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On fuzzy congruence relation in residuated lattices

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Abstract

In this paper, we characterize some properties of fuzzy congruence relations and obtain a fuzzy congruence relation generated by a fuzzy relation in residuated lattices. For this purpose, two various types of fuzzy relations (regular and irregular) are introduced. In order to obtain a fuzzy congruence relation generated by an irregular fuzzy relation it must convert to a regular fuzzy relation.

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1. Introduction

P. Hájek [10] introduced the idea of filters and prime filters in BL-algebras. Many researchers (see[3, 4, 5, 14, 11]), introduced and characterized some types of filters in various logic algebras.

The theory of fuzzy sets which was introduced by Zadeh [19] was not considered at first until researchers realized the importance of this issue and its application in various areas of industry and Mathematics. A fuzzy relation between X and Y as a fuzzy subset of $X \times Y$ was proposed by Zadeh. V. Murali [15] introduced the concept of fuzzy equivalence relations and proved that the set of all fuzzy equivalence relations on nonempty subset X is a complete lattice. The concept of fuzzy congruence relations was studied in various areas such as semigroup, groups and vector spaces (see[1, 13, 12, 17]). Some researches (S. Ghorbani and A. Hassankhani [9], Liu Lianzhen and Li Kaitaia [14]) introduced this concept in residuated lattices and BL-algebras and proved some initial properties of fuzzy congruence relations. In some papers studies show that the relationship of fuzzy congruence relation is related to some specific sub-algebras of these structures. For example, in groups and vector spaces, the relationship between congruence relation and fuzzy congruence relation is related to normal subgroups and sub-vector spaces respectively. In this paper according to this view in residuated lattices, the fuzzy congruence relations by using filters is introduced and some properties of it is proved. In section 4, it is shown that in the theory of filters the union of filters is a filter if and only if at least one of them is contained in the other. But the important question is whether this result is also true for fuzzy congruence relations? Theorem 4 and 4 show that this result need not be true. Also we have this question, if α is a fuzzy relation, how can we construct the smallest fuzzy congruence relation that has α as fuzzy relation. In other words, if α is a fuzzy relation, what is the fuzzy congruence relation generated by α? The study of fuzzy relations shows that not every fuzzy relation can be generalized to a fuzzy congruence relation. For this purpose, the set of fuzzy relations on a residuated lattice is divided into two sets of regular and irregular fuzzy relations. Only a regular fuzzy relation can be extended to a fuzzy congruence relation. Therefore, to obtain a fuzzy congruence relation from an irregular fuzzy relation, it must first be converted to a regular fuzzy relation, as described in this paper.

2. Preliminaries

Definition 2.1. *[2, 8, 10] (i) A residuated lattice is an algebra* $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ *of type (2,2,2,2,0,0) such that*
(a) $(L, \vee, \wedge, 0, 1)$ *is a bounded lattice with the greatest element 1 and the smallest element 0,*
(b) $(L, \odot, 1)$ *is a commutative monoid,*
(c) $(\odot, \rightarrow)$ *is an adjoint couple on* L*,*
(ii) A resituated lattice L *is called an* MTL-algebra, *if it satisfies the prelinearity equation:*
$(x \rightarrow y) \vee (y \rightarrow x) = 1$ *for all* $x, y \in L$*,*
(iii) An MTL-algebra L *is called an* $IMTL$-algebra, *if* $(x \rightarrow 0) \rightarrow 0 = x$*, for all* $x \in L$*,*
(iv) An MTL-algebra L *is called a* BL-algebra *if* $x \wedge y = x \odot (x \rightarrow y)$*, for all* $x, y \in L$*.*

Proposition 2.2. *[2, 8, 10] The following properties hold for any resituated lattice((R1)-(R10)),* MTL-algebra*((R1)-(M8)) and* BL-algebra *((R1)-(B2)).*
(R1): $x \leq y \Leftrightarrow x \rightarrow y = 1$,
(R2): $1 \rightarrow x = x, x \rightarrow 1 = 1, x \rightarrow x = 1, 0 \rightarrow x = 1, x \rightarrow (y \rightarrow x) = 1$,
(R3): $x \leq y \rightarrow z \Leftrightarrow y \leq x \rightarrow z$,
(R4): $x \rightarrow (y \rightarrow z) = (x \odot y) \rightarrow z = y \rightarrow (x \rightarrow z)$,
(R5): $x \leq y$ *implies* $z \rightarrow x \leq z \rightarrow y$ *and* $y \rightarrow z \leq x \rightarrow z$,
(R6): $z \rightarrow y \leq (x \rightarrow z) \rightarrow (x \rightarrow y)$, $z \rightarrow y \leq (y \rightarrow x) \rightarrow (z \rightarrow x)$,
(R7): $(x \rightarrow y) \odot (y \rightarrow z) \leq x \rightarrow z$,
(R8): $x^* = x^{***}, x \leq x^{**}$, *when* $x^* = x \rightarrow 0$,
(B1): $(x \wedge y)^{**} = (x^{**} \wedge y^{**})$,$(x \vee y)^{**} = (x^{**} \vee y^{**})$, $(x \odot y)^{**} = (x^{**} \odot y^{**})$,
(B2): $(x^{**} \rightarrow x)^* = 0$, $(x \rightarrow y)^{**} = (x^{**} \rightarrow y^{**})$.
for any $x, y, z \in L$*.*

For any BL-algebra A, B(A) denotes the Boolean algebra of all complemented elements in L(A). Hence, B(A)= B(L(A)).

Proposition 2.3. *[7] Let* $e \in A$*. The following are equivalent:*
(i) $e \in B(A)$.
(ii) $e \odot e = e$ and $e^{**} = e$.
(iii) $e \odot e = e$ and $e^* \rightarrow e = e$.
(iv) $e \vee e^* = 1$.

Definition 2.4. *[6, 15] Let X be a non-empty set. A fuzzy relation on X is a map η from $X \times X$ to the [0, 1], and $R(X)$ will denote the set of all fuzzy relations on X.*

Definition 2.5. *[15, 16] Let $\varphi, \psi \in R(X)$. Then:*
(i) $\varphi \subseteq \psi$ if and only if $\forall x, y \in X, \varphi(x, y) \leq \psi(x, y)$.
(ii) $(\varphi \cup \psi)(x, y) = \varphi(x, y) \vee \psi(x, y)$.
(ii) $(\varphi \cap \psi)(x, y) = \varphi(x, y) \wedge \psi(x, y)$.
(iii)$\varphi^{-1}(x, y) = \varphi(y, x)$.
(v) $(\varphi \circ \psi)(x, y) = \bigvee_{z \in X}\{\varphi(x, z) \wedge \psi(z, y)\}$.

Definition 2.6. *[15] A fuzzy relation R on X is called a fuzzy equivalence or similarity relation on X if:*
(i) $R(x, x) = 1$ for all $x \in X$ (reflexive).
(ii)$R(x, y) = R(y, x)$ for all $x, y \in X$ (symmetric).
(iii) $R \circ R \leq R$ (transitive).

Definition 2.7. *[9] A fuzzy equivalence relation θ on a residuated lattice L is called a fuzzy congruence relation on L if*
(C1) $\theta(x \odot y, z \odot w) \geq \theta(x, z) \wedge \theta(y, w)$.
(C2) $\theta(x \to y, z \to w) \geq \theta(x, z) \wedge \theta(y, w)$.
(C3) $\theta(x \wedge y, z \wedge w) \geq \theta(x, z) \wedge \theta(y, w)$.
(C4) $\theta(x \vee y, z \vee w) \geq \theta(x, z) \wedge \theta(y, w)$.
for all $x, y, z, w \in L$.

Definition 2.8. *[10]. A filter of residuated lattice L is a nonempty subset F of L such that for all $a, b \in L$,we have:*
(1) $a, b \in F$ implies $a \odot b \in F$.
(2) $a \in F$ and $a \leq b$ imply $b \in F$.

Definition 2.9. *[18] A non empty subset F of residuated lattice L is a called deductive system if*
(a) $1 \in F$ and
(b) $x \in F$ and $x \to y \in F$. Then $y \in F$ for all $x, y \in L$.

Proposition 2.10. *[18] A non empty subset F of L is a deductive system if and only if F is a filter.*

3. Some results of fuzzy congruence relations

Throughout this paper, we consider L to a residuated lattice, unless otherwise stated.

Theorem 3.1. *Suppose that F is a nonempty subset of L and $t \in [0, 1)$. Then F is a filter of L if and only if*

$$\alpha(x, y) = \begin{cases} 1 & \text{if } (x \to y) \odot (y \to x) \in F \\ t & \text{otherwise} \end{cases}$$

is a fuzzy congruence relation on L.

Proof. *Suppose that F is a filter of L. Then it is clear that α is a fuzzy reflexive and symmetric. For transitivity it is sufficient to prove, if $\alpha(x, z) = \alpha(z, y) = 1$, then $\alpha(x, y) = 1$. Therefore $(x \to z) \odot (z \to x) \in F$ and $(y \to z) \odot (z \to y) \in F$, so by Definition 2.8 (2), we have $x \to z$ and $z \to y$ belong to F, on the other hand, since $z \to y \leq (x \to z) \to (x \to y)$ and F is a filter we have $x \to y \in F$. Similarly we obtain $y \to x \in F$. Thus we have $(x \to y) \odot (y \to x) \in F$ and it implies $\alpha(x, y) = 1$. Therefore α is a fuzzy equivalence relation on L so we prove $C_1 - C_4$.*
(C_1) If $\alpha(x, z) = \alpha(y, w) = 1$, then $(x \to z) \odot (z \to x) \in F$ and $(y \to w) \odot (w \to y) \in F$. Now we prove $\theta(x \odot y, z \odot w) = 1$. Since $x \odot y \to y \odot z \geq x \to z$ and $z \odot y \to x \odot y \geq z \to x$, we have $(x \odot y \to y \odot z) \odot (z \odot y \to x \odot y) \geq (x \to z) \odot (z \to x)$, hence $(x \odot y \to y \odot z) \odot (z \odot y \to x \odot y) \in F$ and it implies $\alpha(x \odot y, y \odot z) = 1$. Similarly, $\alpha(y \odot z, w \odot z) = 1$, hence by transitivity

$$\theta(x \odot y, z \odot w) \geq \alpha(x \odot y, y \odot z) \wedge \alpha(y \odot z, w \odot z) = \theta(x, z) \wedge \theta(y, w) = 1.$$

(C_2) The proof is similar to the proof of (C_1).
(C_3) Suppose that $\alpha(x, z) = \alpha(y, w) = 1$. Then by $R12$, we have

$$\begin{aligned} x \wedge y \to y \wedge z &= (x \wedge y \to y) \wedge (x \wedge y \to z) \\ &= 1 \wedge (x \wedge y \to z) \\ &= x \wedge y \to z \\ &\geq x \to z. \end{aligned}$$

Similarly, $y \wedge z \to x \wedge y \geq z \to x$*, hence* $[x \wedge y \to y \wedge z] \odot [y \wedge z \to x \wedge y] \in F$*. Therefore* $\alpha(x \wedge y, y \wedge z) = 1$*. In a similar way we have* $\alpha(y \wedge z, z \wedge w) = 1$*, hence by transitivity we have*

$$\alpha(x \wedge y, z \wedge w) = 1$$

and it completes the proof.

Conversely if α *is a fuzzy congruence relation given by the rule, then by reflexivity we have for all* $x \in L$*,* $\alpha(x,x) = 1$*, hence* $(x \to x) \odot (x \to x) = 1 \odot 1 = 1 \in F$*. If* $x, y \in F$*, then* $x = (x \to 1) \odot (1 \to x) \in F$*, this implies that* $\alpha(x,1) = 1$*, similarly we have* $\alpha(y,1) = 1$*. By Definition 2.7* (C_1)*, we have* $\alpha(x \odot y, 1) = \alpha(x \odot y, 1 \odot 1) \geq \alpha(x,1) \wedge \alpha(y,1) = 1 \wedge 1 = 1$*, hence* $\alpha(x \odot y, 1) = 1$*, and it follows that* $x \odot y \in F$*. On the other hand if* $x \in F$ *and* $x \leq y$*, then we have* $\alpha(x,1) = 1$*, and so by Definition 2.7* (C_2)*, we have* $\alpha(1,y) = \alpha(x \to y, y) = \alpha(x \to y, 1 \to y) \geq \alpha(x,1) \wedge \alpha(y,y) = 1$*, hence* $\alpha(1,y) = 1$*, and this implies that* $y = (y \to 1) \odot (1 \to y) \in F$*. Now by Definition 2.8, we have* F *is a filter.* □

Theorem 3.2. *Let* α *be a fuzzy relation on L. Then* α *is reflexive and transitive if and only if for all* $t \in [0,1]$*,* $\alpha_t = \{x \to y \in L | \alpha(x,y) \geq t\}$ *is a filter of L.*

Proof. *Let* α *be a reflexive and transitive fuzzy relation. Then for* $x \in L$ *and* $t \in [0,1]$*,* $\alpha(x,x) = 1 \geq t$ *so* $x \to x = 1 \in \alpha_t$*. If* $x \in \alpha_t$ *and* $x \to y \in \alpha_t$*, then* $\alpha(1,x) \geq t$ *and* $\alpha(x,y) \geq t$*, hence by transitivity we have* $\alpha(1,y) \geq \alpha(1,x) \wedge \alpha(x,y) \geq t$*, we have* $1 \to y = y \in \alpha_t$*. Conversely since for all* $x \in L$*,* $x \to x = 1 \in \alpha_1$*, it follows that* $\alpha(x,x) \geq 1$*, hence* $\alpha(x,x) = 1$*. Therefore* α *is reflexive. Suppose that* $\alpha(x,y) \wedge \alpha(y,z) = t$*. Then we have* $\alpha(x,y) \geq t$ *and* $\alpha(y,z) \geq t$*, so* $x \to y \in \alpha_t$ *and* $y \to z \in \alpha_t$*.* α_t *is a filter,* $(x \to y) \odot (y \to z) \leq x \to z$ *and* $x \to z \in \alpha_t$*, hence* $\alpha(x,z) \geq t = \alpha(x,y) \wedge \alpha(y,z)$*. Therefore* α *is transitive.* □

Corollary 3.3. *Let* α *be a fuzzy relation on L. Then* α *is a fuzzy equivalence (congruence) relation if and only if for all* $t \in [0,1]$*,* $\alpha_t = \{x \to y \in L | \alpha(x,y) \geq t\}$ *is a filter of L.*

Theorem 3.4. *Let* α *be a fuzzy congruence relation on L. Then:*
(i) $\alpha(x,y) = \alpha(x^*, y^*)$*, for all* $x, y \in B(L)$*.*
(ii)$\alpha^t = \{(x,y) \in L \times L | \alpha(x,y) \geq t\}$ *is a congruence relation on L.*

Proof. *(i)* $\alpha(x, y) = \alpha(x^{**}, y^{**}) = \alpha(x^* \to 0, y^* \to 0) \geq \alpha(x^*, y^*) \geq \alpha(x, y)$.
(ii) It is easy to check that α *is a equivalence relation. Let* $(x, y) \in \alpha^t$ *and* $(z, w) \in \alpha^t$. *Then* $\alpha(x, y) \geq t$ *and* $\alpha(z, w) \geq t$, *hence* $\alpha(x \to z, y \to w) \geq \alpha(x, y) \wedge \alpha(z, w) \geq t$, *consequently we have* $(x \to z, y \to w) \in \alpha^t$. *It is clear that* $I_L = \{(x, x)|x \in L\}$ *and* L *are the smallest and the biggest congruence relations in* L, *respectively.* □

Theorem 3.5. *Let* α *be a fuzzy congruence relation on* L *and* $\alpha(x, y) < \alpha(z, w)$ *such that* $z, w \in B(L)$. *Then:*
(i) $\alpha(x, y) = \alpha(x \wedge z, y \wedge w)$ *or* $\alpha(x, y) = \alpha(x \wedge z^*, y \wedge w^*)$.
(ii) $\alpha(x, y) = \alpha(x \vee z, y \vee w)$ *or* $\alpha(x, y) = \alpha(x \vee z^*, y \vee w^*)$.
(iii) $\alpha(x, y) = \alpha(x \odot z, y \odot w)$ *or* $\alpha(x, y) = \alpha(x \odot z^*, y \odot w^*)$.

Proof. *Suppose that* $\alpha(x, y) < \alpha(z, w)$ *and* $z, w \in B(L)$. *Then:*
(i): By Definition 2.7, $\alpha(x \wedge z, y \wedge w) \geq \alpha(x, y) \wedge \alpha(z, w) = \alpha(x, y)$ *(1).*

$$\begin{aligned} \alpha(x, y) = \alpha(x \wedge 1, y \wedge 1) &= \alpha(x \wedge (z \vee z^*), y \wedge (w \vee w^*)) \\ &= \alpha((x \wedge z) \vee (x \wedge z^*), (y \wedge w) \vee (y \wedge w^*)) \\ &\geq \alpha(x \wedge z, y \wedge w) \wedge \alpha(x \wedge z^*, y \wedge w^*) \end{aligned}$$

If $\alpha(x \wedge z, y \wedge w) \wedge \alpha(x \wedge z^*, y \wedge w^*) = \alpha(x \wedge z, y \wedge w)$, *then by (1) and above inequality we have* $\alpha(x, y) = \alpha(x \wedge z, y \wedge w)$. *If* $\alpha(x \wedge z, y \wedge w) \wedge \alpha(x \wedge z^*, y \wedge w^*) = \alpha(x \wedge z^*, y \wedge w^*)$, *then* $\alpha(x, y) \geq \alpha(x \wedge z^*, y \wedge w^*)$. *On the other hand since* $\alpha(z^*, w^*) \geq \alpha(z, w)$, *we have* $\alpha(x \wedge z^*, y \wedge w^*) \geq \alpha(x, y) \wedge \alpha(z^*, w^*) = \alpha(x, y)$. *Therefore* $\alpha(x, y) = \alpha(x \wedge z^*, y \wedge w^*)$.
(ii) We know $\alpha(x \vee z, y \vee w) \geq \alpha(x, y) \wedge \alpha(z, w) = \alpha(x, y)$ *(2). Also*

$$\begin{aligned} \alpha(x, y) = \alpha(x \vee 0, y \vee 0) &= \alpha(x \vee (z \odot z^*), y \vee (w \odot w^*)) \\ &= \alpha((x \vee z) \odot (x \vee z^*), (y \vee w) \odot (y \vee w^*)) \\ &\geq \alpha(x \vee z, y \vee w) \wedge \alpha(x \vee z^*, y \vee w^*) \end{aligned}$$

In above inequality, if $\alpha(x \vee z, y \vee w) \wedge \alpha(x \vee z^*, y \vee w^*) = \alpha(x \vee z, y \vee w)$, *then* $\alpha(x, y) \geq \alpha(x \vee z, y \vee w)$. *Hence by (2) we have* $\alpha(x, y) = \alpha(x \vee z, y \vee w)$. *If* $\alpha(x \vee z, y \vee w) \wedge \alpha(x \vee z^*, y \vee w^*) = \alpha(x \vee z^*, y \vee w^*)$, *then* $\alpha(x, y) \geq \alpha(x \vee z^*, y \vee w^*)$. *Similar to (i) we have* $\alpha(x \vee z^*, y \vee w^*) \geq \alpha(x, y)$. *Therefore* $\alpha(x, y) = \alpha(x \vee z^*, y \vee w^*)$ *and this completes the proof.* □

Theorem 3.6. *Let α be a fuzzy symmetric relation on L and $1 \in Img(\alpha)$. Then α is a fuzzy congruence relation on L, if*

$$(x \to y) \odot (t \to z) \leq u \to v \;\; implies \;\; \alpha(x,y) \wedge \alpha(t,z) \leq \alpha(u,v)$$

$$for\ all\ x,y,t,z,u,v \in L\ (1)$$

Proof. *If (1) holds and $x,y,z,t,u,v \in L$. Since $(x \to y) \odot (x \to y) \leq u \to u$, and so by (1) $\alpha(x,y) \wedge \alpha(x,y) = \alpha(x,y) \leq \alpha(u,u)$ this implies $\alpha(u,u) = 1$ (reflexivity). Since $(x \to y) \odot (y \to z) \leq x \to z$, we have by (1) $\alpha(x,y) \wedge \alpha(y,z) \leq \alpha(x,z)$ (transitivity). Hence α is equivalence relation. If in (1) we put $t = z$, then we have*

$$x \to y \leq u \to v \;\; implies \;\; \alpha(x,y) \leq \alpha(u,v)\ for\ all\ x,y,u,v \in L\ (1^*)$$

Now, by Definition 2.7 it is sufficient to prove $C1, C2, C3$ and $C4$. Since by $R11$, $x \odot y \to y \odot z \geq x \to z$ and $y \odot z \to w \odot z \geq y \to w$, so by (1^*) we have $\alpha(x \odot y, y \odot z) \geq \alpha(x,z)$ and $\alpha(y \odot z, w \odot z) \geq \alpha(y,w)$ and it follows that

$$\begin{aligned} \alpha(x \odot y, w \odot z) &\geq \alpha(x \odot y, y \odot z) \wedge \alpha(y \odot z, w \odot z) \\ &\quad \text{(by transitivity property)} \\ &\geq \alpha(x,z) \wedge \alpha(y,w) \end{aligned}$$

$(C2)$

Since $(x \to y) \to (z \to y) \geq z \to x$ and $(z \to y) \to (z \to w) \geq y \to w$, so by (1^*) we have $\alpha(x \to y, z \to y) \geq \alpha(z,x)$ and $\alpha(z \to y, z \to w) \geq \alpha(y,w)$. It follows that

$$\begin{aligned} \alpha(x \to y, w \to z) &\geq \alpha(x \to y, z \to y) \wedge \alpha(z \to y, z \to w) \\ &\quad \text{(by transitivity property)} \\ &\geq \alpha(x,z) \wedge \alpha(y,w) \end{aligned}$$

$(C3)$ *By $R12$, we have*

$$\begin{aligned} x \wedge y \to y \wedge z &= (x \wedge y \to y) \wedge (x \wedge y \to z) \\ &= 1 \wedge (x \wedge y \to z) \\ &= x \wedge y \to z \\ &\geq x \to z. \end{aligned}$$

Therefor by (1^*) we have $\alpha(x \wedge y, y \wedge z) \geq \alpha(x,z)$. In a similar way, we have $\alpha(y \wedge z, w \wedge z) \geq \alpha(y,w)$. Hence

$$\begin{aligned}\alpha(x\wedge y, w\wedge z) &\geq \alpha(x\wedge y, y\wedge z)\wedge\alpha(y\wedge z, w\wedge z) \quad \textit{(by transitivity property)}\\ &\geq \alpha(x,z)\wedge\alpha(y,w)\end{aligned}$$

$(C4)$ *The proof is similar to the proof of* $(C3)$. □

4. Union of fuzzy congruence relation

In this section, in Theorem 4.1 is shown that the union of two filters is a filter if and only if one of the filters is contained in the other. Does this theorem apply to fuzzy congruence relations as well? For this purpose, it is enough to answer the question of whether there are two fuzzy congruence relations, neither of which includes the other, but the union of them is a fuzzy congruence relation? This question will be answered as follows.

Theorem 4.1. *Let F_1 and F_2 be filters of L. Then $F_1 \cup F_2$ is a filter if and only if $F_1 \subseteq F_2$ or $F_2 \subseteq F_1$.*

Proof. ($\Rightarrow$) *Suppose that F_1, F_2 and $F_1\cup F_2$ are filters. If $F_1 \nsubseteq F_2$ and $F_2 \nsubseteq F_1$, then by suppose there are $a, b \in L$, such that $a \in F_1$,$a \notin F_2$ and $b \in F_2$, $b \notin F_1$. Since $F_1 \cup F_2$ is a filters, hens $a\wedge b \in F_1 \cup F_2$. Therefore $a\wedge b \in F_1$ or $a \wedge b \in F_2$. If $a\wedge b \in F_1$, then $a\wedge b \to b = 1 \in F_1$, and this implies that $b \in F_1$. This is a contradiction with $b \notin F_1$. Similarly if $a\wedge b \in F_2$, we have $a \in F_2$ and this a contradiction with $a \notin F_2$. Therefore $F_1 \subseteq F_2$ or $F_2 \subseteq F_1$.*
$\Leftarrow$ *The proof is straightforward.* □

Example 4.2. *Let $L = \{0, a, b, c, 1\}$ and operations "$\to$" and "$\odot$" on L are defined as follows:*

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	1	1	1
b	0	a	1	c	1
c	0	a	b	1	1
1	0	a	b	c	1

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	a	a	a
b	0	a	b	a	b
c	0	a	a	c	c
1	0	a	b	c	1

Let $\wedge$ and $\vee$ are defined on L by sup and inf, respectively. Then $(L, \wedge, \vee, \odot, 0, 1)$ is a residuated lattice. It is easy to check that $F_1 = \{1, b\}$

and $F_2 = \{1, c\}$ *are filters. If* $t_i \in [0,1]$, $0 \leq i \leq 2$, *such that* $1 > t_1 > t_2$, *then by Theorem 3.1, the fuzzy relations*

$$\alpha_1(x,y) = \begin{cases} 1 & \text{if } (x \to y) \odot (y \to x) \in F_1 \\ t_1 & \text{otherwise} \end{cases} \quad \text{and}$$

$$\alpha_2(x,y) = \begin{cases} 1 & \text{if } (x \to y) \odot (y \to x) \in F_2 \\ t_2 & \text{otherwise} \end{cases}$$

are fuzzy congruence relations on L, *but the union of these fuzzy congruence relations given by*

$$\alpha(x,y) = (\alpha_1 \cup \alpha_2)(x,y) = \begin{cases} 1 & \text{if } (x \to y) \odot (y \to x) \in F_1 \cup F_2 \\ t_1 & \text{otherwise} \end{cases}$$

is not a fuzzy congruence relations on L. If $b \in F_1$ *and* $c \in F_2$, *then*

$$\alpha(c \odot b, 1) = \alpha(a, 1) = t_1 \alpha(c, 1) \wedge \alpha(b, 1) = 1$$

Let α be a fuzzy congruence relation on L such that $Im\alpha = \{1, t\}$ and $0 < t < 1$. Then there exist fuzzy congruence relations α_1 and α_2 on L such that $\alpha = \alpha_1 \cup \alpha_2$, $\alpha_1 \nsubseteq \alpha_2$ and $\alpha_2 \nsubseteq \alpha_1$.

Proof. Let F_1 and F_2 be filters of L such that $F_1 \subset F_2$ and $1 > t_1 > t_2 \geq 0$. Then by Theorem 3.1, the fuzzy relation α_1 and α_2 on L given by

$$\alpha_1(x,y) = \begin{cases} 1 & \text{if } (x \to y) \odot (y \to x) \in F_1 \\ t_1 & \text{otherwise} \end{cases}$$

$$\alpha_2(x,y) = \begin{cases} 1 & \text{if } (x \to y) \odot (y \to x) \in F_2 \\ t_2 & \text{otherwise} \end{cases}$$

are fuzzy congruence relations on L. Since for all $x, y \in F_2 - F_1$, $\alpha_2(x,y) > \alpha_1(x,y)$, also for all $x, y \in L - F_2$, $\alpha_2(x,y) < \alpha_1(x,y)$. Therefore $\alpha_2(x,y) \neq \alpha_1(x,y)$.

$\alpha_1 \nsubseteq \alpha_2$ and $\alpha_2 \nsubseteq \alpha_1$. Also

$$\alpha(x,y) = (\alpha_1 \cup \alpha_2)(x,y) = \begin{cases} 1 & \text{if } (x \to y) \odot (y \to x) \in F_2 \\ t_1 & \text{otherwise} \end{cases}$$

is a fuzzy congruence relation. □

Now the question is whether theorem 4, also is valid for $|IM\alpha| \leq 3$ or not? The following Theorem responds this question.

Let α be a fuzzy congruence relation on L such that $3 \leq |Im\alpha| < \infty$. Then there exist fuzzy congruence relations α_1 and α_2 such that $\alpha = \alpha_1 \cup \alpha_2$, $\alpha_1 \nsubseteq \alpha_2$ and $\alpha_2 \nsubseteq \alpha_1$.

Proof. Let α be a fuzzy congruence relation on L and $Im\alpha = \{t_0 = 1, t_1, ..., t_n\}$, which $2 \leq n < \infty$ and $1 = t_0 > t_1 > t_2 > ... > t_n$. Then we choose $r_1, r_2 \in [0, 1]$ such that $1 = t_0 > t_1 > r_1 > t_2 > r_2 > t_3.. > t_n$

$$\alpha_1(x, y) = \begin{cases} 1 & \text{if } (x \to y) \odot (y \to x) \in \alpha_{t_0} \\ t_1 & \text{if } (x \to y) \odot (y \to x) \in \alpha_{t_1} - \alpha_{t_0} \\ r_2 & \text{if } (x \to y) \odot (y \to x) \in \alpha_{t_2} - \alpha_{t_1} \\ \alpha(x, y) & \text{otherwise} \end{cases}$$

$$\alpha_2(x, y) = \begin{cases} 1 & \text{if } (x \to y) \odot (y \to x) \in \alpha_{t_0} \\ r_1 & \text{if } (x \to y) \odot (y \to x) \in \alpha_{t_1} - \alpha_{t_0} \\ t_2 & \text{if } (x \to y) \odot (y \to x) \in \alpha_{t_2} - \alpha_{t_1} \\ \alpha(x, y) & \text{otherwise} \end{cases}$$

It is obvious that α_1 and α_2 are fuzzy congruence relations on L and $\alpha = \alpha_1 \cup \alpha_2$, but $\alpha_1 \nsubseteq \alpha_2$ and $\alpha_2 \nsubseteq \alpha_1$ □

Corollary 4.3. *Let α be a fuzzy congruence relation on L such that $2 \leq |Im\alpha| < \infty$. Then there exist fuzzy congruence relations α_1 and α_2 such that $\alpha = \alpha_1 \cup \alpha_2$, $\alpha_1 \nsubseteq \alpha_2$ and $\alpha_2 \nsubseteq \alpha_1$.*

Example 4.4. *In Example 5.3, $\langle\rho\rangle$ is a fuzzy congruence relation, we put $\langle\rho\rangle = \beta$, if $r_1 = .5, r_2 = .2$, then by Theorem 4, fuzzy congruence relations β_1 and β_2 is constructed as follows:*

$$\beta_1(x, y) = \begin{cases} 1 & \text{if } (x \to y) \odot (y \to x) \in \{1\} \\ .7 & \text{if } (x \to y) \odot (y \to x) \in \{1, b\} - \{1\} \\ .2 & \text{if } (x \to y) \odot (y \to x) \in \{1, a, b, c\} - \{1, b\} \\ \beta(x, y) & \text{if } (x \to y) \odot (y \to x) \in L - \{1, a, b, c\} \end{cases}$$

and

$$\beta_2(x, y) = \begin{cases} 1 & \text{if } (x \to y) \odot (y \to x) \in \{1\} \\ .5 & \text{if } (x \to y) \odot (y \to x) \in \{1, b\} - \{1\} \\ .3 & \text{if } (x \to y) \odot (y \to x) \in \{1, a, b, c\} - \{1, b\} \\ \beta(x, y) & \text{if } (x \to y) \odot (y \to x) \in L - \{1, a, b, c\} \end{cases}$$

It is easy to check that β_1 *and* β_2 *are fuzzy congruence relations and* $\beta = \beta_1 \cup \beta_2$, *where* $\beta_1 \nsubseteq \beta_2$ *and* $\beta_2 \nsubseteq \beta_1$.

5. Fuzzy congruence relation generated by a fuzzy relation

In this section, we construct the fuzzy congruence relation generated by a fuzzy relation in residuated lattices. For this purpose, by supposing that α is a fuzzy relation, we must consider two states: $1 \in Im\alpha$ and $1 \notin Im\alpha$. The filter generated by set X, and the fuzzy congruence relation generated by fuzzy relation α are denoted by $\langle X \rangle$ and $\langle \alpha \rangle$, respectively.

Definition 5.1. *Let* α *be a fuzzy relation on L. Then* α *is called a regular fuzzy relation if there exist fuzzy equivalence relation* ρ *such that* $\alpha \subseteq \rho$ *or for all* $(x, y) \in Dom(\alpha)$, $\alpha(x, y) = \rho(x, y)$. *In other words the fuzzy relation* α *is a regular fuzzy relation if it can be extended to a fuzzy equivalence relation. A fuzzy relation is called an irregular fuzzy relation if it is not a regular fuzzy relation.*

Example 5.2. *In Example 5.3, it is clear to check that* α *and* ρ *are regular fuzzy relations, but* β, θ *and* γ *that given by*

$$\beta(u,v) = \begin{cases} .5 & \text{if } u = v = b \text{ or } u = v = c \\ .3 & \text{if } u, v \in \{1, b\} \text{ and } u \neq v \\ .2 & \text{if } u, v \in \{b, c, 0\} \text{ and } u \neq v \end{cases}$$

$$\theta(u,v) = \begin{cases} .3 & \text{if } u = a \text{ and } v = b \\ .2 & \text{if } u = b \text{ and } v = a \\ .1 & \text{if } u, v \in \{b, c, 0\} \text{ and } u \neq v \end{cases}$$

$$\gamma(u,v) = \begin{cases} .3 & \text{if } u = a \text{ and } v = b \\ .2 & \text{if } u = b \text{ and } v = c \\ .1 & \text{if } u, v \in \{a, c, 0\} \text{ and } u \neq v \end{cases}$$

are irregular fuzzy relations. Since $\beta(a, a) = \beta(c, c) = .3 < 1$, $\theta(a, b) \neq \theta(b, a)$ *and* $\gamma(a, b) \wedge \gamma(b, c) = .2 > \gamma(a, c) = .1$, *it follows that* β, θ *and* γ *are not reflexive, symmetric and transitive respectively.*

$^{\dagger}If$ α is an irregular fuzzy relation, then we have at least one of these cases:

Case 1: α contradict reflexive property. So we must correct this part. For example

$$\beta(u,v) = \begin{cases} .5 & \text{if } u,v \in \{a,c\} \\ .3 & \text{if } u,v \in \{1,b\} \ \text{ and } u \neq v \\ .2 & \text{if } u,v \in \{b,c,0\} \ \text{ and } u \neq v \end{cases}$$

Then $\beta(a,a) = \beta(c,c) = .5$ and this contradict $\beta(a,a) = \beta(c,c) = 1$. Therefore we omit this part and we have

$$\overline{\beta}(u,v) = \begin{cases} .5 & \text{if } u,v \in \{a,c\} \text{ and } u \neq v \\ .3 & \text{if } u,v \in \{1,b\} \ \text{ and } u \neq v \\ .2 & \text{if } u,v \in \{b,c,0\} \ \text{ and } u \neq v \end{cases}$$

$\overline{\beta}$ is a regular fuzzy relation and we have $\langle \beta \rangle = \left\langle \overline{\beta} \right\rangle$.

Case 2: α contradict symmetric property. So there exist $a, b \in L$, such that $\alpha(a,b) < \alpha(b,a)$. In order to correct this part we define $\overline{\alpha}(a,b) = \alpha(b,a) = \overline{\alpha}(b,a)$. For instance in Example 5.2, we have

$$\overline{\theta}(u,v) = \begin{cases} .3 & \text{if } u = a \text{ and } v = b \\ .1 & \text{if } u,v \in \{b,c,0\} \ \text{ and } u \neq v \end{cases}$$

$\overline{\theta}$ is a regular fuzzy relation and we have $\langle \theta \rangle = \left\langle \overline{\theta} \right\rangle$.

Case 3: α contradict transitive property. So there exist $a, b, c \in L$, such that $\alpha(a,b) \wedge \alpha(b,c) > \alpha(a,c)$, in order to correct this part we define $\overline{\alpha}(a,c) = \alpha(a,b) \wedge \alpha(b,c)$. For instance in Example 5.2, we define $\overline{\gamma}(a,c) = \gamma(a,b) \wedge \gamma(b,c) = .2$, so

$$\overline{\gamma}(u,v) = \begin{cases} .3 & \text{if } u = a \text{ and } v = b \\ .2 & \text{if } (u,v) = (a,c) \ \textit{or} \ (b,c) \\ .1 & \text{if } u,v \in \{a,c,0\} \ \text{ and } (u,v) \neq (a,c) \end{cases}$$

and we have

$$\langle\overline{\gamma}\rangle(x,y)=\begin{cases}1 & \text{if } (x\to y)\odot(y\to x)\in\langle 1\rangle\\ .3 & \text{if } (x\to y)\odot(y\to x)\in\langle a,b\rangle-\langle 1\rangle=\{1,a,b,c\}-\{1\}\\ .1 & \text{if } (x\to y)\odot(y\to x)\in\langle\{1,a,b,c\},0,a,c\rangle-\langle a,b\rangle\\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad =\text{L-\{ 1,a,b,c \}}\end{cases}$$

Let α be a regular fuzzy relation on L, $Im\alpha=\{t_0,t_1,....,t_n\}$, such that $1\geq t_0>t_1>t_2....>t_n\geq 0$. Then
if $t_0=1$ (In this case if for all $\alpha(u,v)=t_0=1$, $u=v$, then $F_0=\{1\}$)

$$\langle\alpha\rangle(x,y)=\begin{cases}1 & \text{if } (x\to y)\odot(y\to x)\in F_0\\ t_i & \text{if } (x\to y)\odot(y\to x)\in F_i-F_{i-1}\quad 1\leq i\leq n\end{cases}$$

else if $t_0<1$

$$\langle\alpha\rangle(x,y)=\begin{cases}1 & \text{if } (x\to y)\odot(y\to x)\in\langle 1\rangle\\ t_0 & \text{if } (x\to y)\odot(y\to x)\in F_0-\langle 1\rangle\\ t_i & \text{if } (x\to y)\odot(y\to x)\in F_i-F_{i-1}\quad 1\leq i\leq n\end{cases}$$

With these conditions that $X_i=\{x,y\in L\mid\alpha(x,y)=t_i \text{ and } x\neq y\}$, $F_i=\langle X_i\cup F_{i-1}\rangle$ for $0\leq i\leq n$. Where $F_{-1}=\emptyset$ and $F_n=L$.

Proof. Let α be a regular fuzzy relation on L. Then it is clear that $\langle\alpha\rangle$ is reflexive and symmetric. Now we prove $\langle\alpha\rangle$ is transitive. If $\langle\alpha\rangle(x,y)=t_i$ and $\langle\alpha\rangle(y,z)=t_j$ such that $t_j\leq t_i$ $(i\leq j)$, then we have $(x\to y)\odot(y\to x)\in F_i-F_{i-1}$ and $(y\to z)\odot(z\to y)\in F_j-F_{j-1}$. Since $F_i\subseteq F_j$, we have $(x\to y)\odot(y\to x)\in F_j$ and $(y\to z)\odot(z\to y)\in F_j$, hence

$$(x\to y)\odot(y\to x)\odot(y\to z)\odot(z\to y)\in F_j$$

Therefore

$$\begin{aligned}(x\to y)\odot(y\to x)\odot(y\to z)\odot(z\to y) &= [(x\to y)\odot(y\to z)]\\ &\quad\odot[(z\to y)\odot(y\to x)]\\ &\leq (x\to z)\odot(z\to x)\end{aligned}$$

and it implies that $(x\to z)\odot(z\to x)\in F_j$. Since $\cup_{ct=0}^{t=j}(F_t-F_{t-1})=F_j$, hence there exist $k\leq J$ such that $(x\to z)\odot(z\to x)\in F_k-F_{k-1}$ and consequently $\alpha(x,z)=t_k\geq\alpha(x,y)\wedge\alpha(y,z)=t_j$. Now by Definition 2.7, it is sufficient to prove $C_1 - C_4$. The Proof of (C_1) and (C_2) are similar, so we prove (C_2). If $\langle\alpha\rangle(x,z) = t_i$ and $\langle\alpha\rangle(y,w) = t_j$ such that $t_j \leq t_i$ $(i \leq j)$, then we have $(x \to z) \odot (z \to x) \in F_i - F_{i-1}$ and $(y \to w) \odot (w \to y) \in F_j - F_{j-1}$. Since $F_i \subseteq F_j$, it follows that $(x \to z) \odot (z \to x) \in F_j$ and $(y \to w) \odot (w \to y) \in F_j$. On the other hand since $(x \to y) \to (z \to y) \geq z \to x$ and $(z \to y) \to (x \to y) \geq x \to z$, we have

$$[(x \to y) \to (z \to y)] \odot [(z \to y) \to (x \to y)] \geq (x \to z) \odot (z \to x)$$

Hence $[(x \to y) \to (z \to y)] \odot [(z \to y) \to (x \to y)] \in F_J$. Since $\cup_{ct=0}^{t=j}(F_t - F_{t-1}) = F_j$, hence there exist $(k_1 \leq J)$ such that $[(x \to y) \to (z \to y)] \odot [(z \to y) \to (x \to y)] \in F_{k_1} - F_{k_1 - 1}$, hence $\alpha(x \to y, z \to y) = t_{k_1}$. Similarly there exist $(k_2 \leq J)$ such that $\alpha(z \to y, z \to w) = t_{k_2}$. Now since $\langle\alpha\rangle$ is transitive, we have

$$\begin{aligned}
\langle\alpha\rangle(x \to y, z \to w) &\geq \langle\alpha\rangle(x \to y, z \to y) \wedge \langle\alpha\rangle(z \to y, z \to w) \\
&= \min\{t_{k_1}, t_{k_2}\} \\
&\geq t_j = \langle\alpha\rangle(x,z) \wedge \langle\alpha\rangle(y,w)
\end{aligned}$$

(C_4) Similar to (C_2) for $\langle\alpha\rangle(x,z)$ and $\langle\alpha\rangle(y,w)$, there exist t_i and t_j in $Im\rho$ such that $t_j \leq t_i$, so $F_i \subseteq F_j$ and we have $(x \to z) \odot (z \to x) \in F_j$ and $(y \to w) \odot (w \to y) \in F_j$. On the other hand since

$$\begin{aligned}
(x \vee y) \to (z \vee y) &\geq (x \to z \vee y) \wedge (y \to z \vee y) \\
&= (x \to z \vee y) \\
&\geq x \to z
\end{aligned}$$

Similarly we have $(z \vee y) \to (x \vee y) \geq z \to x$, so

$$[(x \vee y) \to (z \vee y)] \odot [(z \vee y) \to (x \vee y)] \geq (x \to z) \odot (z \to x)$$

Therefore $[(x \vee y) \to (z \vee y)] \odot [(z \vee y) \to (x \vee y)] \in F_j$, so there exist $k^{'} \leq J$ such that $[(x \vee y) \to (z \vee y)] \odot [(z \vee y) \to (x \vee y)] \in F_{k'} - F_{k'-1}$, hence $\alpha(x \vee y, z \vee y) = t_{k'}$. Similarly there exist $k^{''} \leq J$ and $\alpha(z \vee y, z \vee w) = t_{k''}$. Now since $\langle\alpha\rangle$ is transitive we have

$$\begin{aligned}
\langle\alpha\rangle(x \vee y, z \vee w) &\geq \langle\alpha\rangle(x \vee y, z \vee y) \wedge \langle\alpha\rangle(z \vee y, z \vee w) \\
&= \min\{t_{k'}, t_{k''}\} \\
&\geq t_j = \langle\alpha\rangle(x,z) \wedge \langle\alpha\rangle(y,w)
\end{aligned}$$

□

Example 5.3. *In Example 4.2, it is easy to check that* $\{1\}$, $\{1,b\}$ *and* $\{1,c\}$ *and* $\{1,a,b,c\}$ *are proper filters. If* $t_i \in [0,1]$, $1 \leq i \leq 2$, *such that* $1 \geq t_0 > t_1 > t_2$, *and fuzzy relations* α *and* ρ *are defined as follows:*

$$\alpha(u,v) = \begin{cases} t_0 = 1 & \text{if } u = v \\ t_1 & \text{if } u,v \in \{1,b\} \text{ and } u \neq v \\ t_2 & \text{if } u,v \in \{b,c,0\} \text{ and } u \neq v \end{cases}$$

$$\rho(u,v) = \begin{cases} .7 & \text{if } u,v \in \{1,b\} \text{ and } u \neq v \\ .3 & \text{if } u,v \in \{c,a\} \text{ and } u \neq v \\ .1 & \text{if } u,v \in \{0,a\} \text{ and } u \neq v \end{cases}$$

Then

$$\langle \alpha \rangle (x,y) = \begin{cases} t_0 = 1 & \text{if } (x \to y) \odot (y \to x) \in \langle 1 \rangle \\ t_1 & \text{if } (x \to y) \odot (y \to x) \in \langle 1,b \rangle - \langle 1 \rangle \\ t_2 & \text{if } (x \to y) \odot (y \to x) \in \langle 1,b,0,c \rangle - \langle 1,b \rangle = L - \{1,b\} \end{cases}$$

$$\langle \rho \rangle (x,y) = \begin{cases} 1 & \text{if } (x \to y) \odot (y \to x) \in \langle 1 \rangle \\ .7 & \text{if } (x \to y) \odot (y \to x) \in \langle 1,b \rangle - \langle 1 \rangle = \{1,b\} - \{1\} \\ .3 & \text{if } (x \to y) \odot (y \to x) \in \langle \langle 1,b \rangle, c, a \rangle - \langle 1,b \rangle \\ & \qquad = \{1,a,b,c\} - \{1,b\} \\ .1 & \text{if } (x \to y) \odot (y \to x) \in \langle \langle 1,a,b,c \rangle, 0 \rangle - \langle 1,a,b,c \rangle \\ & \qquad = L - \{1,a,b,c\} \end{cases}$$

Theorem 5.4. *In Theorem 5, if* $F_n \neq L$, *then we have if* $t_0 = 1$ *(In this case if for all* $\alpha(u,v) = t_0 = 1$, $u = v$, *then* $F_0 = \{1\}$*)*

$$\langle \alpha \rangle (x,y) = \begin{cases} 1 & \text{if } (x \to y) \odot (y \to x) \in F_0 \\ t_i & \text{if } (x \to y) \odot (y \to x) \in F_i - F_{i-1} \quad 1 \leq i \leq n \\ 0 & \text{if } (x \to y) \odot (y \to x) \in L - F_n \end{cases}$$

else if $t_0 < 1$

$$\langle \alpha \rangle (x,y) = \begin{cases} 1 & \text{if } (x \to y) \odot (y \to x) \in \langle 1 \rangle \\ t_0 & \text{if } (x \to y) \odot (y \to x) \in F_0 - \langle 1 \rangle \\ t_i & \text{if } (x \to y) \odot (y \to x) \in F_i - F_{i-1} \quad 1 \leq i \leq n \\ 0 & \text{if } (x \to y) \odot (y \to x) \in L - F_n \end{cases}$$

Proof. The proof is similar to the proof of Theorem 5. □

In the case that α is an irregular fuzzy relation, we must consider different states, for instance, we obtain the fuzzy congruence relations generated by irregular fuzzy relations β and θ in Example 5.3.

Example 5.5. *Let β and θ be fuzzy relations in Example 5.3. Then the fuzzy congruence relations generated by β and θ are denoted as follows:*

$$\langle\beta\rangle(x,y) = \begin{cases} 1 & \textit{if}(x\to y)\odot(y\to x)\in\langle 1\rangle \\ .3 & \textit{if}(x\to y)\odot(y\to x)\in\langle 1,b\rangle-\langle 1\rangle \\ .2 & \textit{if}(x\to y)\odot(y\to x)\in\langle 1,b,0,c\rangle-\langle 1,b\rangle \\ & \qquad =L\text{-}\{1,b\} \end{cases}$$

$$\langle\theta\rangle(x,y) = \begin{cases} 1 & \textit{if}\ (x\to y)\odot(y\to x)\in\langle 1\rangle \\ .3 & \textit{if}\ (x\to y)\odot(y\to x)\in\langle a,b\rangle-\langle 1\rangle=\{1,a,b,c\}-\{1\} \\ .1 & \textit{if}\ (x\to y)\odot(y\to x)\in\langle\langle 1,a,b,c\rangle,0\rangle-\langle 1,a,b,c\rangle \\ & \qquad =L\text{-}\{1,a,b,c\} \end{cases}$$

6. Conclusion

It is well known that a congruence relation is an important subject in algebraic systems. In this paper, contrary to the theory of filters, it is first proved that there are fuzzy congruence relations, none of which includes the other, but union of them is a fuzzy congruence relation. Regular and irregular fuzzy relation is introduced and by this item the fuzzy equivalence generated by a fuzzy relation is investigated.

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References

[1] J. Kim and D. Bae, “Fuzzy congruences in groups”, *Fuzzy Sets and Systems,* vol. 85, no. 1, pp. 115-120, 1997. doi: 10.1016/0165-0114(95)00334-7

[2] R. Bělohlávek, “Some properties of residuated lattices”, *Czechoslovak Mathematical Journal,* vol. 53, no. 1, pp. 161-171, 2003.

[3] R. A. Borzooei, S. Khosravi Shoar and R. Ameri, “Some types of filters in MTL-algebras”, *Fuzzy Sets and systems*, vol. 187, no. 1, pp. 92-102, 2012. doi: 10.1016/j.fss.2011.09.001

[4] D. Buşneag and D. Piciu, “A new approach for classification of filters in residuated lattices”, *Fuzzy Sets and Systems,* vol. 260, no. 1, pp. 121-130, 2015. doi: 10.1016/j.fss.2014.07.022

[5] D. Buşneag and D. Piciu, “Some types of filters in residuated lattices”, *Soft Computing,* vol. 18, pp. 825-837, 2014. doi: 10.1007/s00500-013-1184-6

[6] M. K. Chakraborty and M. Das, “Reduction of fuzzy strict order relations”, *Fuzzy Sets and Systems,* vol. 15, pp. 33-44, 1985. doi: 10.1016/0165-0114(85)90014-4

[7] A. Di Nola, “Boolean Products of BL-Algebras”, *Journal of Mathematical Analysis and Applications,* vol. 251, pp. 106-131, 2000. doi: 10.1006/jmaa.2000.7024

[8] F. Esteva and L. Godo, “Monoidal t-norm based logic Towards a logic for left-continuos t-norms”, *Fuzzy Setes and System,* vol. 124, pp. 271-288, 2001. doi: 10.1016/S0165-0114(01)00098-7

[9] S. Ghorbani and A. Hasankhani, “Fuzzy Convex Subalgebras of Commutative Residuated Lattices”, *Iranian Journal of Fuzzy Systems,* vol. 7, pp. 41-54, 2010. doi: 10.22111/IJFS.2010.171

[10] P. Hájek, *Metamathematics of Fuzzy Logic*. Dordrecht: Kluwer Academic Publisher, 1998.

[11] M. Haveshki, A. Borumand Saeid and E. Eslami, “Some type of filters in BL-algebras”, *Soft Computing,* vol. 10, pp. 657-664, 2006. doi: 10.1007/s00500-005-0534-4

[12] S. Khosravi Shoar, R. A. Borzoeei, R. Moradian, “Fuzzy congruence relation generated by a fuzzy relation in vector spaces”, *Journal of Intelligent and Fuzzy Systems,* vol. 35, pp. 5635-5645, 2018. doi: 10.3233/JIFS-17088

[13] S. Khosravi Shoar, "Fuzzy normal congruence and fuzzy coset relation on group", *International Journal of Pure and Applied Mathematics*, vol. 115, pp. 211-224, 2017. doi: 10.12732/ijpam.v115i2.2

[14] L. Lianzhen and L. Kaitaia, "Fuzzy Boolean and positive implicative filters of BL-algebras", *Fuzzy Sets and Systems*, vol. 152, pp. 333-348, 2005. doi: 10.1016/j.fss.2004.10.005

[15] V. Murali, "Fuzzy equivalence relations", *Fuzzy Sets and Systems*, vol. 30, pp. 155-163, 1989. doi: 10.1016/0165-0114(89)90077-8

[16] V. Murali, Fuzzy congruence relations, *Fuzzy Sets and Systems*, vol. 41, pp. 359-369, 1991. doi: 10.1016/0165-0114(91)90138-G

[17] M. Samhan, "Fuzzy congruences on semigroups", *Information Sciences*, vol. 74, pp. 165-175, 1993. doi: 10.1016/0020-0255(93)90132-6

[18] E. Turunen, "Boolean deductive systems of BL-algebras", *Archive for Mathematical Logic*, vol. 40, pp. 467-473, 2001. doi: 10.1007/s001530100088

[19] L. A. Zadeh, "Fuzzy sets", *Information and Control*, vol. 8, pp. 338-353, 1965. doi: 10.1016/S0019-9958(65)90241-X

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