs} \in A_{rs}$, $(k, t, r, s \in \{1, 2\})$. Thus we will analyze the following steps:

Step 1: It is easy to see that $D(a_{ii}b_{jj}) = 0 = D(a_{ii})b_{jj} + a_{ii}D(b_{jj})$ and $D(a_{ij}b_{ij}) = 0 = D(a_{ij})b_{ij} + a_{ij}D(b_{ij})$, where $i \neq j$.

Step 2: Since D is a Jordan ultra derivation we have

$$\begin{aligned}
&D(a_{ii}b_{ij}) = D(a_{ii}b_{ij} + (-1)^{\overline{a_{ii}}\overline{b_{ij}}}b_{ij}a_{ii}) \\
&= D(a_{ii})b_{ij} + (-1)^{\overline{D(a_{ii})}\overline{b_{ij}}}b_{ij}D(a_{ii}) + (-1)^{\overline{a_{ii}}\overline{D}}((-1)^{\overline{a_{ii}}\overline{D(b_{ij})}}D(b_{ij})a_{ii} + a_{ii}D(b_{ij})) \\
&= D(a_{ii})b_{ij} + (-1)^{\overline{a_{ii}}\overline{D}}a_{ii}D(b_{ij}),
\end{aligned}$$

where $i \neq j$. Similarly, we also have that $D(b_{ji}a_{ii}) = D(b_{ji})a_{ii} + (-1)^{\overline{b_{ji}}\overline{D}}b_{ji}D(a_{ii})$.

Step 3: Note that,
$D(b_{ji}a_{ij} + (-1)^{\overline{b_{ji}}\overline{a_{ij}}}a_{ij}b_{ji}) = D(b_{ji})a_{ij} + (-1)^{\overline{a_{ij}}\overline{D(b_{ji})}}a_{ij}D(b_{ji}) + (-1)^{\overline{b_{ji}}\overline{D}}(b_{ji}D(a_{ij}) + (-1)^{\overline{b_{ji}}\overline{D(a_{ij})}}D(a_{ij})b_{ji})$, where $i \neq j$. It follows that $D(b_{ji}a_{ij}) = D(b_{ji})a_{ij} + (-1)^{\overline{b_{ji}}\overline{D}}b_{ji}D(a_{ij})$.

Step 4: On the one hand, for $c_{ij} \in A_{ij}$ with $i \neq j$ we have

$$\begin{aligned}
&D((c_{ji}a_{ii})b_{ii} + (-1)^{\overline{b_{ii}}\overline{(c_{ji}a_{ii})}} b_{ii}(c_{ji}a_{ii}))\\
&= D(c_{ji}a_{ii})b_{ii} + (-1)^{\overline{b_{ii}}\overline{D(c_{ji}a_{ii})}} b_{ii}D(c_{ji}a_{ii})\\
&+(-1)^{\overline{(c_{ji}a_{ii})}\overline{D}}((c_{ji}a_{ii})D(b_{ii}) + (-1)^{\overline{b_{ii}}\overline{D(c_{ji}a_{ii})}} D(b_{ii})(c_{ji}a_{ii}))\\
&= D(c_{ji}a_{ii})b_{ii} + (-1)^{\overline{(c_{ji}a_{ii})}\overline{D}}((c_{ji}a_{ii})D(b_{ii})\\
&= D(c_{ji})a_{ii}b_{ii} + (-1)^{\overline{c_{ji}}\overline{D}} c_{ji}D(a_{ii})b_{ii} + (-1)^{\overline{(c_{ji}a_{ii})}\overline{D}}(c_{ji}a_{ii})D(b_{ii}).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
D(c_{ji}(a_{ii}b_{ii}) \quad &+(-1)^{\overline{a_{ii}b_{ii}}\overline{c_{ji}}}(a_{ii}b_{ii})c_{ji})\\
&= D(c_{ji})(a_{ii}b_{ii}) + (-1)^{\overline{a_{ii}b_{ii}}\overline{D(c_{ji})}}(a_{ii}b_{ii})D(c_{ji})\\
&+(-1)^{\overline{c_{ji}}\overline{D}}(c_{ji}D(a_{ii}b_{ii}) + (-1)^{\overline{c_{ji}}\overline{D(a_{ii}b_{ii})}} D(a_{ii}b_{ii})c_{ji}).
\end{aligned}$$

Therefore,
$c_{ji}\left[(-1)^{\overline{c_{ji}}\overline{D}}(D(a_{ii}b_{ii}) - (-1)^{\overline{c_{ji}}\overline{D}} D(a_{ii})b_{ii} - (-1)^{\overline{(c_{ji}a_{ii})}\overline{D}} a_{ii}D(b_{ii})\right] = 0$, for all $c_{ji} \in A_{ji}$, hence by (♣) we get $D(a_{ii}b_{ii}) = D(a_{ii})b_{ii} + (-1)^{\overline{a_{ii}}\overline{D}} a_{ii}D(b_{ii})$. It follows from the four steps above that D is an ultra derivation, and therefore Theorem 1 holds true. □

It is easy to see that any prime ultra algebra satisfies (♣) and (♢), hence we have

Corollary 2. *Let A be an unital prime ultra algebra over ground field of characteristic not 2, such that A_{00} containing a nontrivial idempotent e_1. Then any Jordan ultra derivation of A is an ultra derivation.*

Clearly, by Lemma 3, we have

Theorem 3. *Let A be an unital prime ultra algebra over ground field of characteristic not 2 without zerodivisors and satisfying the ultra commutator identity such that A containing a nontrivial idempotent. Then any Jordan ultra derivation of A is an ultra derivation.*

4. Conclusion

The natural question that appears is how to generalize some classical structures. In this paper we have introduced Jordan ultra algebras, and then we have the following question: What is the connection between the structure of ultra algebras and Jordan ultra algebras? In addition, the study about map structures on nonassociative algebras has become an area of great interest of pure math in the last years, we can quote some recent works [1, 2, 3, 4, 5, 6, 7]. Therefore other line of research that appears here is to know when a map is additive on nonassociative ultra algebras. Another open problem that appear in this work is: If A is a nonassociative ultra algebra (e.g. alternative ultra algebra, Jordan ultra algebra) with nontrivial idempotent then the Theorem 1 holds true ?

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