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Bounds for absolute values and imaginary parts of matrix eigenvalues via traces

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Abstract

Let $\lambda_1(A), \lambda_2(A), ..., \lambda_n(A)$ be the eigenvalues of an $n \times n$-matrix A taken with their algebraic multiplicities. We suggest new bounds for $|\lambda_j(A) - \frac{\text{trace}(A)}{n}|$ and $|\text{Im } \lambda_j(A) - \frac{\text{Im trace}(A)}{n}|$ $(j = 1, ..., n)$, which refine the previously published results.

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1. Introduction and statement of the main result

Let $\mathbf{C}^{n\times n}$ be the set of all $n\times n$-complex matrices. For an $A\in\mathbf{C}^{n\times n}$, A^* is the adjoint matrix, $\lambda_1(A),\lambda_2(A),...,\lambda_n(A)$ are the eigenvalues of A repeated according to their algebraic multiplicities, and $|A|_F=(\text{trace}\,AA^*)^{1/2}$ is the Frobenius norm.

In the paper [14] it has been shown that

$$|\lambda_j(A)-\frac{\text{trace}\,A}{n}|^2\le\frac{n-1}{n}(|A|_F^2-\frac{|\text{trace}\,A|^2}{n})\ \ (j=1,...,n).$$

In [13] that inequality has been refined. The refinement asserts that

$$|\lambda_j(A)-\frac{\text{trace}\,A}{n}|^2\le\frac{n-1}{n}(|A|_F^2-\frac{|A^*A-AA^*|_F^2}{6|A|_F^2}-\frac{|\text{trace}\,A|^2}{n})\ \ (j=1,...,n). \tag{1.1}$$

This inequality has caused a great interest of many mathematicians [2, 8, 9, 10, 12, 17]. One of the best results connected with inequality (1.1) has been established in [8]. Namely, in that paper it is proved that

$$|\lambda_j(A)-\frac{\text{trace}\,A}{n}|^2\le\frac{n-1}{n}(|A|_F^2-\frac{|A^*A-AA^*|_F^2}{4|A|_F^2}-\frac{|\text{trace}\,A|^2}{n})\ \ (j=1,...,n). \tag{1.2}$$

In addition, in [13, Theorem 6] the following inequalities have been proved:

$$|\text{Im}\,\lambda_j(A)-\frac{\text{Im trace}\,A}{n}|^2\le\frac{n-1}{n}(|A_I|_F^2-\frac{1}{n}(\text{Im trace}\,A)^2-\frac{|A^*A-AA^*|_F^2}{12|A|_F^2}) \tag{1.3}$$

and

$$|\text{Re}\,\lambda_j(A)-\frac{\text{Re trace}\,A}{n}|^2\le\frac{n-1}{n}(|A_R|_F^2-\frac{1}{n}(\text{Re trace}\,A)^2-\frac{|A^*A-AA^*|_F^2}{12|A|_F^2})$$

$$(j=1,...,n), \tag{1.4}$$

where $A_I=(A-A^*)/2i$ and $A_R=(A+A^*)/2$.

In the present paper we suggest new estimates for the eigenvalues, which refine inequalities (1.2), (1.3) and (1.4). To this end put $|A|_1=$

$\operatorname{trace}(AA^*)^{1/2}$. Recall that for any matrix $T \in \mathbf{C}^{n\times n}$ whose entries in an orthogonal normal basis are t_{jk} $(j,k=1,...,n)$ we have

$$\sum_{k=1}^{m} |\lambda_k(T)| \geq \sum_{k=1}^{m} |t_{kk}| \ \ (m=1,...,n),$$

cf. [7, Section II.4.3]. Since the self-commutator $A^*A - AA^*$ of A is a Hermitian matrix, we can write

$$|A^*A - AA^*|_1 \geq \sum_{k=1}^{m} |c_{kk}| \ \ (m=1,...,n),$$

where $c_{jj}, j=1,...,n$ are the diagonal entries of the commutator in an orthogonal normal basis. Now we are in a position to formulate the main result of the paper.

Theorem 1.1. *For any $A \in \mathbf{C}^{n\times n}$ and all $j=1,...,n$, the inequalities*

$$|\lambda_j(A) - \frac{\operatorname{trace} A}{n}|^2 \leq \frac{n-1}{n}\left[(|A|_F^4 - \frac{1}{4}|A^*A - AA^*|_1^2)^{1/2} - \frac{1}{n}|\operatorname{trace} A|^2\right] \tag{1.5}$$

and

$$|\operatorname{Im}\lambda_j(A) - \frac{1}{n}\operatorname{Im}\operatorname{trace} A|^2$$
$$\leq \frac{n-1}{n}\left[\frac{1}{2}(|A|_F^4 - \frac{1}{4}|AA^* - A^*A|_1^2)^{1/2} + |A_I|_F^2 - \frac{1}{2}|A|_F^2 - \frac{1}{n}(\operatorname{Im}\operatorname{trace} A)^2\right] \tag{1.6}$$

are valid.

Due to Lemma 2.1 from [5] for any Hermitian matrix S with $\operatorname{trace} S = 0$ we have $|S|_1 \geq \sqrt{2}|S|_F$. Thus,

$$|A^*A - AA^*|_1 \geq \sqrt{2}|A^*A - AA^*|_F. \tag{1.7}$$

Theorem 1.1 and (1.7) imply

Corollary 1.2. *For any $A \in \mathbf{C}^{n\times n}$ and all $j=1,...,n$, the inequalities*

$$|\lambda_j(A) - \frac{\operatorname{trace} A}{n}|^2 \leq \frac{n-1}{n}\left[(|A|_F^4 - \frac{1}{2}|A^*A - AA^*|_F^2)^{1/2} - \frac{1}{n}|\operatorname{trace} A|^2\right] \tag{1.8}$$

and

$$|\operatorname{Im}\lambda_j(A) - \frac{\operatorname{Im}\operatorname{trace} A}{n}|^2$$
$$\leq \frac{n-1}{n}\left[\frac{1}{2}(|A|_F^4 - \frac{1}{2}|AA^* - A^*A|_F^2)^{1/2} + |A_I|_F^2 - \frac{1}{2}|A|_F^2 - \frac{1}{n}(\operatorname{Im}\operatorname{trace} A)^2\right] \tag{1.9}$$

are valid.

Furthermore, replacing matrix A by iA ($i^2 = -1$) in Theorem 1.1 and Corollary 1.2, we obtain

$$|\mathrm{Re}\ \lambda_j(A) - \frac{\mathrm{Re\ trace}\, A}{n}|^2$$

$$\leq \frac{n-1}{n}\left[\frac{1}{2}(|A|_F^4 - \frac{1}{4}|AA^* - A^*A|_1^2)^{1/2} + |A_R|_F^2 - \frac{1}{2}|A|_F^2 - \frac{1}{n}(\mathrm{Re\ trace}\, A)^2\right]$$

$$\leq \frac{n-1}{n}\left[\frac{1}{2}(|A|_F^4 - \frac{1}{2}|AA^* - A^*A|_F^2)^{1/2} + |A_R|_F^2 - \frac{1}{2}|A|_F^2 - \frac{1}{n}(\mathrm{Re\ trace}\, A)^2\right] \tag{1.10}$$

($j = 1, ..., n$). Below we check that

$$|A|_F^2 - \frac{|A^*A - AA^*|_F^2}{4|A|_F^2} \geq (|A|_F^4 - \frac{1}{2}|A^*A - AA^*|_F^2)^{1/2} \tag{1.11}$$

and therefore (1.8) and (1.5) refine (1.2). In addition, from (1.11) it follows that

$$|A|_F^2 - \frac{|A^*A - AA^*|_F^2}{6|A|_F^2} \geq (|A|_F^4 - \frac{1}{2}|AA^* - A^*A|_F^2)^{1/2} \tag{1.12}$$

and therefore inequalities (1.9) and (1.6) refine (1.3). Similarly, one can show that (1.10) improves inequality (1.4).

In the next section we also check that

$$|A_I|_F^2 = \frac{1}{2}|A|_F^2 - \frac{1}{2}\mathrm{Re\ trace}\, A^2 \tag{1.13}$$

and

$$|A_R|_F^2 = \frac{1}{2}|A|_F^2 - \frac{1}{2}\mathrm{Im\ trace}\, A^2. \tag{1.13}$$

Now (1.9) and (1.10) yield the inequalities

$$|\mathrm{Im}\ \lambda_j - \frac{\mathrm{Im\ trace}\, A}{n}|^2$$

$$\leq \frac{n-1}{n}\left[\frac{1}{2}(|A|_F^4 - \frac{1}{2}|AA^* - A^*A|_F^2)^{1/2} - \frac{1}{2}\mathrm{Re\ trace}\, A^2 - \frac{1}{n}(\mathrm{Im\ trace}\, A)^2\right] \tag{1.14}$$

and

$$|\mathrm{Re}\ \lambda_j - \frac{\mathrm{Im\ trace}\, A}{n}|^2$$

$$\leq \frac{n-1}{n}\left[\frac{1}{2}(|A|_F^4 - \frac{1}{2}|AA^* - A^*A|_F^2)^{1/2} - \frac{1}{2}\mathrm{Im\ trace}\, A^2 - \frac{1}{n}(\mathrm{Re\ trace}\, A)^2)\right] \tag{1.15}$$

($j = 1, ..., n$).

2. Proofs

Throughout the rest of the paper for the simplicity we put $\lambda_j(A) = \lambda_j$.

Proof of Theorem 1.1: due to Theorem 2.2 from [14]

$$(2.1)\quad |\lambda_j - \frac{\operatorname{trace} A}{n}|^2 \le \frac{n-1}{n}(\sum_{k=1}^{n} |\lambda_k|^2 - \frac{|\operatorname{trace} A|^2}{n}) \ \ (j = 1, ..., n).$$

According to Theorem 1.1 [4] (see also the book [6, Section 8.1]) we have

$$\sum_{k=1}^{n} |\lambda_k|^2 \le (|A|_F^4 - \frac{1}{4}|A^*A - AA^*|_1^2)^{1/2}.$$

Now (2.1) yields (1.5). To prove (1.6) we need the inequality

$$|\operatorname{Im} \lambda_j - \frac{\operatorname{Im} \operatorname{trace} A}{n}|^2 \le \frac{n-1}{n}(\sum_{k=1}^{n} (\operatorname{Im} \lambda_k)^2 - \frac{1}{n}(\operatorname{Im} \operatorname{trace} A)^2) \ \ (j = 1, ..., n),$$
(2.2)

cf. [13, Inequality (7)]. By Theorem 1.2 from [4], we have

$$\sum_{k=1}^{n} (\operatorname{Im} \lambda_k)^2 \le |A_I|_F^2 - \frac{1}{2}|A|_F^2 + \frac{1}{2}(|A|_F^4 - \frac{1}{4}|AA^* - A^*A|_1^2)^{1/2}.$$

Now (2.2) implies the inequality

$$|\operatorname{Im} \lambda_j - \frac{\operatorname{Im} \operatorname{trace} A}{n}|^2 \le \frac{n-1}{n}(|A_I|_F^2 - \frac{1}{2}|A|_F^2 + \frac{1}{2}(|A|_F^4 - \frac{1}{4}|AA^* - A^*A|_1^2)^{1/2}$$
$$- \frac{1}{n}(\operatorname{Im} \operatorname{trace} A)^2) \ \ (j = 1, ..., n).$$

So (1.6) is also valid, as claimed. □

Proofs of inequality (1.11): Inequality (1.11) is due to the following relations:

$$(|A|_F^2 - \frac{|A^*A - AA^*|_F^2}{4|A|_F^2})^2 = |A|_F^4 - \frac{|A^*A - AA^*|_F^2}{2} + \frac{|A^*A - AA^*|_F^4}{16|A|_F^4}$$
$$\ge |A|_F^4 - \frac{1}{2}|A^*A - AA^*|_F^2.$$

□

Proof of equalities (1.13) and (1.14): Obviously,

$$|A_I|_F^2 = \frac{1}{4}|A - A^*|_F^2 = -\frac{1}{4}\text{trace}\,(A - A^*)^2$$

$$= \frac{1}{2}\text{trace}\,(AA^*) - \frac{1}{4}\text{trace}\,(A^2) - \frac{1}{4}\text{trace}\,(A^*)^2.$$

Or

$$|A_I|_F^2 = \frac{1}{2}|A|_F^2 - \frac{1}{4}\text{trace}\,(A^2) - \frac{1}{4}\text{trace}\,(A^*)^2.$$

Let $a_k = \text{Re}\,\lambda_k, b_k = \text{Im}\,\lambda_k$. Then

$$\text{trace}\,(A^2 + (A^*)^2) = \sum_{k=1}^{n}(a_k + ib_k)^2 + (a_k - ib_k)^2 = 2\sum_{k=1}^{n} a_k^2 - b_k^2.$$

Take into account that

$$\text{Re trace}\,A^2 = \sum_{k=1}^{\infty} a_k^2 - b_k^2.$$

Thus

$$\text{trace}\,(A^2 + (A^*)^2) = 2\text{Re trace}\,A^2$$

and

$$|A_I|_F^2 = \frac{1}{2}|A|_F^2 - \frac{1}{2}\text{Re trace}\,A^2.$$

So (1.13) is valid.

Similarly (1.14) can be proved.□

3. Additional inequalities and an example

Making use of [5, Theorem 1.1], we can write

$$\sum_{k=1}^{n}(\text{Im}\,\lambda_k)^2 \le |A_I|_F^2 - \frac{1}{8}\left(\text{spread}(A) - \sqrt{\text{spread}^2(A) + 2\sqrt{2}|AA^* - A^*A|_F}\right)^2, \tag{3.1}$$

where

$$\text{spread}(A) := \sup_{j,k=1,\dots,n;\ j\neq k} |\lambda_j - \lambda_k|.$$

That result refines the classical inequality

$$\sum_{k=1}^{\infty}(\text{Im } \lambda_k)^2 \leq |A_I|_F^2.$$

Combining (3.1) with (2.2), we arrive at the inequality

$$|\text{Im } \lambda_j - \frac{\text{Im trace } A}{n}|^2 \leq$$

$$\frac{n-1}{n}(|A_I|_F^2 - \frac{1}{8}\left(\text{spread}(A) - \sqrt{\text{spread}^2(A) + 2\sqrt{2}|AA^* - A^*A|_F}\right)^2$$

$$- \frac{1}{n}(\text{Im trace } A)^2) \tag{3.2}$$

for all $j = 1, ..., n$.

Note that the literature on the bounds for the spread is rather rich, cf. [1, 3, 11, 15, 16, 18, 19].

Example 3.1. *Consider the matrix*

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Simple calculations show that $|A^*A - AA^*|_F^2 = 14$, trace $A = 3$ and $|A|_F^2 = 8$. Now (1.8) yields $|\lambda_j - 1| \leq 1.746$ $(j = 1, 2, 3)$.

In addition, $|A_I|_F^2 = 2.5$ and Im trace $A = 0$. Making use of (1.9), we obtain

$$|\text{Im } \lambda_j(A)| \leq 1.236.$$

The eigenvalues of A are $\lambda_1 = 2$ and $\lambda_{2,3} = \frac{1}{2}(1 \pm i\sqrt{3})$ $(i^2 = -1)$. I.e. $|\lambda_j - 1| = 1$ $(j = 1, 2, 3)$, and

$$|\text{Im } \lambda_j(A)| = \frac{1}{2}\sqrt{3} \approx 0.866. \quad (j = 2, 3).$$

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