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Existence of weak solutions for some quasilinear degenerated elliptic systems in weighted Sobolev spaces

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Abstract

We consider, for a bounded open domain Ω in R^n; $(n \geq 1)$ and a function $u : \Omega \to I\!R^m$; $(m \geq 1)$ the quasilinear elliptic system:

$$(QESw)_{(f,g)} \begin{cases} -div\sigma\left(x, u\left(x\right), Du\left(x\right)\right) & = & v(x) + f(x,u) + divg(x,u) \ in \Omega \\ u & = & 0 \ on \ \partial\Omega, \end{cases}$$

(0.1)

Which is a Dirichlet problem. Here, v belongs to the dual space $W^{-1,p'}(\Omega, \omega^*, R^m)$, $\left(\frac{1}{p} + \frac{1}{p'} = 1, \ p > 1\right)$, f and g satisfy some standard continuity and growth conditions. we will show the existence of a weak solution of this problem in the four following cases: σ is monotonic, σ is strictly monotonic, σ is quasi montone and σ derives from a convex potential.

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1. Introduction

In this paper, the main point is that we not require monotonicity in the or strict monotonicity of a typical Leray-Lions operators as it is usually assumed in previous papers. The aims of this paper is to prove analogous existence results under relaxed monotonicity, in particular under strict quasi-monotonicity. The main technical tool we handle and use throughout the proof are Young measures. By applying a Galerkin schema, we obtain easily an approximating sequence u_k. The Ball theorem [1] and especially the resulting tools mode available by Hungerbühler to partial differential equation theory give then a sufficient control on the gradient approximating sequence Du_k to pass to the limit.This method is used by Dolzmann [4], Muller [8], L. Boccardo; F. Murat [11], M. Candela [12] and Mainly by Hungurbühler to get the existence of a weak solution for the quasilinear elliptic system [14]. This paper can be seen as generalization of Hungerbühler and as a continuation of Y- Akdim [17]and [18]. we also based on all the following references [3], [5], [6], [7], [9], [10] and [13].

2. Preliminary

Let $\omega = \{\omega_{ij} : \quad 0 \leq i \leq n; \quad 1 \leq j \leq m\}$ and $\omega_0 = (\omega_{0j})_{1 \leq j \leq m}$. weight function systems defined in Ω and satisfying the following integrability conditions:

$$\omega_{ij} \in L^1_{loc}(\Omega), \quad \omega_{ij}^{\frac{-1}{p-1}} \in L^1_{loc}(\Omega), \text{ for some } p \in]1,\infty[\text{ and } \exists s > Max(\tfrac{n}{p}, \tfrac{1}{p-1}) \text{ such that } \omega_{ij}^{-s} \in L^1(\Omega). \tag{2.1}$$

with $\omega^* = \{\omega^*_{ij} = \omega_{ij}^{1-p'} \quad : \quad 0 \leq i \leq n, \quad 1 \leq j \leq m\}$, $\sigma = (\sigma_{rs})$ with $1 \leq s \leq n, \quad 1 \leq r \leq m$ and which satisfies some hypotheses (see below).
We denote by $\mathbb{M}^{m \times n}$ the real vector space of $m \times n$ matrices equipped with the inner product $M : N = \sum_{ij} M_{ij} N_{ij}$.
The Jacobian matrix of a function $u : \Omega \longrightarrow \mathbb{R}^m$ is denoted by $Du(x) = (D_1u(x), D_2u(x), \ldots.., D_nu(x))$ with $D_i = \partial/\partial(x_i)$.

The space $W^{1,p}(\Omega, \omega, \mathbb{R}^m)$ is the set of functions

$$\{u = u(x) | \ u \in L^p(\Omega, \overline{\omega_0}, \mathbb{R}^m)\}, \ D_{ij}u = \frac{\partial u^i}{\partial x_j} \in L^p(\Omega, \omega_{ij}, \mathbb{R}^m),$$

$$1 \leq i \leq n, \ 1 \leq j \leq m\},$$

with

$$L^p(\Omega, \omega_{ij}, I\!R^m) = \{u = u(x)| \quad | u | \omega_{ij}{}^{\frac{1}{p}} \in L^p(\Omega, I\!R^m)\}.$$

The weighted space $W^{1,p}(\Omega, \omega, I\!R^m)$ can be equipped by the norm :

$$\|u\|_{1,p,\omega} = \left(\sum_{j=1}^{m} \int_{\Omega} |u_j|^p \omega_{0j} dx + \sum_{1\le i\le n, 1\le j\le m} \int_{\Omega} |D_{ij} u|^p \omega_{ij} dx \right)^{\frac{1}{p}},$$

the norm $\|.\|_{1,\omega,p}$ is equivalent to the norm $|||\ .\ |||$, on $W_0^{1,p}(\Omega, \omega, I\!R^m)$, such that, $|||\ u\ |||= (\sum_{1\le i\le n, 1\le j\le m} \int_{\Omega} |D_{ij} u|^p \omega_{ij} dx)^{\frac{1}{p}}$

Proposition 2.1. *The weighted Sobolev space $W^{1,p}(\Omega, \omega, I\!R^m)$ is a Banach space, separable and reflexive. The weighted Sobolev space $W_0^{1,p}(\Omega, \omega, I\!R^m)$ is the closure of $C_0^{\infty}(\Omega, \omega, I\!R^m)$ in $W^{1,p}(\Omega, \omega, I\!R^m)$ equipped by the norm $\|.\|_{1,p,\omega}$.*

Proof: The prove of proposition is a slight modification of the analogous one in [15] [Kufner-Drabek].

Definition 2.1. *A Young measure $(\vartheta_x)_{x\in\Omega}$ is called $W^{1,p}$-gradient young measures ($1 \le p < \infty$) if it is associated to a sequence of gradients Du_k such that u_k is bounded in $W^{1,p}(\Omega)$.*
The $W^{1,p}$-gradient young measures $(\vartheta_x)_{x\in\Omega}$ is called homogeneous, if it doesn't depend on x, i-e, if $\vartheta_x = \vartheta$ for a.e. $x \in \Omega$.

Theorem 2.1. *(Kinderlehrer-Pedregal) Let $(\upsilon_x)_{x\in\Omega}$, be a family of probability measures in $(C(M^{m\times n}))'$, then $(\upsilon_x)_{x\in\Omega}$ are $W^{1,p}$ Young measures if and only if:*
(i)There is a $u \in W^{1,p}(\Omega, I\!R^m)$ such that $Du(x) = \int_{M^{m\times n}} A d\vartheta_x(A)$, a.e in Ω.
(ii)Jensen's inequality: $\phi(Du(x)) \le \int_{M^{m\times n}} \phi(A) d\vartheta_x(A)$ hold for all $\phi \in X^p$ quasi-convex, and
(iii)The function: $\psi(x) = \int_{M^{m\times n}} |A|^p d\vartheta_x(A) \in L^1(\Omega)$. here, X^p denotes the (not separable) space : $X^p = \{\psi \in C(M^{m\times n}) : |\psi(A)| \le c \times (1 + |A|^p)$, for all $A \in M^{m\times n}\}$.

See [14].

Theorem 2.2. *(Ball) Let $\Omega \subset I\!R^n$ be Lebesgue measurable, let $K \subset I\!R^m$ be closed, and let $u_j : \Omega \to I\!R^m, j \in I\!N$, be a sequence of Lebesgue measurable functions satisfying*
$u_j \to K$, as $j \to \infty$, i-e. given any open neighborhood U of $K \in I\!R^m$
$\lim_{j\to\infty} |x \in \Omega : u_j(x) \in U| = 0$. Then there exist a subsequence u_k of u_j and a family $\vartheta_x, x \in \Omega$, of positive measures on $I\!R^m$, depending measurably on x, such that
(i) $\|\vartheta_x\|_M = \int_{I\!R^m} d\vartheta_x \leq 1$, for a.e $x \in \Omega$.
(ii)$Supp\vartheta_x \subset K$ for a.e. $x \in \Omega$.
(iii) $f(u_k) \rightharpoonup^* \langle \vartheta_x, f\rangle = \int_{I\!R^m} f(\lambda)d\vartheta_x(\lambda)$ in $L^\infty(\Omega)$, for each continuous functions $f : I\!R^m \to I\!R$ satisfying
$\lim f(\lambda) = 0, |\lambda| \to \infty$ [1].

Theorem 2.3. *(Vitali) Let $\Omega \in I\!R^n$ be an open bounded domain and let u_n be a sequence in $L^p(\Omega, I\!R^m)$ with $1 \leq p < \infty$,*
then u_n is a Cauchy sequence in the L^p- norm if and only if the two following conditions holds:
(i) u_n is Cauchy in measure (i-e.: $\forall\varepsilon > 0, | \{x \in \Omega | u_n(x) - u_m(x)| \geq \varepsilon\} |= 0$ as $m, n \to \infty$.
(ii) $(|u_n|^p)$ is equi-integrable i-e :
($sup_n \int_\Omega |u_n|^p dx < \infty$ and $\forall\varepsilon > 0, \exists\delta > 0$ such that $\int_E |u_n|^p dx < \varepsilon$ for all n whenever $E \subset \Omega$ and $|E| < \delta$.) Note that if u_n converges pointwise, then u_n is cauchy in measure.

Hypotheses (H_0) (Hardy inequality): There exist a constant $c > 0$, a weighted function γ and a real q $(1 < q < \infty)$ such that,

$$\left(\sum_{j=1}^{m} \int_\Omega |u_j(x)|^q \gamma_j(x)dx\right)^{\frac{1}{q}} \leq c \left(\sum_{1\leq i\leq n;\ 1\leq j\leq m} \int_\Omega |D_{ij}u|^p \omega_{ij}\right)^{\frac{1}{p}},$$

for all $u \in W_0^{1,p}(\Omega, \omega, I\!R^m)$, with $\gamma = \{\gamma_j / 1 \leq j \leq m\}$.
The injection $W_0^{1,p}(\Omega, \omega, I\!R^m) \hookrightarrow\hookrightarrow L^q(\Omega, \gamma, I\!R^m)$ is compact.
By the conditions (2.1) we have $W_0^{1,p}(\Omega, \omega, I\!R^m) \hookrightarrow W^{1,p_s}(\Omega, I\!R^m)$, with $(p_s = \frac{ps}{s+1})$, and $W_0^{1,p}(\Omega, \omega, I\!R^m) \hookrightarrow\hookrightarrow L^r(\Omega, I\!R^m)$ is compact, (by [15]) with

$$\begin{cases} 1 \leq r \prec \frac{nps}{n(s+1)-ps} & \text{if} \quad ps \prec n(s+1) \\ \qquad r \geq 1 & \text{if} \quad n(s+1) \prec ps \end{cases}$$

(H_1) Continuity: $\sigma : \Omega \times I\!R^m \times I\!M^{m\times n} \longrightarrow I\!M^{m\times n}$ is a Carathéodory function (i-e $x \longmapsto \sigma(x,u,F)$ is measurable for every $(u,F) \in I\!R^m \times I\!M^{m\times n}$ and $(u,F) \longmapsto \sigma(x,u,F)$ is continuous for almost every $x \in \Omega$).
(H_2) There exist $c_1 \geq 0, c_2 > 0$ $\lambda_1 \in L^{p'}(\Omega)$, $\lambda_2 \in L^1(\Omega)$, $\lambda_3 \in L^{(p/\alpha)'}(\Omega)$, $0 < \alpha < p$, $1 < q < \infty$ and $\beta > 0$ such that for all $1 \leq r \leq n$; $\quad 1 \leq s \leq m$, we have the Growths conditions:

$$|\sigma_{rs}(x,u,F)| \leq \beta \omega_{rs}^{1/p}[\lambda_1(x) + c_1 \sum_{j=1}^{m} |\gamma_j|^{1/p'} \cdot |u_j|^{q/p'} + c_1 \sum_{1\leq i\leq n;\ 1\leq j\leq m} \omega_{ij}^{1/p'} |F_{ij}|^{p-1}]$$

(2.2)
and coercivity conditions:

$$\sigma(x,u,F) : F \geq -\lambda_2(x) - \sum_{j=1}^{m} \omega_{0j}(x)^{\alpha/p} \lambda_3(x)|u_j|^{\alpha} + c_2 \sum_{1\leq i\leq n;\ 1\leq j\leq m} \omega_{ij}(x) \cdot |F_{ij}|^p$$

(2.3)
(H_3) Monotonicity conditions: σ satisfies one of the following conditions:

a) For all $x \in \Omega$, and all $u \in I\!R^m$, the map $F \longmapsto \sigma(x,u,F)$ is a C^1-function and is monotone(i-e, $(\sigma(x,u,F) - \sigma(x,u,G)) : (F - G) \geq 0$, for all $x \in \Omega$, all $u \in I\!R^m$ and all $F, G \in I\!M^{m\times n}$).

b) There exists a function $W : \Omega \times I\!R^m \times I\!M^{m\times n} \longrightarrow I\!M^{m\times n}$ such that $\sigma(x,u,F) = \dfrac{\partial W}{\partial F}(x,u,F)$ and $F \longmapsto W(x,u,F)$ is convex and C^1 function.

c) For all $x \in \Omega$, and for all $u \in I\!R^m$ the map $F \longmapsto \sigma(x,u,F)$ is strictly monotone (i.e, $\sigma(x,u,.)$ is monotone and : $[(\sigma(x,u,F) - \sigma(x,u,G)) : (F - G) = 0] \Longrightarrow F = G$).

d) $\sigma(x,u,F)$ is strictly p-quasi-monotone in F, i.e,

$$\int_{I\!M^{m\times n}} (\sigma(x,u,\lambda) - \sigma(x,u,\overline{\lambda})) : (\lambda - \overline{\lambda}))d\vartheta(\lambda) > 0,$$

for all homogeneous $W^{1,p}$-gradient young measures ϑ with center of mass $\overline{\lambda} = \langle \vartheta, id \rangle$ which are not a single Dirac mass.
The main point is that we do not require strict monotonicity or monotonicity in the variables (u,F) in (H_3) as it is usually assumed in previous work see ([2] or [16]). (F_0): (continuity) $f : \Omega \times I\!R^m \to I\!R^m$

is a Carathéodory function i-e: $x \mapsto f(x,u)$ is measurable for every $u \in I\!R^m$, and, $u \mapsto f(x,u)$ is continuous for almost every $x \in \Omega$.
(F_1): (growth condition): There exist : $b_1 \in L^{p'}(\Omega)$ such that :

$$|f_j(x,u)| \leq [b_1(x) + \gamma_j^{\frac{1}{p'}} |u_j|^{\frac{q}{p'}}]\omega_{0j}^{\frac{1}{P}}; \quad \forall j = 1; ...; m.$$

(G_0): (continuity) the map $g : \Omega \times I\!R^m \to I\!M^{m\times n}$ is a Carathéodory function.
(G_1): (growth condition) there exist : $b_2 \in L^{p'}(\Omega)$

$$|g_{rs}| \leq \omega_{rs}^{\frac{1}{p}}[b_2 + \sum_j \gamma_j^{\frac{1}{p'}} |u_j|^{\frac{q}{p'}}]$$

For all $1 \leq r \leq n$ and $1 \leq s \leq m$.
Under the previous hypotheses $H_0, H_1, H_2, F_0, F_1, G_0, G_1$ and in each condition of $H_3; a, b, c$ and d we will demonstrate the existence of a weak solution of system $(QESw)_{f,g}$ in the space $W_0^{1,P}(\Omega, \omega, I\!R^m)$.

Remark 2.1. *The conditions (F_0) and (G_0) ensure the measurability of f and g for all measurable function u.*
(F_1) and (G_1) ensure that growths conditions, in particularly: if $u \in W_0^{1,P}(\Omega, \omega, I\!R^m)$ then $f(.,u).u$ and $g(.,u) : Du$ is in $L^1(\Omega, \omega)$.
If $g = 0$ we denote the system $(QES)_{f,g}$ by $(QES)_f$.

Theorem 2.4. *If $p \in (1, \infty)$ and σ satisfies the conditions $(H_1) - (H_3)$, f satisfies (F_0) and (F_1) and g satisfies (G_0) and (G_1), then the Dirichlet problem $(QESW)_{f,g}$ has a weak solution $u \in W_0^{1,p}(\Omega, \omega, I\!R^m)$, for every $v \in W^{-1,p'}(\Omega, \omega^*, I\!R^m)$.*

Lemma 2.1. *For arbitrary $u \in W_0^{1,p}(\Omega, \omega, I\!R^m)$ and $v \in W^{-1,p'}(\Omega, \omega^*, I\!R^m)$, the functional*

$$\begin{aligned}
&F(u) : W_0^{1,p}(\Omega, \omega, I\!R^m) \longrightarrow I\!R \\
&\varphi \longmapsto \int_\Omega \sigma(x, u(x), Du(x)) : D\varphi(x)dx \\
&- \langle v, \varphi \rangle - \int_\Omega f(x,u) : \varphi dx + \int_\Omega g(x,u) : D\varphi dx
\end{aligned}$$

is well defined, linear and bounded.

Proof For all $\varphi \in W_0^{1,p}(\Omega, \omega, I\!R^m)$, we denote

$$F(u)(\varphi) = I_1 + I_2 + I_3 + I_4$$

with

$$I_1 = \int_\Omega \sigma(x, u(x), Du(x)) : D\varphi(x)dx,$$

$$I_2 = -\langle v, \varphi \rangle,$$

$$I_3 = -\int_\Omega f(x, u) : \varphi dx$$

and

$$I_4 = \int_\Omega g(x, u) : D\varphi dx.$$

We define

$$I_{rs} = \int_\Omega \sigma_{rs}(x, u(x), Du(x)) : D_{rs}\varphi(x)dx$$

Firstly, by virtue of the growth conditions (H_2) and the Hölder inequality, one has

$$\begin{aligned}
|I_{rs}| &\leq \int_\Omega |\sigma_{rs}(x, u(x), Du(x))| : |D_{rs}\varphi(x)|dx \\
&\leq \int_\Omega \beta\omega_{rs}^{1/p}(x)[\lambda_1(x) + c_1 \sum_{j=1}^m |\gamma_j(x)|^{1/p'}|u(x)|^{q/p'} + c_1 \\
&\quad \sum_{1\leq i\leq n;\ 1\leq j\leq m} \omega_{ij}^{1/p'}|D_{ij}|^{p-1}]|D_{rs}\varphi|dx. \\
&\leq \beta[(\int_\Omega |\lambda_1(x)|^{p'}dx)^{1/p'}(\int_\Omega |D_{rs}\varphi(x)|^p\omega_{rs}dx)^{1/p} \\
&\quad +(\int_\Omega |D_{rs}\varphi(x)|^p\omega_{rs})^{1/p}(\sum_{j=1}^m \int_\Omega |u_j|^q\gamma_j dx)^{1/p'} \\
&\quad +(\sum_{1\leq i\leq n;\ 1\leq j\leq m} \int_\Omega |D_{ij}u|^p\omega_{ij}dx)^{1/p'}(\int_\Omega |D_{rs}\varphi|^p\omega_{rs}dx)^{1/p}]
\end{aligned}$$

with $(p = p'(p-1))$, and thanks to Hardy-Type inequalities we have:

$$\begin{aligned}
|I_{rs}| &\leq c\beta\Bigg[\|\lambda_1\|_{p'} \|\varphi\|_{1,p,\omega_{rs}} + c_1 \|D\varphi\|_{p,\omega_{rs}} (\int_\Omega |u|^q\gamma dx)^{1/p'} \\
&\quad + c_1 \sum_{ij} \|D\varphi\|_{p,\omega_{ij}} \|Du\|_{p,\omega_{rs}} \Bigg] \\
&\leq c'\beta[\|\lambda_1\|_{p'} \|\varphi\|_{1,p,\omega_{rs}} + \|\varphi\|_{1,p,\omega_{rs}} \|u\|_{q,\gamma} + \|u\|_{1,p,\omega} \|\varphi\|_{1,p,\omega_{rs}}
\end{aligned}$$

with $c' = \max(c, 1)$. Which gives

$$|I_1| \leq c'\beta[\|\lambda_1\|_{p'} + \|u\|_{1,p,\omega}^{q/p'} + \|u\|_{1,p,w}]\|\varphi\|_{1,p,\omega} < \infty.$$

and

$$|I_2| \leq \int_\Omega |v||\varphi|dx \leq \|v\|_{-1,p',\omega*}\|\varphi\|_{1,p,\omega} < \infty.$$

$$I_3 = \sum_j \int_\Omega f_j(x,u)\varphi_j(x)dx.$$

We denote $I_{3,j} = |\int_\Omega f_j(x,u)\varphi_j(x)dx|$.

$$I_{3,j} \leq \int_\Omega |f_j(x,u)||\varphi_j(x)|dx$$

$$\leq \int_\Omega b_1(x)|\varphi_j(x)|\omega_{0j}^{\frac{1}{P}}dx + \int_\Omega \gamma_j^{\frac{1}{p'}}|u_j|^{\frac{q}{p'}}|\varphi_j(x)|\omega_{0j}^{\frac{1}{P}}dx$$

$$\leq (\int_\Omega |b_1(x)|^{p'})^{\frac{1}{p'}}(\int_\Omega |\varphi_j(x)|^p\omega_{0j}dx)^{\frac{1}{p}} + (\int_\Omega \gamma_j(x)|u_j|^q dx)^{\frac{1}{p'}}(\int_\Omega |\varphi_j(x)|^p\omega_{0j}dx)^{\frac{1}{p}}$$

$$\leq \| b_1 \|_{p'} \| \varphi \|_{1,p,\omega} + (\sum_j \int_\Omega \gamma_j(x)|u_j|^q dx)^{\frac{1}{p'}} \| \varphi \|_{1,p,\omega}$$

$$\leq \| b_1 \|_{p'} \| \varphi \|_{1,p,\omega} + c \| Du \|_{1,p,\omega} \| \varphi \|_{1,p,\omega}$$

$$\leq (\| b_1 + c \| Du \|_{1,p,\omega}) \| \varphi \|_{1,p,\omega} .$$

$$I_4 = \sum_{rs} \int_\Omega g_{rs}(x,u)D_{rs}\varphi dx$$

$$\int_\Omega |g_{rs}| : |D_{rs}\varphi|dx \leq \int_\Omega b_2\omega_{rs}^{\frac{1}{p}}D_{rs}\varphi dx + \sum_j \int_\Omega \gamma_j^{\frac{1}{p'}}(x)|u_j|^{\frac{q}{p'}}\omega_{rs}^{\frac{1}{p}}D_{rs}\varphi dx$$

$$\leq (\int_\Omega |b_2|^{p'} dx)^{\frac{1}{p'}} (\int_\Omega |D_{rs}\varphi|^p \omega_{rs} dx)^{\frac{1}{p}} + \sum_j (\int_\Omega |u_j|^q \gamma_j(x) dx)^{\frac{1}{p'}} (\int_\Omega |D_{rs}\varphi|^p \omega_{rs}(x) dx)^{\frac{1}{p}}$$

$$\leq \|b_2\|_{p'} \|D_{rs}\varphi\|_{1,p,\omega_{rs}} + \|u\|_{q,\gamma}^{\frac{q}{p'}} (\int_\Omega |D_{rs}\varphi|^p \omega_{rs} dx)^{\frac{1}{p}}$$

$$I_4 \leq \| b_2 \|_{p'} \| D_{rs}\varphi \|_{1,p,\omega_{rs}} + \|u\|_{q,\gamma}^{\frac{q}{p'}} (\int_\Omega |D_{rs}\varphi|^p \omega_{rs} dx)^{\frac{1}{p}}$$

$$\leq \| b_2 \|_{p'} \| D\varphi \|_{1,p,\omega} + \|u\|_{q,\gamma}^{\frac{q}{p'}} \| D\varphi \|_{1,p,\omega}$$

$$\leq c'' \| \varphi \|_{1,p,\omega},$$

hence $I \leq c_4 \| \varphi \|_{1,p,\omega}$ with $c_4 < \infty$

Finally the functional $F(.)$ is bounded.

Lemma 2.2. *The restriction of F to a finite dimensional linear subspace V of $W_0^{1,p}(\Omega, \omega, I\!R^m)$ is continuous.*

Proof Let d be the dimension of V and $(e_1, e_2, ..., e_d)$ a basis of V. Let $u_j = \sum_{1 \leq i \leq d} a_j^i . e_i$ be a sequence in V which converges to $u = \sum_{1 \leq i \leq d} a^i e_i$ in V. The sequence (a_j) converge to $a \in I\!R^d$, so $u_j \longrightarrow u$ and $Du_j \longrightarrow Du$ a.e., on the other hand $\|u_j\|_p$ and $\|Du_j\|_p$ are bounded by a constant c. Thus, it follows by the continuity conditions (H_1), that

$$\sigma(x, u_j, Du_j) : D\varphi \longrightarrow \sigma(x, u, Du) : D\varphi$$

for all $\varphi \in W_0^{1,p}(\Omega, \omega, I\!R^m)$ and a.e. in Ω. Let Ω' be a measurable subset of Ω and let $\varphi \in W_0^{1,p}(\Omega, \omega, I\!R^m)$.
Thanks to the condition (H_2), we get

$$\int_{\Omega'} |\sigma(x, u_j, Du_j) : D\varphi| dx < \infty.$$

By the continuity conditions (F_0) and (G_0) we have:

$$f(x,u_j).\varphi \to f(x,u).\varphi$$

and

$$g(x,u_j).D\varphi \to g(x,u).D\varphi$$

almost everywhere. Moreover we infer from the growth conditions (F_1) and (G_1) that the sequences:
$(\sigma(x,u_j,Du_j):D\varphi)$? $(f(x,u_j).\varphi)$ and $(g(x,u_j).D\varphi)$
are equi-integrable. Indeed, if $\Omega' \subset \Omega$ is a measurable subset and $\varphi \in W_0^{1,p}(\Omega,\omega,I\!R^m)$ then: $\int_{\Omega'} |f(x,u_j).\varphi|dx < \infty$ (by (F_1) and Hölder inequality),
$\int_{\Omega'} |g(x,u_j).D\varphi|dx < \infty$ (by (G_1) and Hölder inequality),
$\int_{\Omega'} |\sigma(x,u_j,Du_j):D\varphi|dx < \infty$ (by Hölder inequality),
which implies that $\sigma(x,u_j,Du_j):D\varphi$ is equi-integrable. And by applying the Vitali's theorem, it follows that

$$\int_\Omega \sigma(x,u_j,Du_j):D\varphi dx \longrightarrow \int_\Omega \sigma(x,u,Du):D\varphi dx,$$

for all $\varphi \in W_0^{1,p}(\Omega,\omega,I\!R^m$.
Finally

$$\lim_{j\to\infty} \langle F(u_j),\varphi\rangle = \langle F(u),\varphi\rangle\,,$$

which means that

$$F(u_j) \longrightarrow F(u) \text{ in } W^{-1,p'}(\Omega,\omega^*,I\!R^m).$$

Remark 2.2. *Now, the problem $(QES)_{f,g}$ is equivalent to find a solution $u \in W_0^{1,p}(\Omega,\omega,I\!R^m)$ such that $\langle F(u),\varphi\rangle = 0$, for all $\varphi \in W_0^{1,p}(\Omega,\omega,I\!R^m)$. In order to find such a solution we apply a Galerkin Schema.*

3. Galerkin approximation

Remark 3.1. (*Galerkin Schema*)
Let $V_1 \subset V_2 \subset ..... \subset W_0^{1,p}(\Omega,\omega,I\!R^m)$ be a sequence of finite dimensional subspaces with $\bigcup_{k\in I\!N} V_k$ dense in $W_0^{1,p}(\Omega,\omega,I\!R^m)$. The sequence V_k exists since $W_0^{1,p}(\Omega,\omega,I\!R^m)$ is separable.

Let us fix some k, we assume that V_k has a dimension d and that $(e_1, e_2, ..., e_d)$ is a basis of V_k, then we define the map

$$\begin{array}{llll} G: & \mathbb{R}^k & \longrightarrow & \mathbb{R}^k \\ & (a_1, ..., a_k) & \longmapsto & \left(\left\langle F(\sum_{i=1}^d a_i e_i), e_1 \right\rangle, ..., \left\langle F(\sum_{i=1}^d a_i e_i), e_d \right\rangle \right)^t . \end{array}$$

Proposition 3.1. *The map G is continuous and $G(a) \cdot a$ tends to infinity when $\|a\|_{\mathbb{R}^k}$ tends to infinity.*

Proof. Since F restricted to V_k is continuous by Lemma 2.2, so G is continuous.
let $a \in \mathbb{R}^d$ and $u = \sum_{1 \le i \le d} a^i \cdot e_i$ in V_k, then $G(a) \cdot a = \langle F(u), u \rangle$ and which implies that $\|a\|_{\mathbb{R}^d}$ tends to infinity if $\|u\|_{1,p,\omega}$ tends to infinity.

$$G(a) \cdot a = \sum_{1 \le i \le d} \langle F(u), a^i \cdot e_i \rangle = \langle F(u), u \rangle$$

and

$$\begin{aligned} \|u\|_{1,p,\omega}^p = \left\| \sum_{1 \le i \le d} a^i \cdot e_i \right\|_{1,p,\omega}^p & \le \left(\sum_{1 \le i \le d} |a^i| \cdot \|e_i\|_{1,p,\omega} \right)^p \\ & \le \max_{1 \le i \le d} \left(\|e_i\|_{1,p,\omega}^p \right) \cdot \left(\sum_{1 \le i \le d} |a^i| \right)^p \\ & \le c \cdot \|a\|_{\mathbb{R}^p} , \end{aligned}$$

which implies that $\|a\|_{\mathbb{R}^p}$ tends to infinity if $\|u\|_{1,p,\omega}$ tends to infinity.

Now, it suffices to prove that

$$\langle F(u), u \rangle \to \infty \quad \text{when} \quad \|u\|_{1,p,\omega} \to \infty.$$

Indeed, thanks to the first coercivity condition and the Hölder inequality, we obtain

$$I = \int_\Omega \sigma(x, u, Du) : Dudx \ge - \|\lambda_2\|_1 - \int_\Omega \lambda_3 \omega_{0j}^{\alpha/p} |u_j|^\alpha dx + c_2 \sum_{1 \le i,j \le n,m} \int_\Omega |D_{ij} u|^p \omega_{ij} dx.$$

By the Hölder inequality, we have

$$\begin{aligned} \int_\Omega \lambda_3 |u_j|^\alpha \omega_{0j}^{\alpha/p} dx & \le \|\lambda_3\|_{(p/\alpha)'} \left(\int_\Omega \omega_{0j} |u_j|^{(p/\alpha)\cdot\alpha} \, dx \right)^{\alpha/p} \\ & \le c' \|\lambda_3\|_{(p/\alpha)'} \|u_j\|_{1,p,\omega_{0j}} . \end{aligned}$$

where c' is a constant positive.
For $\|u\|_{1,p,\omega}$ large enough, we can write

$$\begin{aligned} |\ I\ | &\geq -\|\lambda_2\|_1 - c'\,\|\lambda_3\|_{(p/\alpha)'} \cdot \|u_j\|^{\alpha}_{1,p,\omega_{0j}} + c_2 \cdot \sum_{1\leq i,j\leq n,m} \|Du_j\|^p_{1,p,\omega_{ij}} \\ &\geq -\|\lambda_2\|_1 - c'\,\|\lambda_3\|_{(p/\alpha)'} \cdot \|u\|^{\alpha}_{1,p,\omega} + c_2 c' \cdot \|u\|^p_{1,p,\omega}\,, \end{aligned}$$

and since

$$|I'| = |\,\langle v, u\rangle\,| \leq \|v\|_{-1,p',\omega^*} \cdot \|u\|_{1,p,\omega}\,.$$

Finally, it follows from the growth condition F_1 and G_1 that:

$$|I''| = |\int_\Omega f(x,u).udx| \leq (\|b_1\|_{p'} + c.\|Du\|_{1,p,\omega}).\|u\|_{1,p,\omega}$$

.

$$\leq c_3.\|u\|_{1,p,\omega}$$

$$|I'''| = |\int_\Omega g(x,u).Dudx| \leq (\|b_2\|_{p'} + \|u\|^{\frac{q}{p'}}_{q,\gamma}).\|Du\|_{1,p,\omega} \leq c_4.\|u\|_{1,p,\omega};$$

with c_4 is a constant, $0 < \alpha < p$ and $p > 1$, we get:

$$\begin{aligned} I - I' - I'' \geq c_2.c'. \cdot \|u\|^p_{1,p,\omega} - \|v\|_{-1,p',\omega^*} \cdot \|u\|_{1,p,\omega} - c'\,\|\lambda_3\|_{(p/\alpha)'} \cdot \|u\|^{\alpha}_{1,p,\omega} \\ - \|\lambda_2\|_1 - c_3.\,\|u\|_{1,p,\omega} \end{aligned} \tag{3.1}$$

Consequently, by using (**??**), we deduce

$$I - I' - I'' \longrightarrow \infty \text{ as } \|u\|_{1,p,\omega} \longrightarrow \infty$$

. and

$$I''' \longrightarrow \infty \text{ as } \|u\|_{1,p,\omega} \longrightarrow \infty$$

.

$$\langle F(u), u\rangle \longrightarrow \infty \text{ as } \|u\|_{1,p,\omega} \longrightarrow \infty$$

.

Remark 3.2. *The properties of G allows us to construct our Galerkin approximations.*

Corollary 3.1. *For all $k \in I\!N$, there exists $(u_k) \subset V_k$ such that $\langle F(u_k), \varphi\rangle = 0$, for all $\varphi \in V_k$.*

Proof By the proposition 3.1, there exists $R > 0$, such that for all $a \in \partial B_R(0) \subset I\!R^d$, we have $G(a) \cdot a > 0$, and the usual topological argument see.e.g [Zei 86 proposition 2.8] [19] implies that $G(x) = 0$ has a solution $x \in B_R(0)$. So, for all $k \in I\!h$, there exists $(u_k) \subset V_k$, such that

$$\left\langle F(x^j e_j), e_j \right\rangle = 0 \ \text{ for all } 1 \leq j \leq d, \text{ with } d = \dim V_k.$$

Taking $u_k = (x_k^i e_i),\ e_i \in V_k$, so we obtain:

$$\langle (F(u_k)), \varphi \rangle = 0, \text{ for all } \varphi \in V_k.$$

Proposition 3.2. *The Galerkin approximations sequence constructed in corollary 3.1 is uniformly bounded in $W_0^{1,p}(\Omega, \omega, I\!R^m)$; i.e.,*

there exists a constant $R > 0$, such that $\|u_k\|_{1,p,\omega} \leq R$, for allk $\in I\!N$.

Proof Like in the proof of proposition 3.1, we can see that

$$\langle F(u), u \rangle \longrightarrow \infty \text{ as } \|u\|_{1,p,\omega} \longrightarrow \infty.$$

Then, there exists R satisfying $\langle F(u), u \rangle > 1$ when $\|u\|_{1,p,\omega} > R$. Now, for the sequence of Galerkin approximations $(u_k) \subset V_k$ of corollary 3.1, which satisfying $\langle F(u_k), u_k \rangle = 0$, we have the uniform bound $\|u_k\|_{1,p,\omega} \leq R$, for all $k \in I\!N$.

Remark 3.3. *There exists a subsequence (u_k) of the sequence $(u_k) \subset V_k$, such that:*

$$u_k \rightharpoonup u \text{ in } W_0^{1,p}(\Omega, \omega, I\!R^m)$$

and

$$u_k \longrightarrow u \text{ in measure in } L^r(\Omega, I\!R^m);$$

with

$$\begin{cases} 1 \leq r \prec \frac{nps}{n(s+1)-ps} & \text{if } \ ps \prec n(s+1) \\ \qquad\quad r \geq 1 & \text{if } \ n(s+1) \prec ps \end{cases}$$

The gradient sequence (Du_k) generates the Young measure ϑ_x, Since $u_k \longrightarrow u$ in measure, then (u_k, Du_k) generates the Young measure $(\delta_{u(x)} \otimes \vartheta_x)$, see e.g [4] . Moreover, for almost x in Ω, we have,

(i) *ϑ_x is the probability measure, i.e, $\|\vartheta_x\|_{mes} = 1$.*

(ii) *ϑ_x is the $W^{1,p,\omega}$ - gradient homogeneous Young measure.*

(iii) *$\langle \vartheta_x, id \rangle = Du(x)$, see e.g [3].*

Proof. See [4].(Dolzmann, N.Humgerbühler s Muller. Non linear elliptic system....)

4. Passage to the limit

Now, we are in a position to prove our main result under convenient hypotheses.
Let

$$I_k = (\sigma(x, u_k, Du_k) - \sigma(x, u, Du)) : (Du_k - Du). \tag{4.1}$$

Lemma 4.1. *(Fatou lemma type) (See [4]) Let : $F : \Omega \times I\!R^m \times I\!M^{m\times n} \longrightarrow I\!R$ be a Carathéodory function, and $u_k : \Omega \longrightarrow I\!R^m$ a measurable sequence, such that (Du_k) generates the Young measure ϑ_x, with $\|\vartheta_x\|_{mes} = 1$, for a.e. $x \in \Omega$. Then:*

$$\liminf_{k\to\infty} \int_\Omega F(x, u_k, Du_k)dx \geq \int_\Omega \int_{I\!M^{m\times n}} F(x, u, \zeta)d\vartheta_x(\zeta)dx, \tag{4.2}$$

which provided that the negative part of $F(x, u_k, Du_k)$ is equi-integrable.

Lemma 4.2. *Let $p > 1$ and u_k be a sequence which is uniformly bounded in $W_0^{1,p}(\Omega, \omega, I\!R^m)$. There exists a subsequence of u_k (for convenience not relabeled) and a function $u \in W_0^{1,p}(\Omega, \omega, I\!R^m)$ such that $u_k \rightharpoonup u$ in $W_0^{1,p}(\Omega, \omega, I\!R^m)$*
And such that $u_k \to u$ in measure on Ω and in $L^r(\Omega, I\!R^m)$, with :

$$\begin{cases} 1 \leq r \prec \frac{nps}{n(s+1)-ps} & if \;\; ps \prec n(s+1) \\ \qquad\quad r \geq 1 & if \;\; n(s+1) \prec ps \end{cases}$$

Proof. see [14], with a slight modification.

Lemma 4.3. *The sequence (I_k) is equi-integrable.*

Proof

We have

$$\begin{aligned} I_k &= (\sigma(x, u_k, Du_k) - \sigma(x, u, Du)) : (Du_k - Du) \\ &= [\sigma(x, u_k, Du_k) : Du_k] - [\sigma(x, u_k, Du_k) : Du] - [\sigma(x, u, Du) : Du_k] \\ &\quad +[\sigma(x, u, Du) : Du] \\ &= I_k^1 + I_k^2 + I_k^3 + I_k^4 \end{aligned} \tag{4.3}$$

We denote $(I_k^1)^- = -[\sigma(x, u_k, Du_k : Du_k]^-$. Thanks to the coercivity condition (H_2), we have

$$\begin{aligned}\int_{\Omega'} |(I_k^1)^-|dx &\leq \int_{\Omega} |\lambda_2| + c_2 \sum_{1\leq j\leq m} \omega_{0j}^{\frac{\alpha}{p}} |\lambda_3| \cdot |u_{kj}|^{\alpha} + c \sum_{1\leq i,j\leq n,m} \omega_{ij} |D_{ij} u_k|^p dx \\ &\leq \|\lambda_2\|_1 + \int_{\Omega'} (\sum_{1\leq j\leq m} \omega_{0j}^{\alpha/p} |u_{kj}|^{\alpha})^{p/\alpha} \|\lambda_3\|_{(p/\alpha)'} + c_2 \|u_k\|_{1,\omega,p}^p\end{aligned}$$

(4.4)
with $p/\alpha \geq 1$. Therefore,

$$\begin{aligned}\int_{\Omega'} |(I_k^1)^-|dx &\leq \|\lambda_2\|_1 + \Big(\sum_{1\leq j\leq m} \omega_{0j} |u_{kj}|^p\Big)^{\alpha/p} \|\lambda_3\|_{(p/\alpha)'} + c_2 \|u_k\|_{1,\omega,p}^p \\ &\leq \|\lambda_2\|_1 + \|u_k\|_{p,\overline{\omega}_{00}}^{\alpha} \|\lambda_3\|_{(p/\alpha)'} + c_2 \|u_k\|_{1,\omega,p}^p \\ &< \infty,\end{aligned}$$

for all $\Omega' \subset \Omega$.
Similarly for $(|(I_k^4)^-|$.

Now, by using the growth condition (H_2) and the Hardy -Type inequalities (H_0), we have

$$\begin{aligned}\int_{\Omega'} |(I_k^2)^-|dx &= \int_{\Omega'} |\sigma(x, u_k, Du_k) : Du_k|dx \\ &\leq \beta \int_{\Omega'} \omega_{rs}^{1/p} \Bigg(\lambda_1 + c_1 \sum_{1\leq j\leq m} \gamma_j^{1/p'} . |u_{kj}|^{q/p'} \\ &\quad + c_1 \sum_{1\leq i,j\leq n,m} \omega_{ij}^{1/p'} |D_{ij} u_k|^{p-1}\Bigg) D_{rs} u_k dx.\end{aligned} \tag{4.5}$$

Thus, by the Hölder inequality, we obtain

$$\int_{\Omega'} |(I_k^2)^-|dx \leq \beta \Bigg[\|\lambda_1\|_{p'} \left(\int_{\Omega'} |D_{rs} u_k|^p \omega_{rs} dx\right)^{1/p}$$

$$+ c_1 \left(\int_{\Omega'} |D_{rs} u_k|^p \omega_{rs} dx\right)^{1/p} \left(\int_{\Omega'} \left(\sum_{1\leq j\leq m} \gamma_j^{1/p'} |u_{kj}|^{q/p'}\right)^{p'} dx\right)^{1/p'}$$

$$+ c_1 \left(\sum_{1\leq j\leq m} \int_{\Omega'} \left(|D_{ij} u_k(x)|^{p'(p-1)} \omega_{ij} dx\right)^{1/p'}\right)$$

$$\left(\int_{\Omega'} |D_{rs}u_k|^p \omega_{rs} dx\right)^{1/p}\Bigg]. \tag{4.6}$$

So, by combining (4.5) and (??), we deduce that

$$\int_{\Omega'} |\sigma(x, u_k, Du_k) : Du_k| dx \leq c'\beta\left(\|\lambda_1\|_{p'}\|\|u_k\|_{1,p,\omega} + \|u_k\|_{1,p,\omega}\right) < \infty. \tag{4.7}$$

Similarly to $(|(I_k^2)^-|$, we obtain $(|(I_k^3)^-|$. Finally: I_k is equi-integrable. We choose a sequence φ_k such that φ_k belongs to the same space V_k and $\varphi_k \to \varphi$ in $W_0^{1,p}(\Omega, \omega, I\!R^m)$ this allows us in particular, to use $u_k - \varphi_k$ as a test function in (3.1). We have:

$$\begin{aligned} &\int_{\Omega} |\sigma(x, u_k, Du_k) : (Du_k - D\varphi_k)| dx = \langle v, u_k - \varphi_k\rangle \\ &+ \int_{\Omega} f(x, u_k)(u_k - \varphi_k) dx - \int_{\Omega} g(x, u_k) : (Du_k - D\varphi_k) dx. \end{aligned} \tag{4.8}$$

The first term on the right in (4.8) converge to zero since $(u_k - \varphi_k) \rightharpoonup 0$ in $W_0^{1,p}(\Omega, \omega, I\!R^m)$.
By the choice of φ_k, the sequence φ_k uniformly bounded in $W_0^{1,p}(\Omega, \omega, I\!R^m)$.
And lemma (4.2) Next, for the second term: $II_k = \displaystyle\int_{\Omega} f(x, u_k)(u_k - \varphi_k) dx$ in (4.8) it follows from the growth condition F_1 and the Hölder inequality that :

$$\begin{aligned} |II_k| &\leq (\|b_1\|_{p'} + c.\|D(u_k - \varphi_k)\|_{1,p,\omega})\|u_k - \varphi_k\|_{1,p,\omega} \\ &\leq (\|b_1\|_{p'} + c.\|D(u_k - \varphi_k)\|_{1,p,\omega}).\|u_k - \varphi_k\|_{1,p,\omega}. \end{aligned}$$

By the equivalence of the norm in $W_0^{1,p}(\Omega, \omega, I\!R^m)$ and the sequence (u_k is uniformly bounded in $W_0^{1,p}(\Omega, \omega, I\!R^m)$, $\|u_k\|_{1,p,\omega}$ is bounded. Moreover, by the construction of φ_k, and lemma (4.2) we have:

$$\|u_k - \varphi_k\|_{1,p,\omega} \leq \|u_k - u\|_{1,p,\omega} + \|u - \varphi_k\|_{1,p,\omega}$$

$$(\|u_k - u\|_{1,p,\omega} + \|u - \varphi_k\|_{1,p,\omega}) \to 0$$

We infer that the second term in (4.8) vanishes as $k \to \infty$. Finally, for the last term

$$III_k = \int_{\Omega} g(x, u_k) : D(u_k - \varphi_k) dx$$

in (4.8), we note that

$$g(x, u_k) \to g(x, u)$$

Strongly in $L^{p'}(\Omega, M^{m\times n})$ by (G_0), (G_1) and lemma (4.2).
Indeed we may assure that $u_k \to u$ almost everywhere.

$$III_k \le (\|b_2\|_{p'} + \|u_k - \varphi_k\|_{q,\gamma}^{\frac{q}{p'}}).\|D(u_k - \varphi_k)\|_{1,p,\omega}$$

$$\le c'.(\|b_2\|_{p'} + \|u_k - \varphi_k\|_{q,\gamma}^{\frac{q}{p'}}).\|(u_k - \varphi_k)\|_{1,p,\omega}$$

$$\le c'.(\|b_2\|_{p'} + \|u_k - \varphi_k\|_{q,\gamma}^{\frac{q}{p'}}).(\|u_k - u\|_{1,p,\omega} + \|\varphi_k - u\|_{1,p,\omega})$$

$\|\varphi_k - u\|_{1,p,\omega} \to 0$, $\|u_k - u\|_{1,p,\omega} \to 0$ and $\|u_k - \varphi_k\|_{q,\gamma}^{\frac{q}{p'}} \to 0$.

Now, we consider $(I_k)' = (\sigma(x, u_k, Du_k) : (Du_k - Du)$. We have, I'_k is equi-integrable because I_k it is. So, we define

$$X = \liminf \int_\Omega I_k dx = \liminf \int_\Omega (I_k)' dx \ge \int_\Omega \int_{I\!M^{m\times n}} (\sigma(x, u, \lambda) : (\lambda - Du) d\vartheta_x(\lambda)$$

So to prove (4.2), it suffices to prove that:

$$X \le 0. \tag{4.9}$$

Let $\varepsilon > 0$, so there exists $k_0 \in I\!N$ such that, for all $k > k_0$, we have $dist(u, V_k) < \varepsilon$ since: $(\liminf_{\varphi_k \in V_k}(\|u - \varphi_k\|_{1,p,\omega} < \varepsilon$, $(u_k \rightharpoonup u)$
Or in an equivalent manner $dist(u_k - u, V_k) < \varepsilon$; $\forall k > k_0$ then for all $v_k \in V_k$, we have

$$\begin{aligned} X &= \liminf_{k\to\infty} \int_\Omega (\sigma(x, u_k, Du_k) : (Du_k - Du)dx \\ &= \liminf_{k\to\infty} \left[\int_\Omega (\sigma(x, u_k, Du_k) : D(u_k - u - \varphi_k)dx + \int_\Omega (\sigma(x, u_k, Du_k) : D(\varphi_k)\right] \end{aligned}$$

Combining (H_2) and (0.1), we get

$$X \le \lim \inf_{k\to\infty} \int_\Omega \beta\omega_{rs}^{1/p} \left[\lambda_1 + c_1 \sum_{1\le j\le m} \gamma_j^{1/p'} |u_{k_j}|^{q/p'} + c_1 \sum_{1\le i,j\le n,m} \omega_{ij}^{1/p'} |D_{ij}u_k|^{p-1}\right]$$
$$\times |D_{rs}(u_k - u - \varphi_k)|dx + \langle v, \varphi_k\rangle.$$

For all $\varepsilon > 0$, we choice $\varphi_k \in V_k$ such that

$$\|u_k - u - \varphi_k\|_{1,p,\omega} \le 2\varepsilon, \tag{4.10}$$

for all $k \geq k_0$. Which implies that

$$|\langle v, \varphi_k \rangle| \leq |\langle v, \varphi_k + (u - u_k) \rangle| + |\langle v, u_k - u \rangle| \leq 2\varepsilon \|v\|_{-1,p',\omega^*} + o(k)$$

Hence $\lim_{k\to\infty} \langle v, u_k - u\rangle = 0$. According to Hölder and Hardy inequalities, and by (4.10) we deduce that

$$\begin{aligned} X \quad \leq \quad & \liminf_{k\to\infty} c\beta \Bigg(\|\lambda_1\|_{p'} (\int_\Omega |D_{rs}(u_k - u - \varphi_k)|^p.\omega_{rs} dx \Big)^{1/p} \\ & + c_1 \left(\int_\Omega |u_k|^q.\gamma \right)^{1/p'} . \left(\int_\Omega |D_{rs}(u_k - u - \varphi_k)|^p \omega_{rs} dx \right)^{1/p} \\ & + c_1 \left(\sum \int_\Omega \omega_{ij} |D_{ij} u|^{p'(p-1)} \right)^{1/p'} . \left(\int_\Omega \omega_{rs} |D_{rs}(u_k - u - \varphi_k)|^p \right)^{1/p} \Bigg) \\ & + |\langle v, \varphi_k \rangle| \\ \leq \quad & \liminf_{k\to\infty} c \left(\|\lambda_1\|_{p'} . \|u_k - u - \varphi_k\|_{1,p,\omega} \right) + \|u_k\|^q_{1,p,\omega} \|u_k - u - \varphi_k\|_{1,p,\omega} \\ & + 2\varepsilon \|v\|_{-1,p',\omega^*} + o(k) \end{aligned}$$

Therefore,

$$X \leq 2\varepsilon c\beta \left(\|\lambda_1\|_{p'} + \|u\|^q_{1,p,\omega} + \|v\|_{-1,p',\omega*} \right).$$

Which proves that $X \leq 0$, and finally

$$\int_\Omega \int_{\mathbb{M}^{mn}} \sigma(x, u, \lambda) : \lambda d\vartheta_x dx \leq \int_\Omega \int_{\mathbb{M}^{mn}} \sigma(x, u, \lambda) : Du d\vartheta_x(\lambda) dx.$$

Proof of theorem2.4

For arbitrary φ in $W_0^{1,p}(\Omega, \omega, \mathbb{R}^m)$. It follows from the continuity condition (F_0) and (G_0) that

$$f(x, u_k).\varphi(x) \to f(x, u).\varphi(x)$$

and

$$g(x, u_k) : D\varphi(x) \to g(x, u) : D\varphi(x)$$

almost everywhere. Since, by the growth conditions (F_1), (G_1) and the uniform bound of u_k, $f(x, u_k).\varphi(x)$ and $g(x, u_k) : D\varphi(x)$ are equi-integrable, it follows that the Vitali's theorem. This implies that:

$$\lim_{k\to\infty} \int_\Omega f(x, u_k).\varphi(x) dx = \int_\Omega f(x, u).\varphi(x) dx$$

for all $\varphi \in \cup_{k=1}^{\infty} V_k$
and

$$\lim_{k\to\infty} \int_\Omega g(x,u_k) : D\varphi(x)dx = \int_\Omega g(x,u) : D\varphi(x)dx$$

for all $\varphi \in \cup_{k=1}^{\infty} V_k$ We will start with the easiest case

$$(4.11) \qquad (d): \qquad F \longmapsto \sigma(x,u,F) \text{ is stictly p-quasi-monotone.}$$

Indeed, we assume that ϑ_x is not a Dirac mass on the set M with $x \in M$ of positive Lebesgue measure $|M| > 0$. Moreover, by the strict p-quasi-monotonicity of $\sigma(x,u,\cdot)$ and ϑ_x is an homogeneous $W^{1,p}$ gradient Young measure for a.e. $x \in M$. So, for a.e. $x \in M$, with $\bar{\lambda} = \langle \vartheta_x, Id\rangle = apDu(x))$, with $apDu(x))$ is the differentiable approximation in x. We get

$$\begin{aligned}
\int_{\mathbb{M}^{m\times n}} \sigma(x,u,\lambda) : (\lambda - Du)d\vartheta_x(\lambda) &> \int_{\mathbb{M}^{m\times n}} \sigma(x,u,Du) : (\lambda - Du)d\vartheta_x(\lambda) \\
&> \sigma(x,u,Du) : \int_{\mathbb{M}^{m\times n}} \lambda d\vartheta_x(\lambda) - \\
&\quad\ \sigma(x,u,Du) : Du \textstyle\int_{\mathbb{M}^{m\times n}} d\vartheta_x(\lambda) \\
&> (\sigma(x,u,Du) : Du - \sigma(x,u,Du) : Du)) = 0 \\
&> 0
\end{aligned}$$

On the other hand (4.9), integrating over Ω, and using the div-cul inequality we have:

$$\begin{aligned}
\int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x,u,\lambda) : \lambda d\vartheta_x(\lambda)dx &> \int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x,u,\lambda) : Dud\vartheta_x(\lambda)dx \\
&\geq \int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x,u,\lambda) : \lambda d\vartheta_x(\lambda)dx.
\end{aligned}$$

Which is a contradiction with (4.8). Thus $\vartheta_x = \delta_{\overline{\lambda}} = \delta_{Du(x)}$ for a.e. $x \in \Omega$. Therefore, $Du_k \longrightarrow Du$ in measure when k tends to infinity. Then, we get $\sigma(x,u_k,Du_k) \longrightarrow \sigma(x,u,Du)$ for all $x \in \Omega$. In the other hand, for all $\varphi \in \bigcup_{k\in\mathbb{N}} \vartheta_k$; $\sigma(x,u_k,Du_k) : D\varphi \longrightarrow \sigma(x,u,Du) : D\varphi$ a.e. $x \in \Omega$. Moveover, for all $\Omega' \subset \Omega$ measurable, it is easy to see that:

$$\int_{\Omega'} \sigma(x,u_k,Du_k) : D\varphi dx \leq c\beta \left(\|\lambda_1\|_{p'} + \|u_k\|_{1,p,\omega}^{q/p'} + \|u_k\|_{1,p,\omega}^{p/p'} \right) \|u\|_{1,p,\omega} < \infty,$$

because $\|u_k\|_{1,p,\omega} \leq R$. And thanks to Vitali's theorem, we obtain:

$$\langle F(u), \varphi\rangle = 0, \text{ for all } \varphi \in \bigcup_{k\in\mathbb{N}} \vartheta_k.$$

Which proves the theorem in this case.

Remark 4.1. *Before treating the cases* (a), (b) *and* (c) *of* (H_3), *we note that*

$$\int_\Omega \int_{\mathbb{M}^{m\times n}} (\sigma(x,u,\lambda) - \sigma(x,u,Du)) : (\lambda - Du) d\vartheta_x(\lambda) dx \leq 0 \tag{4.12}$$

Since

$$\int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x,u,\lambda) : (\lambda - Du) d\vartheta_x(\lambda) dx = 0,$$

thanks to the div-Curl inequality in (4.9). On the other hand, the integrand in (4.12) is non negative, by the monotonicity of σ. Consequently, the integrating should be null, a.e., with respect to the product measure $d\vartheta_x \otimes dx$, which mean

$$(\sigma(x,u,\lambda) - \sigma(x,u,Du)) : (\lambda - Du) = 0 \text{ in } spt\vartheta_x. \tag{4.13}$$

Thus,

$$spt\vartheta_x \subset \left\{\lambda \in \mathbb{M}^{m\times n} / \left(\sigma(x,u,\lambda) - \sigma(x,u,Du)\right) : (\lambda - Du) = 0\right\}. \tag{4.14}$$

Case c: We prove that, the map $F \longmapsto \sigma(x,u,F)$ is strictly monotone, for all $x \in \Omega$ and for all $u \in \mathbb{R}^m$.
Sine σ is strict monotone, and according to (4.14),

$$spt\vartheta_x = \{Du\}, \text{ i.e, } \vartheta_x = \delta_{Du}, \quad \text{a.e. in } \Omega,$$

which implies that, $Du_k \longrightarrow Du$ in measure. For the rest of our prove is similarly to case d.

Case b: We start by showing that for almost all $x \in \Omega$, the support ofϑ_x is contained in the set where W agrees with the supporting hyper-plane.

$$L = \left\{(\lambda, W(x,u,\overline{\lambda}) + \sigma(x,u,\overline{\lambda}) : (\lambda - \overline{\lambda}))\right\} \text{ with } \overline{\lambda} = Du(x).$$

So, it suffices to prove that

$$spt\vartheta_x \subset K_x = \left\{\lambda \in \mathbb{M}^{m\times n} / W(x,u,\lambda) = W(x,u,\overline{\lambda}) + \sigma(x,u,\overline{\lambda}) : (\lambda - \overline{\lambda})\right\} \tag{4.15}$$

If $\lambda \in spt\vartheta_x$, thanks to (4.14), we have

$$(1-t).\,(\sigma(x,u,Du) - \sigma(x,u,\lambda)) : (Du - \lambda) = 0, \text{ for all } t \in [0,1]. \tag{4.16}$$

On the other hand, since σ is monotone, for all $t \in [0,1]$ We have:

(4.17) $(1-t).\,(\sigma(x,u,Du+t.(\lambda - Du)) - \sigma(x,u,\lambda)) : (Du - \lambda) \geq 0.$

By subtracting (4.16) from (4.17), we get

$$(1-t)\left[\sigma(x,u,\overline{\lambda}+t(\lambda-\overline{\lambda})) - \sigma(x,u,\overline{\lambda})\right] : (\overline{\lambda}-\lambda) \geq 0, \tag{4.18}$$

for all $t \in [0,1]$. Doing the same by the monotonicity in (4.18), we obtain

$$(1-t)\left[\sigma(x,u,\overline{\lambda}+t(\lambda-\overline{\lambda})) - \sigma(x,u,\overline{\lambda})\right] : (\overline{\lambda}-\lambda) \leq 0. \tag{4.19}$$

Combining (4.18) and (4.19), we conclude that

$$(1-t)\left[\sigma(x,u,\overline{\lambda}+t(\lambda-\overline{\lambda})) - \sigma(x,u,\overline{\lambda})\right] : (\overline{\lambda}-\lambda) = 0, \tag{4.20}$$

for all $t \in [0,1]$, and for all $\lambda \in spt\vartheta_x$.
Now, it follows from (4.19) that

$$\begin{aligned} W(x,u,\lambda) &= W(x,u,\overline{\lambda}) + (W(x,u,\lambda) - W(x,u,\overline{\lambda})) \\ &= W(x,u,\overline{\lambda}) + \int_0^1 [\sigma(x,u,\overline{\lambda}+t(\lambda-\overline{\lambda}))] : (\lambda-\overline{\lambda})dt \\ &= W(x,u,\overline{\lambda}) + \sigma(x,u,\overline{\lambda}) : (\lambda-\overline{\lambda}) \end{aligned}$$

Which prove (4.15).
Now, by the coercivity of W, we get

$$W(x,u,\lambda) \geq W(x,u,\overline{\lambda}) + \sigma(x,u,\overline{\lambda}) : (\lambda-\overline{\lambda}),$$

for all $\lambda \in \mathbb{M}^{m\times n}$. Therefore,

(4.21) $\quad L$ is a supporting hyper-plane, for all $\lambda \in K_x$.

Moveover, the mapping $\lambda \longmapsto W(x,u,\lambda)$ is continuously differentiable, so we obtain

$$\sigma(x,u,\lambda) = \sigma(x,u,\overline{\lambda}), \text{ for all } \lambda \in K_x. \tag{4.22}$$

Thus,

$$\overline{\sigma}(x) = \int_{\mathbb{M}^{m\times n}} \sigma(x,u,\lambda)d\vartheta_x(\lambda) = \sigma(x,u,\overline{\lambda}). \tag{4.23}$$

Now, we consider the Carathéodory function

$$\psi(x,u,\rho) = |(\sigma(x,u,\rho) - \overline{\sigma}(x))|,$$

and lets $\psi_k(x) = \psi(x, u_k, Du_k)$ is equi-integrable. Thus, thanks to Ball's theorem, see [8] $\psi_k \rightharpoonup \overline{\psi}$ weakly in $L^1(\Omega)$, and the weakly limit of ψ_k is given by

$$\begin{aligned}\overline{\psi}(x) &= \int\int_{\mathbb{R}^m\times \mathbb{M}^{m\times n}} |\sigma(x,\eta,\lambda) - \overline{\sigma}(x)| d\delta_{u(x)}(\eta)\otimes d\vartheta_x(\lambda) \\ &= \int_{spt\vartheta_x} |\sigma(x,u(x),\lambda) - \overline{\sigma}(x)| d\vartheta_x(\lambda) \\ &= 0.\end{aligned}$$

According to (4.22) and (4.23), and since $\psi_k \geq 0$, it follow that $\psi_k \longrightarrow 0$ strongly in $L^1(\Omega)$ by Fatou lemma, which gives

$$\lim_{k\to\infty}\int_\Omega \sigma(x,u_k,Du_k) : D\varphi.dx = \int_\Omega \sigma(x,u,Du) : D\varphi dx.$$

Thus

$$\langle F(u),\varphi\rangle = 0, \quad \forall\varphi \in \bigcup_{k\in\mathbb{N}} V_k.$$

This completes the proof of the case (b).

Case (a): In this case, on $spt\vartheta_x$, we affirm that,

$$\sigma(x,u,\lambda) : M = \sigma(x,u,Du) : M + (\nabla_F\sigma(x,u,Du) : M) : (Du-\lambda), \tag{4.24}$$

for all $M \in \mathbb{M}^{m\times n}$, where ∇_F is the derivative with respect to the third variable of σ and $\overline{\lambda} = Du(x)$.
Thanks to the monotonicity of σ, we have

$$(\sigma(x,u,\lambda) - \sigma(x,u,Du+tM)) : (\lambda - Du - tM) \geq 0, \text{ for all } t \in \mathbb{R}.$$

By invoking (4.19), we obtain

$$-\sigma(x,u,\lambda) : (tM) \geq -(\sigma(x,u,Du) : (\lambda - Du) + \sigma(x,u,Du+tM) : (\lambda - Du - t.M).$$

On the other hand, $F \longmapsto \sigma(x,u,F)$ is a C^1 function, so

$$\sigma(x,u,Du+tM) = \sigma(x,u,Du) + \nabla_F(x,u,Du).(tM) + o(t).$$

Thus

$$-\sigma(x,u,\lambda) : (t.M) \geq -\sigma(x,u,Du) : (tM) + \nabla_F\sigma(x,u,Du)(t.M) : (\lambda - Du) + o(t),$$

which gives

$$-\sigma(x,u,\lambda):(t.M)\geq t[(\nabla_F\sigma(x,u,Du):(M):(\lambda-Du)-\sigma(x,u,Du):(M)]+o(t),$$

t is arbitrary in (4.24). Finally for all $\varphi\in\bigcup_{k\in I\!N}V_k$ the sequence $\sigma(x,u_k,Du_k):D\varphi$ is equi-integrable. Then, by the Ball's theorem, see [1] the weak limit is $\int_{spt\vartheta_x}\sigma(x,u,\lambda):D\varphi d\vartheta_x(\lambda)$.

By choosing $M=Du$ in (4.24), we obtain:

$$\int_{spt\vartheta_x}(Du-\lambda)(\sigma(x,u,\lambda):D\varphi):D\varphi d\vartheta_x(\lambda)$$

$$=\int_{spt\vartheta_x}(\sigma(x,u,Du):D\varphi d\vartheta_x(\lambda)+(\nabla_F\sigma(x,u,Du):D\varphi)^t\int_{spt\vartheta_x}(Du-\lambda)d\vartheta_x(\lambda)$$

$$=(\sigma(x,u,Du):D\varphi)\int_{spt\vartheta_x}d\vartheta_x(\lambda)=\sigma(x,u,Du):D\varphi.$$

Hence,

$$\sigma(x,u_k,Du_k):D\varphi\longrightarrow\sigma(x,u,Du):D\varphi \text{ strongly}$$

This proves that

$$\langle F(u),\varphi\rangle=0 \text{ for all } \varphi\in\bigcup V_k.$$

And since $\bigcup V_k$ is dense in $W_0^{1,P}(\Omega,\omega,I\!R^m)$, so u is a weak solution of $(QES)_{f,g}$, as desired.

Remark 4.2. *In case (b) $\sigma(x,u_k,Du_k):D\varphi\longrightarrow\sigma(x,u,Du):D\varphi$ strongly, but in the case (c) and (d) $Du_k\longrightarrow Du$ in measure.*

Exemple 4.1. *We shall suppose that the weight functions satisfy: $\omega_{i_0j}=0,\ j=1,2,...,m$ for some $i_0\in I^c$; and $\omega_{ij}(x)=\omega(x); x\in\Omega$, with $I^c\bigcup I=\{0;1;2;.....;n\}$, for all $i\in I\sqcup I^c,\ j=1,2,...,m$ and $i\neq i_0$ with $\omega(x)>0$ a.e. in Ω then, we can consider the Hardy-Type inequalities in the form:*

$$\left(\sum_{j=1}^m\int_\Omega|u_j(x)|^q\gamma_j(x)dx\right)^{\frac{1}{q}}\leq c\left(\sum_{1\leq i\leq N1\leq j\leq m}\int_\Omega|D_{ij}u|^p\omega_{ij}\right)^{\frac{1}{p}},$$

for every $u\in W_0^{1,p}(\Omega,\omega,I\!R^m)$ with a constant $c>0$ independent of u and for some $q>p'$. Let us consider the Carathéodory functions: $(\star)$

$$\sigma_{ij}(x,\eta,\xi_I)=\omega(x)|\xi_{ij}|^{p-1}sng(\xi_{ij}),\ j=1,2,....m\, i\in I$$

$$\sigma_{ij}(x,\eta,\xi_{I^c}) = \omega(x)|\xi_{ij}|^{p-1}sng(\xi_{ij}),\ j=1,2,....m\, i \in I^c, i \neq i_0$$
$$\sigma_{i_0j}(x,\eta,\xi_{I^c}) = 0\, j=1,2,...,m.$$

The above functions defined by $(\star)$ *satisfies the growth conditions* (H_2). *In particular, let use the special weight function* ω. γ *expressed in term of the distance to the boundary* $\partial\Omega$ *denote* $d(x) = dist(x;\partial\Omega)$ *and* $\omega(x) = d^\lambda(x)$, $\gamma_j(x) = d^\mu(x)$ *the hardy inequality reads:*

$$\left(\sum_{j=1}^{m}\int_\Omega |u_j(x)|^q d^\mu(x)dx\right)^{\frac{1}{q}} \leq c\left(\sum_{1\leq i\leq N 1\leq j\leq m}\int_\Omega |D_{ij}u|^p d^\lambda(x)\right)^{\frac{1}{p}},$$

and the corresponding $W_0^{1,p}(\Omega;\omega;I\!R^m) \hookrightarrow L^q(\Omega;\gamma;I\!R^m)$ *is compact if:*
i)- For, $1 < p \leq q < \infty$

$$\lambda < p-1; \frac{n}{q} - \frac{n}{p} + 1 \geq 0; \quad \frac{\mu}{q} - \frac{\lambda}{p} + \frac{n}{q} - \frac{n}{p} + 1 > 0.$$

ii)- For, $1 \leq q < p < \infty$

$$\lambda < p-1; \frac{n}{q} - \frac{n}{p} + 1 \geq 0; \quad \frac{\mu}{q} - \frac{\lambda}{p} + \frac{1}{q} - \frac{1}{p} + 1 > 0.$$

iii)- For, $q > 1$
$\mu(q'-1) < 1$, *by the simple modifications of the example in [17].*

Moreover, the monotonicity condition are satisfied:
$\sum_{ij}(\sigma_{ij}(x,\eta,\xi_I) - \sigma_{ij}(x,\eta,\xi'_I)(\xi_{ij} - \xi'_{ij})$

$$= \omega(x)\sum_{ij}(|\ \xi_{ij}\ |^{p-1}\ sng(\xi_{ij}) -|\ \xi'_{ij}\ |^{p-1}\ sng(\xi'_{ij}))(\xi_{ij} - \xi'_{ij}) \geq 0$$

for almost all $x \in \Omega$ *and for all,* $\xi, \xi' \in I\!M^{m\times n}$. *This last inequality can not be strict, since for* $\xi_{I^c} \neq \xi'_{I^c}$ *with* $\xi_{i_0j} \neq \xi'_{i_0j}$ *for all* $j = 1,2,...,m$. *But* $\xi_{ij} = \xi'_{ij}$ *for* $i \in I^c$, $i \neq i_0$, $j = 1,2,...,m$ *the corresponding expression is Zero.*

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