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Jordan product and fixed points preservers

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Abstract

Let $\mathcal{B}(X)$ be the space of all bounded linear operators on complex Banach space X. For $A \in \mathcal{B}(X)$, we denote by $F(A)$ the subspace of all fixed points of A. In this paper, we study and characterize all surjective maps ϕ on $\mathcal{B}(X)$ satisfying

$$F(\phi(T)\phi(A) + \phi(A)\phi(T)) = F(TA + AT)$$

for all $A, T \in B(X)$.

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1. Introduction

In the following, X is a complex Banach space, and $\mathcal{B}(X)$ denotes the space of all bounded linear operators on X. Let X^* be the dual space of X. Given a vector $x \in X$ and a linear functional $f \in X^*$, the rank at most one operator, $x \otimes f$, defined by $(x \otimes f)z = f(z)x$ for all $z \in X$. Note that

$$x \otimes f \text{ is nilpotent if and only if } f(x) = 0, \tag{1.1}$$

and

$$x \otimes f \text{ is idempotent if and only if } f(x) = 1. \tag{1.2}$$

We denote by $\mathcal{F}_1(X)$ and $\mathcal{N}_1(X)$ the set of all rank at most one operators and the set of all rank one nilpotent operators, respectively. For any subspace $Y \subset X$, the dimension of Y will be denoted by $\dim Y$. For every operator $T \in \mathcal{B}(X)$, let $N(T)$ be the kernel of T, and $R(T)$ be its range. For an operator $A \in \mathcal{B}(X)$, a vector $x \in X$ is a fixed point of A if $Ax = x$. Let $F(A)$ be the set of all fixed points of A. The lattice of A, $Lat(A)$, is the set of all invariant subspaces of A. Recall that $F(A) \in Lat(A)$ for every $A \in \mathcal{B}(X)$. Recall also that the set of fixed points of rank one operator is given by

$$F(x \otimes f) = \begin{cases} span\{x\} & \text{if } x \otimes f \text{ is idempotent,} \\ \{0\} & \text{if } x \otimes f \text{ is not idempotent.} \end{cases}$$

The study of maps on operators or matrices that leave some properties invariant, is the most active problems in the last decades, see for instance [1, 3, 5, 6, 8].

Recently, many authors have studied the subspace of fixed points preservers. For example, in [9] A. Taghavi and R. Hosseinzadeh characterized all surjective maps on $\mathcal{B}(X)$ preserving the dimension of the vector space containing of all fixed points of products of operators, they showed that if X is a complex Banach space with $\dim X \geq 3$ and $\phi : \mathcal{B}(X) \longrightarrow \mathcal{B}(X)$ is a surjective map satisfies

$$\dim F(\phi(A)\phi(B)) = \dim F(AB)$$

for all $A, B \in \mathcal{B}(X)$, then there exists an invertible operator $S \in \mathcal{B}(X)$ such that $\phi(A) = \pm SAS^{-1}$ for all $A \in \mathcal{B}(X)$. In [10] A. Taghavi, R. Hosseinzadeh and V. Darvish described the forms of surjective maps ϕ on $\mathcal{B}(X)$ satisfying

$$F(\phi(A)\phi(B)\phi(A)) = F(ABA)$$

for all $A, B \in \mathcal{B}(X)$, where X is a complex Banach space with $\dim X \geq 3$, they proved that there exists a nonzero scalar $\alpha \in \mathbf{C}$ with $\alpha^3 = 1$ such that $\phi(A) = \alpha A$ for every $A \in \mathcal{B}(X)$. In [2] Y. Bouramdane et al. proved the previous result for the generalized product of operators.

This paper is motivated by the ideas from [2], but the proofs of our main results require new agruments. The statements of our main result can be stated as follows.

Theorem 1. *Let X be a complex Banach space with $\dim X \geq 4$ and $\phi : \mathcal{B}(X) \longrightarrow \mathcal{B}(X)$ be a surjective map. Then the following assertions are equivalent.*

(i) *For all $T, S \in \mathcal{B}(X)$, we have*

$$F(\phi(T)\phi(S) + \phi(S)\phi(T)) = F(TS + ST). \tag{1.3}$$

(ii) *There exists a nonzero scalar $\alpha \in \mathbf{C}$ with $\alpha^2 = 1$ such that $\phi(T) = \alpha T$ for all $T \in \mathcal{B}(X)$.*

2. Preliminaries and Notations

To formulate the next lemma, we use this notation

$$\mathcal{F}_{1,\sqrt{2}}(X) := \{x \otimes f : x \in X \text{ and } f \in X^* \text{ with } f(x) = \frac{1}{\sqrt{2}}\}.$$

In the following lemma, we will give a condition for two operators to be the same.

Lemma 1. *Let A and B be non-scalar operators. The following statements are equivalent.*

(i) $A = B$.

(ii) $F(AT + TA) = F(BT + TB)$ *for all $T \in \mathcal{F}_{1,\sqrt{2}}(X)$.*

Proof. Since $(i) \Longrightarrow (ii)$ is obvious, we need only to prove the implication $(ii) \Longrightarrow (i)$. Let us set $R = AT + TA$ and $S = BT + TB$. Assume that $A \neq B$, so we shall distinguish two cases.

Case 1. x, Ax and Bx are linearly independent for certain nonzero vector $x \in X$. We will discuss two cases.

Case 1.1. If x, Ax and A^2x are linearly independent. It follows that there exists $f \in X^*$ such that $f(x) = f\left(A^2x\right) = \frac{1}{\sqrt{2}}$ and $f(Ax) = 1 - \frac{1}{\sqrt{2}}$. Hence we get that $x + Ax \in F(R) = F(S) \subset span\{x, Bx\}$, this is a contradiction.

Case 1.2. If not, then there exist $a, b \in \mathbf{C}$ such that $A^2x = aAx + bx$. Let $f \in X^*$ such that $f(x) = \frac{1}{\sqrt{2}}$ and $f(Ax) = \mu$ with $-\sqrt{2}\mu^2 + (a + 2\sqrt{2})\mu + (\frac{b}{\sqrt{2}} - \sqrt{2}) = 0$. Consider an operator $T \in \mathcal{B}(X)$ such that $T = x \otimes f$. Hence we have

$$\begin{aligned} R(\sqrt{2}(1-\mu)x + Ax) &= (-\sqrt{2}\mu^2 + (a+\sqrt{2})\mu + \tfrac{b}{\sqrt{2}})x + Ax \\ &= \sqrt{2}(1-\mu)x + Ax. \end{aligned} \tag{2.1}$$

Thus by (2.1) we obtain that $\sqrt{2}(1-\mu)x + Ax \in F(R) = F(S) \subset span\{x, Bx\}$, a contradiction.

Case 2. x, Ax and Bx are linearly dependent for all $x \in X$. Lemma 2.4 in [7] tell us that there exist $\alpha, \lambda \in \mathbf{C}$ such that $B = \lambda A + \alpha I$. By hypothesis, A is a non-scalar operator, there exists a nonzero vector $x \in X$ such that Ax and x are linearly independent. On the other hand we have

$$\begin{cases} Rx = f(Ax)x + f(x)Ax \\ RAx = f(A^2x)x + f(Ax)Ax \end{cases} \quad \text{and} \quad \begin{cases} Sx = \lambda Rx + 2\alpha f(x)x \\ SAx = \lambda RAx + 2\alpha f(Ax)x. \end{cases}$$

We discuss two cases.

Case 2.1. If x, Ax and A^2x are linearly independent for certain $x \in X$. Then there exists $f \in X^*$ such that $f(x) = f\left(A^2x\right) = \frac{1}{\sqrt{2}}$ and $f(Ax) = 1 - \frac{1}{\sqrt{2}}$. Hence we get that $x + Ax \in F(R) = F(S)$. Since

$$\begin{aligned} S(x + Ax) &= \lambda(x + Ax) + 2\alpha x \\ &= x + Ax, \end{aligned}$$

we obtain $\lambda = 1$ and $\alpha = 0$.

Case 2.2. If x, Ax and A^2x are linearly dependent for all $x \in X$. It follows that there exist $a, b \in \mathbf{C}$ such that $A^2x = aAx + bx$. As in the *Case 1.2*, we can get that $\sqrt{2}(1-\mu)x + Ax \in F(R) = F(S)$. On the other hand we have

$$S(\sqrt{2}(1-\mu)x + Ax) = \lambda(\sqrt{2}(1-\mu)x + Ax) + 2\alpha x = \sqrt{2}(1-\mu)x + Ax, \tag{2.2}$$

Hence by (2.2) we obtain that $\lambda = 1$ and $\alpha = 0$, as desired. This finishes the proof.

□

In the next lemma, we characterize rank one non-nilpotent operators by the dimension of fixed points of Jordan porduct of operators.

Lemma 2. *For a nonzero operator $A \in \mathcal{B}(X)$. The following statements are equivalent.*

(i) $A \in \mathcal{F}_1(X) \backslash \mathcal{N}_1(X)$.

(ii) $dim F(AT + TA) \leq 1$ *for all* $T \in \mathcal{B}(X)$.

Proof. $(i) \Longrightarrow (ii)$ Let $T \in \mathcal{B}(X)$ be an arbitrary operator, and consider a non-nilpotent operator $A = x \otimes f$ where $x \in X$ and $f \in X^*$. Note that

$$\begin{cases} (AT + TA)x = f(Tx)x + f(x)Tx \\ (AT + TA)Tx = f\left(T^2x\right)x + f(Tx)Tx. \end{cases}$$

Now, if Tx and x are linearly independent, then we have $x \notin F(AT+TA) \subset span\{x, Tx\}$. If not, we get that $F(AT + TA) \subset span\{x\}$. Hence in both cases, we obtain that $dim F(AT + TA) \leq 1$, as desired.

$(ii) \Longrightarrow (i)$ Suppose that there exists a vector $x \in X$ such that x, Ax and A^2x are linearly independent. Let $T \in \mathcal{B}(X)$ such that

$$Tx = 0, \ \ TAx = x \text{ and } TA^2x = 0.$$

Then

$$\begin{aligned} (AT + TA)x &= x \\ (AT + TA)Ax &= Ax, \end{aligned}$$

which implies that $span\{x, Ax\} \subseteq F(AT + TA)$, a contradiction. Hence x, Ax and A^2x are linearly dependent for all $x \in X$. It follows from lemma 2.4 in [7] that there exists a complex minimal polynomial Q of degree less than 2 such that $Q(A) = 0$. We will distinguish two cases.

Case 1. If $d^\circ(Q) = 1$, then $A = \lambda I$ where λ is a nonzero scalar. Consider an operator $T = \frac{1}{2\lambda}I$, it follows that $AT+TA = I$, hence $F(AT+TA) = X$, a contradiction.

Case 2. If $d^\circ(Q) = 2$, then we discuss the following points.

- If Q admits two single nonzero roots $\lambda_1, \lambda_2 \in \mathbf{C}$, then

$Q(A) = (A - \lambda_1 I)(A - \lambda_2 I)$. It follows that there exist $x_1, x_2 \in X$ linearly independent vectors such that $Ax_1 = \lambda_1 x_1$ and $Ax_2 = \lambda_2 x_2$. We choose $T \in \mathcal{B}(X)$ to be an operator satisfying

$$Tx_1 = \frac{1}{2\lambda_1}x_1 \text{ and } Tx_2 = \frac{1}{2\lambda_2}x_2.$$

Then

$$\left\{ \begin{array}{l} (AT + TA)x_1 = x_1 \\ (AT + TA)x_2 = x_2, \end{array} \right. \tag{2.3}$$

Hence by (2.3) we get $span\,\{x_1, x_2\} \subset F(AT + TA)$, a contradiction.

- If Q has a single nonzero root $\lambda \in \mathbf{C}$, it follows that $Q(A) = A(A - \lambda I)$. If $dimN(A - \lambda I) = 1$, then, since $R(A) \subset N(A - \lambda I)$, we have $A \in \mathcal{F}_1(X)\backslash\mathcal{N}_1(X)$, because if $A \in \mathcal{N}_1(X)$, we obtain that $\lambda = 0$. Now, if $dimN(A - \lambda I) \geq 2$, then there exist $x_1, x_2 \in X$ linearly independent vectors such that $Ax_1 = \lambda x_1$ and $Ax_2 = \lambda x_2$. Just as before we can get a contradiction.

- If Q admits a double nonzero root $\lambda \in \mathbf{C}$, then $Q(A) = (A - \lambda I)^2$, and so there exist $x_1, x_2 \in X$ linearly independent vectors such that $Ax_1 = \lambda x_1$ and $Ax_2 = x_1 + \lambda x_2$. Let $T \in \mathcal{B}(X)$ satisfying $Tx_1 = \frac{1}{2\lambda}x_1$ and $Tx_2 = -\frac{1}{2\lambda^2}x_1 + \frac{1}{2\lambda}x_2$. Hence, we get that $span\,\{x_1, x_2\} \subset F(AT + TA)$, which is a contradiction.

- If zero is a double root of Q, hence $Q(A) = A^2$. If $dimN(A) = 1$, then, since $R(A) \subset N(A)$, we have $A \in \mathcal{N}_1(X)$. Let $A = y \otimes f$ where $y \in X$, $f \in X^*$ and $f(y) = 0$. Consider an operator $T \in \mathcal{B}(X)$ such that

(y, Ty, T^2y) are linearly independent, $f(Ty) = 1$ and $f(T^2y) = 0$. Hence we obtain that $span\{y, Ty\} \subset F(AT + TA)$, this is a contradiction. Now, if $dimN(A) \geq 2$ and $dimR(A) \geq 2$, then there exist $x_1, x_2, x_3, x_4 \in X$ linearly independent vectors such that $Ax_1 = 0, Ax_2 = 0, Ax_3 = x_1$ and $Ax_4 = x_2$. Take an operator $T \in \mathcal{B}(X)$ satisfying

$$Tx_1 = x_3, Tx_2 = x_4, Tx_3 = 0 \text{ and } Tx_4 = 0.$$

Thus we get that $span\{x_1, x_2, x_3, x_4\} \subset F(AT + TA)$, a contradiction. This ends the proof. □

3. Proof of Theorem 1

The implication $(ii) \Longrightarrow (i)$ is obvious. We only need to show that $(i) \Longrightarrow (ii)$. Let us discuss the several steps.

Step 1. For every $A \in \mathcal{B}(X)$, we have $\phi(A) = 0$ if and only if $A = 0$.

If $\phi(0) = \alpha I$, then $F(2\alpha T) = \{0\}$ for all $T \in \mathcal{B}(X)$, thus $\alpha = 0$. Assume that $\phi(0) \neq 0$. Let $x \in X$ be a nonzero vector such that $\phi(0)x$ and x are linearly independent. It follows that there is a linear functional $f \in X^*$ such that $f(x) = 0$ and $f(\phi(0)x) = 1$. Since ϕ is surjective, we take an operator $T \in \mathcal{B}(X)$ such that $\phi(T) = x \otimes f$. This implies that

$$\begin{aligned} \{0\} &= F(0T + T0) \\ &= F(\phi(0)\phi(T) + \phi(T)\phi(0)) \\ &= F(\phi(0)x \otimes f + x \otimes f\phi(0)). \end{aligned}$$

On the other hand we have

$$\begin{aligned} (\phi(0)x \otimes f + x \otimes f\phi(0))x &= f(x)\phi(0)x + f(\phi(0)x)x \\ &= x, \end{aligned}$$

and so $x \in F(\phi(0)x \otimes f + x \otimes f\phi(0))$, a contradiction. Thus $\phi(0) = 0$.

Next, assume that $\phi(A) = 0$ for certain $A \in \mathcal{B}(X)$. If $A = \beta I$, then we have $F(2\beta T) = \{0\}$ for all $T \in \mathcal{B}(X)$, hence $\beta = 0$. Suppose that $A \neq 0$, it follows that there is a nonzero vector $x \in X$ such that Ax and x are linearly independent. Then there exists $f \in X^*$ such that $f(x) = 0$ and $f(Ax) = 1$. For $T = x \otimes f$, we have

$$\begin{aligned} \{0\} &= F(\phi(x \otimes f)\phi(A) + \phi(A)\phi(x \otimes f)) \\ &= F(x \otimes fA + Ax \otimes f). \end{aligned} \tag{3.1}$$

Since

$$(x \otimes fA + Ax \otimes f)x \quad = f(x)Ax + f(Ax)x \\ = x, \tag{3.2}$$

Hence by (3.1) and (3.2) we get a contradiction. Therefore $A = 0$.

Step 2. For every operator $R \in \mathcal{B}(X)$, we have $\phi(R) \in \mathcal{F}_1(X) \backslash \mathcal{N}_1(X)$ if and only if $R \in \mathcal{F}_1(X) \backslash \mathcal{N}_1(X)$.

By using lemma 2 and the surjectivity of ϕ, we can easly get that ϕ preserves non-nilpotent rank one operators in both directions.

Step 3. There exists a nonzero scalar $\alpha \in \mathbf{C}$ with $\alpha^2 = 1$ such that $\phi(A) = \alpha A$ for every $A \in \mathcal{F}_{1,\sqrt{2}}(X)$.

Let $x \in X$ and $f \in X^*$ such that $f(x) = \frac{1}{\sqrt{2}}$. From Step 2, there exist $y \in X$ and $g \in X^*$ such that $\phi(x \otimes f) = y \otimes g$. Hence we have

$$\begin{aligned} span\{x\} &= F(\frac{2}{\sqrt{2}} x \otimes f) \\ &= F(x \otimes f x \otimes f + x \otimes f x \otimes f) \\ &= F(\phi(x \otimes f)\phi(x \otimes f) + \phi(x \otimes f)\phi(x \otimes f)) \\ &= F(y \otimes g y \otimes g + y \otimes g y \otimes g) \\ &= F(2g(y) y \otimes g). \end{aligned}$$

Thus we get that $2g(y)^2 = 1$ and $span\{x\} = span\{y\}$.

Without loss of generality, we may assume that $\phi(x \otimes f) = x \otimes g_{x,f}$ where $g_{x,f} \in X^*$.

Now, suppose that f and $g_{x,f}$ are linearly independent. Let $z \in X$ be a nonzero vector such that x and z are linearly independent with $f(z) = \frac{1}{\sqrt{2}}$ and $g_{x,f}(z) = 0$, then $x + z \in F(x \otimes f z \otimes f + z \otimes f x \otimes f)$. On the other hand we have

$$\begin{aligned} F(x \otimes f z \otimes f + z \otimes f x \otimes f) &= F(\phi(x \otimes f)\phi(z \otimes f) + \phi(z \otimes f)\phi(x \otimes f)) \\ &= F\left(x \otimes g_{x,f} z \otimes g_{z,f} + z \otimes g_{z,f} x \otimes g_{x,f}\right) \\ &= F\left(g_{z,f}(x) z \otimes g_{x,f}\right) \\ &= \{0\}, \end{aligned}$$

which is a contradiction. Hence $g_{x,f}$ and f are linearly dependent, thus $\phi(x \otimes f) = \lambda_{x,f} x \otimes f$ for some nonzero scalar $\lambda_{x,f}$. Therefore, there exists a nonzero scalar $\lambda_A \in \mathbf{C}$ such that $\phi(A) = \lambda_A A$ for all $A \in \mathcal{F}_{1,\sqrt{2}}(X)$. It follows from this that

$$\begin{aligned} F(AA + AA) &= F(\phi(A)\phi(A) + \phi(A)\phi(A)) \\ &= F\left(\lambda_A^2(AA + AA)\right). \end{aligned} \tag{3.3}$$

by (3.3) and the fact that $F(AA + AA) \neq \{0\}$, we obtain that $\lambda_A^2 = 1$. Now, let $x \in X$ be a nonzero vector and pick $f \in X^*$ such that $f(x) = \dfrac{1}{\sqrt{2}}$. For $A = x \otimes f$, we have

$$(\frac{1}{\sqrt{2}}IA + \frac{1}{\sqrt{2}}AI)x = x$$

and so

$$(\lambda_A\phi(\frac{1}{\sqrt{2}}I)A + \lambda_A A\phi(\frac{1}{\sqrt{2}}I))x = x. \tag{3.4}$$

This proves that $\phi(\frac{1}{\sqrt{2}}I)x$ and x are linearly dependent for all $x \in X$. Hence $\phi(\frac{1}{\sqrt{2}}I)$ and I are linearly dependent. Thus, from (3.4) we easily get that there exists a nonzero scalar $\alpha \in \mathbf{C}$ such that $\phi(\frac{1}{\sqrt{2}}I) = \alpha\frac{1}{\sqrt{2}}I$ and we have $\lambda_A = \alpha$. We conclude that $\phi(A) = \alpha A$ for every $A \in \mathcal{F}_{1,\sqrt{2}}(X)$ with $\alpha^2 = 1$, as desired.

Step 4. ϕ takes the desired form.

Let α be the nonzero scalar in Step 3. For every $A \in \mathcal{F}_{1,\sqrt{2}}(X)$ and $T \in \mathcal{B}(X) \setminus \mathbf{C}.I$, we have

$$\begin{aligned} F(TA + AT) &= F(\phi(T)\phi(A) + \phi(A)\phi(T)) \\ &= F\left(\alpha\phi(T)A + \alpha A\phi(T)\right). \end{aligned}$$

Lemma 1 ensures that $\phi(T) = \alpha T$ for all $T \in \mathcal{B}(X) \setminus \mathbf{C}.I$.

Now, if $T = \gamma I$ where γ is a nonzero scalar. Consider an operator $A = x \otimes f$ such that $f(x) = \frac{1}{2\gamma}$. Since A is a non-scalar operator, then $\phi(A) = \alpha A$. Just as in Step 3 when $\gamma = \frac{1}{\sqrt{2}}$, we obtain that $\phi(\gamma I) = \alpha\gamma I$.

Therefore, we conclude that $\phi(T) = \alpha T$ for every $T \in \mathcal{B}(X)$.

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