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Fuzzy sequential topology

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Abstract

We define the sequential fuzzy closure and sequential fuzzy interior of fuzzy subsets of I^X by convergence of sequences of fuzzy points. We characterize the fuzzy sequential topology with the sequential fuzzy closure. Furthermore, we compare this topology with the usual fuzzy topology, and prove some basic properties of these concepts.

Keywords: *fuzzy sets; fuzzy points; fuzzy neighborhoods; fuzzy continuity; separated; fuzzy sequential closure; fuzzy sequential interior; sequentially fuzzy closed; sequentially fuzzy open.*

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1. Introduction

Among the various paradigmatic changes in science and mathematics, one such change concerns the concept of uncertainty. In science, this change has been manifested by a gradual transition from the traditional view, which insists that uncertainty is undesirable in science and should be avoided by all possible means, to an alternative view, which is tolerant of uncertainty which is considered essential to science. Zadeh [47], in 1965, introduced fundemantal idea of fuzzy sets by defining them in terms of maps from a set into the unit interval on the real line. Fuzzy sets were introduced to provide means to describe collections of objects for which there is no precise criteria for membership. Collections of this type have vague or "fuzzy" boundaries; there are objects for which it is impossible to determine whether or not they belong to the collection. The theory of fuzzy is frequently confused with probability theory but, the specialty of the fuzzy sets is to capture the idea of partial membership [6] and so, they can be seen like a generalization of the classic sets.

The concept of fuzzy topology was first defined by Chang [2] and later redefined in a somewhat different way by Lowen [9], Hutton and Reilly [18] and others. Several authors have expansively developed the theory of fuzzy sets and its applications. Many abstract structures were generalized using fuzzy sets. Fuzzy topological spaces, fuzzy uniform spaces, fuzzy proximity spaces, fuzzy groups, fuzzy vector spaces, fuzzy topological groups and fuzzy topological vector spaces were introduced in [2], [25], [20], [33], [19], [11], [19] respectively.

Fuzzy topology and fuzzy continuity were first studied in [2]. Since then, many investigations have been done in this field as in [22, 32]. Different types of fuzzy continuity have been defined by many authors taking different approaches, some of which can be found in [1, 2, 10, 21, 9, 34, 36, 46, 47]. Closure and interior are alternative approaches to open sets in topology. So, many researchers have studied for these notions. The notions of fuzzy δ-closure and fuzzy θ-closure of a fuzzy set in a fuzzy topological space were introduced by Ganguly and Saha[11] and Mukherjee and Sinha [14], respectively. Furthermore, the notion of strong δ-continuity was investigated in [15] and the concept of δ-continuity on function spaces was studied in [17]. Tripathy and Ray in [43] studied fuzzy weak continuity in mixed fuzzy topological space. These authors in [44] investigate fuzzy δ-I-continuity in mixed fuzzy ideal topological spaces. They also in [45] introduced fuzzy $\delta^\star$-almost continuous and fuzzy $\delta^\star$-continuous functions in mixed fuzzy ideal topological spaces.

The notion of convergence is one of the basic notion in analysis. In 1979 Lowen [26] introduced and studied the theory of convergence for fuzzy filters in fuzzy topological spaces and applied the results to describe fuzzy compactness and fuzzy continuity. This was followed by an extensive study of the convergence theory of fuzzy filters in fuzzy topological spaces by several authors [16, 7, 12, 13, 27, 29, 5, 4, 24]. In 1980 Pu and Liu [31], by means of the concept of a Q-neighborhood of a fuzzy point in a fuzzy topological space, gave the notion of convergence for fuzzy nets. Tripathy and Borgohain in [39, 40] introduced different type of difference sequence spaces and studied the classes of generalized difference bounded, convergent, and null sequences of fuzzy real numbers defined by an Orlicz function. These authors presented in [41] the classes of statistically convergent difference sequence spaces of fuzzy real numbers defined by Orlicz functions and they introduced in [42] the class of n-normed sequences related to the p-absolutely summable sequence space.

In the setting of general topological spaces, sequences are not sufficient to describe the topology, and convergence of sequences doesn't give us full information on the topology and that more general nets or filters must be used. Sequential closure relative to a given topology τ provides a method for constructing a new topology τ^s which is associated in a natural way with the original one. This topology, which we call sequential topology, is formed by taking open sets those sets whose complements are sequentially closed relative to the original topology, and which is finer [8].

In this paper, we define the fuzzy sequential closure and investigate their properties by using the convergence of sequences of fuzzy points in a fuzzy topological space. We will construct a new fuzzy topology called fuzzy sequential topology that is finner that fuzzy topology. In addition, we prove some basic properties of these concepts.

2. Preliminaries

Throughout this paper, I is the unit interval on the real line and X is a set.

Definition 2.1. *[38, Definition 1.1] A fuzzy set in X is a function with domain X and values in I, that is, an element of I^X.*

Definition 2.2. *[38, Definition 1.3] Let $\mu, \nu \in I^X$. We say μ includes ν (written $\nu \subset \mu$) if $\nu \leq \mu$ (i.e. $\forall x \in X, \nu(x) \leq \mu(x)$).*

Definition 2.3. *[36, 37] A fuzzy point or a fuzzy singleton $p(x_0, t)$ in X is a fuzzy set with membership function μ_p defined by,*

$$\mu_p(x_0) = \begin{cases} t, & \text{for } x = x_0 \\ 0, & \text{otherwise,} \end{cases}$$

where $t \in]0,1[$. x_0 is called the support of p and t its value. Also, p is in a fuzzy set μ or $p \in \mu$ if and only if $\mu_p(x_0) < \mu(x_0)$. So, $p \notin \mu$ if and only if $\mu_p(x_0) \geq \mu(x_0)$. We denote by $\mathcal{P}(X)$ the collection of all fuzzy points in X.

Definition 2.4. *[38, Definition 1.4] Let $\mu, \nu \in I^X$. We define:*

(i) $\mu \wedge \nu \in I^X$, by $(\mu \wedge \nu)(x) = \min\{\mu(x), \nu(x)\}$ *for each x in X.*

(ii) $\mu \vee \nu \in I^X$, by $(\mu \vee \nu)(x) = max\{\mu(x), \nu(x)\}$ *for each $x \in X$.*

(iii) $\mu^c \in I^X$, by $\mu^c(x) = 1 - \mu(x)$ for each $x \in X$.

(iv) $p' \in \mathcal{P}(X)$, by $p'(x, t') = p(x, 1 - t)$.

(v) $c_t \in I^X$, by $c_t(x) = t$ for each $x \in X$ and $t \in I$.

Proposition 2.1. *[28] Let μ, η be fuzzy subsets in I^X and $(\mu_i)_{i \in J}$ a family of fuzzy subsets in I^X. The following properties hold:*

(i) $c_0^c = c_1$ and $c_1^c = c_0$.

(ii) $(\mu^c)^c = \mu$.

(iii) $\mu \leq \eta \Rightarrow \eta^c \leq \mu^c$.

(iv) $(\bigvee_{i \in J} \mu_i)^c = \bigwedge_{i \in J} \mu_i^c$.

(v) $(\bigwedge_{i \in J} \mu_i)^c = \bigvee_{i \in J} \mu_i^c$.

(vi) $\mu(x) \vee \mu^c(x) \geq \frac{1}{2}$, for all $x \in X$.

(vii) $\mu(x) \wedge \mu^c(x) \leq \frac{1}{2}$, for all $x \in X$.

Remark 2.1. *The complementation μ^c of a fuzzy set μ is the Zadeh's complement [47], it is not a complement in the order-theoretic sense. Because $\mu \vee (1-\mu) \neq c_1$ and $\mu \wedge (1-\mu) \neq c_0$. For this, we can consider the constant fuzzy set*

$$\mu : X \to I : x \to \frac{1}{2},$$

then $\mu^c = \mu$, $\mu \vee (1-\mu) \neq c_1$ and $\mu \wedge (1-\mu) \neq c_0$, which of course is a most unusual situation from the point of view of classical set theory. One therefore has to be careful with the interpretation of this pseudocomplement[28].

Definition 2.5. *[38, Definition 1.5] Let $\tau \subset I^X$ satisfy the following conditions:*

(i) $c_0, c_1 \in \tau$.

(ii) *If $\mu_1, \mu_2 \in \tau$, then $\mu_1 \wedge \mu_2 \in \tau$.*

(iii) *If $\{\mu_j : j \in J\} \subset \tau$, then $\bigvee_{j\in J}\{\mu_j\} \in \tau$.*

τ is called a fuzzy topology on X and (X, τ) a fuzzy topological space (or f.t.s, for brevity). The pair (X, τ) is called a fuzzy topological space. The elements of τ are called fuzzy open sets in X.

Definition 2.6. *[38, Definition 1.6] Let $p \in \mathcal{P}(X)$ and (X, τ) be a f.t.s, $\nu \in I^X$ is said to be a fuzzy neighborhood of p, if there is some $\mu \in \tau$ such that $p \in \mu$ and $\mu \subset \nu$. The collection of all fuzzy neighborhoods of a fuzzy point p in an f.t.s. (X, τ) is denoted by $\mathcal{N}_p^\tau$. A collection $\mathcal{B}_p^\tau$ of subsets of $\mathcal{N}_p^\tau$ is a local base at p if for all $\nu \in \mathcal{N}_p^\tau$, there is some $\mu \in \mathcal{B}_p^\tau$ with $\mu \subset \nu$.*

Theorem 2.2. *[37, Theorem 2.4] A fuzzy set in (X, τ) is fuzzy open if it is an fuzzy neighborhood of each of its fuzzy points.*

Definition 2.7. *[38, Definition 1.7] Let (X, τ) and (Y, γ) be f.t.s. and f a map from X into Y. f is said to be a fuzzy continuous map if for each $p \in \mathcal{P}(X)$ and each $\nu \in \mathcal{N}_{f(p)}^\gamma$, there is some $\mu \in \mathcal{N}_p^\tau$ such that $f(\mu) \subset \nu$.*

Theorem 2.3. *[38, Theorem 1.1 (5)] Let (X, τ) and (Y, γ) be f.t.s. and f a map from X into Y. Then f is fuzzy continuous iff for each $\nu \in \gamma$, we have $f^{-1}(\nu) \in \tau$.*

Definition 2.8. *[38, Definition 1.8] Let (X, τ) be a f.t.s., $p(x, t) \in \mathcal{P}(X)$ and $\mu \in I^X$: p is said to be an adherence value of μ if for every $\nu \in \mathcal{N}_{p'}^\tau$, then $\nu \not\subset \mu^c$.*

Proposition 2.4. *[35, Proposition 1.2]*
$\overline{\mu} = \bigvee\{p : p \text{ is an adherence value of } \mu\}$.

Definition 2.9. *[23, Definition 3.1]. A fuzzy topological space (X, τ) is said to be separated (or T_2) if and only if to each pair $p_1(x_1, \lambda_1)$, $p_2(x_2, \lambda_2)$ of fuzzy points in X, with $x_1 \neq x_2$, there exists fuzzy open sets μ and η in X such that $p_1 \in \mu$, $p_2 \in \eta$ and $(\mu \wedge \eta)(z) = 0$ for every $z \in X$.*

Definition 2.10. *[3] Let X and Y be any two non-empty sets, $f : X \to Y$ be a map and μ be a fuzzy subset in I^X. Then $f(\mu)$ is a fuzzy subset of Y defined by*

$$f(\mu)(y) = \begin{cases} \sup_{x \in f^{-1}(y)} \mu(x) & \text{if } f^{-1}(y) \neq 0, \\ 0 & \text{else,} \end{cases}$$

for all $y \in Y$, where $f^{-1}(y) = \{x : f(x) = y\}$. If η is a fuzzy subset of Y, then the fuzzy subset $f^{-1}(\eta)$ of X is defined by $f^{-1}(\eta)(x) = \eta(f(x))$ for all $x \in X$.

3. Main results

Definition 3.1. *Let $(p_n(x_n, t_n))$ (briefly, (p_n)) be a sequence of fuzzy points in a f.t.s. (X, τ) with supports (x_n) and values (t_n). Let $p(x, t)$ (briefly, p) be a fuzzy point with support x and value t. Then (p_n) is said to be converge to p, written $p_n \underset{n \to +\infty}{\longrightarrow} p$, iff for every member μ of τ such that $p \in \mu$, there exists a number m, such that $p_n \in \mu$ for all $n > m$. In this case $p(x, t)$ is denoted by $\lim(p_n)$.*

The following properties are the most familiar axioms of convergence (see [30]):

(a) Every constant sequence (p_n) (i.e. $p_n = p, \forall n \in \mathbf{N}$) converges to p.

(b) If a sequence $(p_n)_n$ has the limit p, then each of its subsequences has the same limit.

(c) If each subsequence of the sequence (p_n) has a subsequence which converges to p, then (p_n) converges to p.

(d) Every sequence does not converges to two fuzzy points with different supports.

Theorem 3.1. *The convergence of sequences in a fuzzy topological space (X, τ) satisfies (a), (b) and (c). Furthermore, it satisfies (d) if X is separated.*

Proof. The proof of (a) and (b) is straightforward.

(c) Suppose, to derive a contradiction, that (p_n) is not convergent to p. Then, there is $\mu \in \tau$, such that $p \in \mu$ and for all $n \in \mathbf{N}$, there is $m \in \mathbf{N}$ such that $m > n$ and $p_m \notin \mu$. The set $M = \{m \in \mathbf{N} : p_m \notin \mu\}$ is infinite. Let $M = \{m_1 < m_2 < ... < m_k < ...\}$. The sequence (p_{m_k}) is subsequence of (p_n) without a subsequence convergent to p, which is not the case.

(d) Suppose, towards a contradiction, that a sequence (p_n) of fuzzy points converges to two different fuzzy points $p_1(x_1, t_1)$ and $p_2(x_2, t_2)$ $(x_1 \neq x_2)$. Since $p_n \underset{n\to+\infty}{\longrightarrow} p_1$ and $p_n \underset{n\to+\infty}{\longrightarrow} p_2$, then

$$\forall \mu \in \tau \text{ with } p_1 \in \mu, \exists N_1 \in \mathbf{N} \text{ such that, } p_n \in \mu, \forall n > N_1.$$

$$\forall \eta \in \tau \text{ with } p_2 \in \eta, \exists N_2 \in \mathbf{N} \text{ such that, } p_n \in \eta, \forall n > N_2.$$

Therefore $\mu \wedge \eta \neq c_0$, which is contradiction since X is a fuzzy separated space.

□

Definition 3.2. *Let X be a fuzzy topological space.*

(i) For each fuzzy subset μ in I^X, we define the fuzzy sequential closure of μ by:

$$\overline{\mu}^s = \{p \in X : \exists (p_n) \subset \mu \text{ and } p'_n \underset{n\to+\infty}{\longrightarrow} p'\}.$$

We note $\overline{\mu}^s = \overline{\eta}^s$ where $\eta = \mu$ for all μ in I^X.

(ii) A subset $\mu \in I^X$ is said to be sequentially fuzzy closed if $\overline{\mu}^s = \mu$.

(iii) A subset $\nu \in I^X$ is said to be sequentially fuzzy open (fuzzy s-open) if $1 - \nu$ is sequentially fuzzy closed (fuzzy s-closed).

Theorem 3.2. *Let μ be a fuzzy subset in I^X then,*

(i) $\mu \leq \overline{\mu}^s \leq \overline{\mu}$.

(ii) $\overline{c}_0^s = c_0$ and $\overline{c}_1^s = c_1$.

Proof.

(i) Let $p(x,t) \in \mu$ and $p_n(x_n,t_n) = p(x,t), \forall n \in \mathbf{N}$. It is obvious that $p'_n(x_n, 1-t_n) = p'(x, 1-t)$. Since $p'_n \underset{n\to+\infty}{\longrightarrow} p'$ then $p \in \overline{\mu}^s$. Now, if $p \in \overline{\mu}^s$, then there is a fuzzy sequence $(p_n)_n$ in μ such that $p'_n \underset{n\to+\infty}{\longrightarrow} p'$ i.e. for all $\nu \in \mathcal{N}^{\tau}_{p'}$ there is $N \in \mathbf{N}$ such that $p'_n \in \nu, \forall n > N$. Therefore, for all $n > N$, $t_n < \mu(x_n)$ and $1 - t_n < \nu(x_n)$. Hence, $\nu(x_n) > 1 - \mu(x_n)$ $(\nu \not\subset \mu^c)$ i.e. $p \in \overline{\mu}$.

(ii) it is straightforward.

□

We have also the following properties :

Theorem 3.3. *Let μ and η be fuzzy subsets in I^X and $(\mu_i)_{i\in J}$ a family of fuzzy subsets in I^X. We have the following relations:*

(i) $\mu \le \eta \Rightarrow \overline{\mu}^s \le \overline{\eta}^s$.

(ii) $\overline{\bigwedge_{i\in J} \mu_i}^s \le \bigwedge_{i\in J} \overline{\mu_i}^s$.

(iii) $\bigvee_{i\in J} \overline{\mu_i}^s \le \overline{\bigvee_{i\in J} \mu_i}^s$.

(iv) $\overline{\mu}^s \vee \overline{\eta}^s = \overline{\mu \vee \eta}^s$.

Proof.

(i) Let p a fuzzy point in $\overline{\mu}^s$, then there is a sequence (p_n) in μ such that $p'_n \underset{n\to+\infty}{\longrightarrow} p'$, i.e., for every member ζ of τ such that $p' \in \zeta$, there exists a number m, such that $p'_n \in \zeta$ for all $n > m$. Since $\mu \le \eta$ then for all $n \in \mathbf{N}$, p_n is in η. Then p is in $\overline{\eta}^s$.

(ii) It is straightforward by (i).

(iii) It is straightforward by (i).

(iv) By (iii), $\overline{\mu}^s \vee \overline{\eta}^s \le \overline{\mu \vee \eta}^s$. Conversely, let p a fuzzy point in $\overline{\mu \vee \eta}^s$. Then, there is a sequence (p_n) in $\mu \vee \eta$ such that $p'_n \underset{n\to+\infty}{\longrightarrow} p'$. Suppose, towards a contradiction, that $\{n \in \mathbf{N} : p_n \in \eta\}$ and $\{n \in \mathbf{N} : p_n \in \mu\}$ are finite. Then, there is $N \in \mathbf{N}$ such that for all $n > \mathbf{N}$, $p_n \notin \mu \vee \eta$ which is not the case. Assume that $M = \{n \in \mathbf{N} : p_n \in \mu\}$ is infinite and set $M' = \{n_1 < n_2 < ... < n_k < ...$ such that $n_k \in M\}$. Hence, p_{n_k} is a subsequence of P_n. By Theorem 3.1 $p'_n \underset{n\to+\infty}{\longrightarrow} p'$ i.e. $p \in \overline{\mu}^s$. So, $p \in \overline{\mu}^s \vee \overline{\eta}^s$.

□

Definition 3.3. *Let μ be a fuzzy subset in I^X. The fuzzy sequential interior of μ is the set*

$$\overset{\circ s}{\mu} = \mu \wedge (1 - \overline{1-\mu}^s)$$

Theorem 3.4. *Let μ and η be fuzzy subsets in I^X and $(\mu_i)_{i\in J}$ a family of fuzzy subsets in I^X. We have the following relations:*

(i) $\mu \leq \eta \Rightarrow \overset{\circ s}{\mu} \leq \overset{\circ s}{\eta}$.

(ii) $\overset{\circ s}{\mu} \vee \overset{\circ s}{\eta} \leq \mu \overset{\circ s}{\vee} \eta$.

(iii) $\overset{\circ s}{\mu} \wedge \overset{\circ s}{\eta} = \mu \overset{\circ s}{\wedge} \eta$.

Proof.

(i) Since $\mu \leq \eta$, then $\overline{1-\eta}^s \leq \overline{1-\mu}^s$, i.e., $1 - \overline{1-\mu}^s \leq 1 - \overline{1-\eta}^s$. Therefore, $\min(\mu(x), 1 - \overline{1-\mu}^s(x)) \leq \min(\eta(x), 1 - \overline{1-\eta}^s(x))$ for all $x \in X$, i.e., $\mu \wedge (1 - \overline{1-\mu}^s) \leq \eta \wedge (1 - \overline{1-\eta}^s)$. Thus, $\overset{\circ s}{\mu} \leq \overset{\circ s}{\eta}$.

(ii) We have $\overset{\circ s}{\mu} \leq \mu \overset{\circ s}{\vee} \eta$ and $\overset{\circ s}{\eta} \leq \mu \overset{\circ s}{\vee} \eta$. Hence, $\overset{\circ s}{\mu} \vee \overset{\circ s}{\eta} \leq \mu \overset{\circ s}{\vee} \eta$.

(iii) Since $1-\mu\wedge\eta = (1-\mu)\vee(1-\eta)$ then $\overline{1-\mu\wedge\eta}^s = \overline{(1-\mu)\vee(1-\eta)}^s$. By Theorem 3.3 $\overline{1-\mu\wedge\eta}^s = \overline{1-\mu}^s \vee \overline{1-\eta}^s$. Therefore $1 - (\overline{1-\mu\wedge\eta}^s) = 1 - (\overline{1-\mu}^s \vee \overline{1-\eta}^s)$, i.e. $1 - (\overline{1-\mu\wedge\eta}^s) = (1 - \overline{1-\mu}^s) \wedge (1 - \overline{1-\eta}^s)$. Hence,

$$\begin{aligned}(\mu\wedge\eta)\wedge[1-(\overline{1-\mu\wedge\eta}^s)] &= (\mu\wedge\eta)\wedge[(1-\overline{1-\mu}^s)\wedge(1-\overline{1-\eta}^s)] \\ &= [\mu\wedge(1-\overline{1-\mu}^s)]\wedge[\eta\wedge(1-\overline{1-\eta}^s)].\end{aligned}$$

Thus, $\overset{\circ s}{\mu} \wedge \overset{\circ s}{\eta} = \mu \overset{\circ s}{\vee} \eta$.

□

Definition 3.4. *Let ν a subset in I^X. ν is said to be a fuzzy s-neighborhood of $p(x,t)$, if there is a fuzzy s-open subset μ, such that $p \in \mu$ and $\mu \leq \nu$.*

Theorem 3.5. *(i) Each fuzzy closed (resp. fuzzy open) set is fuzzy s-closed (resp. fuzzy s-open).*

(ii) μ is fuzzy s-open if, and only if $\overset{\circ s}{\mu} = \mu$.

Proof. (i) Let μ a fuzzy closed subset of I^X. By Theorem 3.2 $\mu \leq \overline{\mu}^s \leq \overline{\mu}$. Since $\overline{\mu} = \mu$ then $\overline{\mu}^s = \mu$ i.e. μ is s-closed. If μ is fuzzy open subset of I^X then $1-\mu$ is fuzzy closed, that is fuzzy s-closed. So, μ is s-open.
(ii) Since μ is fuzzy s-open, then $1-\mu$ is fuzzy s-closed. Therefore, $\overset{\circ s}{\mu} = \mu \wedge (1-(1-\mu))$. Thus, By Theorem 2.1, $\overset{\circ s}{\mu} = \mu$.

Conversely, let $\overset{\circ s}{\mu} = \mu = \mu \wedge (1-\overline{1-\mu}^s)$. Then,

$$\begin{aligned} 1-\mu &= 1-[\mu \wedge (1-\overline{1-\mu}^s)] \\ &= (1-\mu) \vee [1-(1-\overline{1-\mu}^s)] \\ &= (1-\mu) \vee \overline{1-\mu}^s \end{aligned}$$

Therefore, $\overline{1-\mu}^s \leq 1-\mu$. Hence, $\overline{1-\mu}^s = 1-\mu$, i.e. μ is fuzzy s-open. □

Remark 3.1. *A fuzzy subset ν of X is fuzzy s-open if, and only if, ν is a fuzzy s-neighbourhood of each of its point.*

Theorem 3.6. *Let μ be a fuzzy subset in I^X. Then, we have the following inclusions:*

$$\overset{\circ}{\mu} \leq \overset{\circ s}{\mu} \leq \mu \leq \overline{\mu}^s \leq \overline{\mu}.$$

Proof. For $\mu \leq \overline{\mu}^s \leq \overline{\mu}$ see Theorem 3.2. By [2, Definition 2.4] $\overset{\circ}{\mu} \leq \mu$. Then, $\overline{1-\mu}^s \leq \overline{1-\overset{\circ}{\mu}}^s$. Therefore, $1-\overline{1-\overset{\circ}{\mu}}^s \leq 1-\overline{1-\mu}^s$. Hence, $\mu \wedge (1-\overline{1-\overset{\circ}{\mu}}^s) \leq \mu \wedge (1-\overline{1-\mu}^s)$. Thus, $\overset{\circ}{\mu} \leq \overset{\circ s}{\mu}$. Furthermore, by definition of $\overset{\circ s}{\mu}$, $\overset{\circ s}{\mu} \leq \mu$. So, $\overset{\circ}{\mu} \leq \overset{\circ s}{\mu} \leq \mu \leq \overline{\mu}^s \leq \overline{\mu}$. □

Theorem 3.7. *Let μ be a fuzzy subset in I^X. The following assertions are equivalent:*

(i) μ is fuzzy s-closed.

(ii) $\mu = \wedge\{\kappa/\kappa$ is s-closed and $\mu \leq \kappa\}$.

Proof. $(i) \Rightarrow (ii)$ It is obvious.
$(ii) \Rightarrow (i)$ Suppose that $\mu = \wedge\{\kappa/\kappa$ is s-closed and $\mu \leq \kappa\}$. Let show that μ is fuzzy s-closed. It is sufficient to show that $\overline{\mu}^s \leq \mu$. Let $p(x,t)$ a fuzzy point which is not in μ. Then, there exists a fuzzy s-closed fuzzy subset κ such that $\mu \leq \kappa$ and $p \notin \kappa$. Suppose towards a contradiction that $p \in \overline{\mu}^s$, so there exists a sequence (p_n) of fuzzy points in μ such that $p_n \underset{n\to+\infty}{\longrightarrow} p$. Since $\mu \leq \kappa$, then (p_n) is also in κ. Therefore, $p \in \kappa$, which is not the case. Thus, $p \in \overline{\mu}^s$. □

Corollary 3.8. *Let μ be a fuzzy subset in I^X. The following assertions are equivalent:*

(i) μ is fuzzy s-open.

(ii) $\mu = \vee\{\zeta/\zeta$ is s-open and $\zeta \leq \mu\}$.

Proof. $(i) \Rightarrow (ii)$ It is obvious.
$(ii) \Rightarrow (i)$ If $\mu = \vee\{\zeta/\zeta$ is s-open and $\zeta \leq \mu\}$, then $1-\mu = \wedge\{1-\zeta/1-\zeta$ is s-closed and $1-\mu \leq 1-\zeta\}$. By previous Theorem, $1-\mu$ is s-closed. Hence, μ is s-open. □

Theorem 3.9. *Let τ^s be the subfamily of I^X defined by*

$$\tau^s = \{\mu \in I^X / \text{ for all sequence } (p_n)_n \text{ in } 1-\mu \text{ such that } p'_n \underset{n\to+\infty}{\longrightarrow} p'$$

$$\text{then } p \text{ is in } 1-\mu\},$$

i.e. τ^s is the collection of all fuzzy s-open sets in X, it is a fuzzy topology on X with $\tau^s \succeq \tau$. We call it the fuzzy sequential topology.

Proof. Let $\mu, \eta \in \tau^s$ and $(\mu_i)_{i\in I}$ a family in τ^s. It is obvious that $c_0, c_1 \in \tau^s$. Since $(\mu \wedge \eta)^c = \mu^c \vee \eta^c$, then by Theorem 3.3 and Theorem 3.7 $\mu^c \vee \eta^c$ is fuzzy s-closed, so $\mu \wedge \eta \in \tau^s$. On the other hand, $(\underset{i\in I}{\vee} \mu_i)^c = \underset{i\in I}{\wedge} \mu_i^c$. By the same Theorems once again $\underset{i\in I}{\wedge} \mu_i^c$ is fuzzy s-closed, so $\underset{i\in I}{\vee} \mu_i \in \tau^s$.

Therefore τ^s is a fuzzy topology on X. Now, let $\mu \in \tau$, μ^c is fuzzy closed in X. Therefore, by Theorem 3.5, it is fuzzy s-closed i.e. $\mu \in \tau^s$. Then $\tau^s \succeq \tau$. □

Theorem 3.10. *Let (p_n) be a sequence of fuzzy point in X and p is fuzzy point in X. If, (p_n) is τ^s-convergent to p, then it is also τ-convergent to p.*

Proof. Since $\tau^s \succeq \tau$, the result is true. □

Theorem 3.11. *Let (X, τ), (Y, ρ) be fuzzy topological spaces and $f : X \to Y$ a function. If f is fuzzy continuous. Then, f is fuzzy sequentially continuous.*

Proof. Suppose that f is fuzzy continuous. Let (p_n) be a sequence of fuzzy points in X and p a fuzzy point in X such that $p_n \underset{n\to+\infty}{\longrightarrow} p$ in (X, τ). We must show that $f(p_n) \underset{n\to+\infty}{\longrightarrow} f(p)$ in (X, ρ). Let $\mu \in \rho$ such that $f(p) \in \mu$. By fuzzy continuity of f, $f^{-1}(\mu) \in \tau$, we denote $f^{-1}(\mu)$ by η. Therefore, $p \in \eta$. On the other hand, for all $y \in Y$:

$$\begin{aligned} f(\eta)(y) &= \sup_{x\in f^{-1}(y)} \eta(x) \quad \text{(Definition 2.10)} \\ &= \sup_{x\in f^{-1}(y)} f^{-1}(\mu)(x) \\ &= \sup_{x\in f^{-1}(y)} \mu(f(x)) \\ &= \begin{cases} \mu(y), \text{ if } y = f(x) \\ 0, \text{ if not} \end{cases} \end{aligned}$$

Then, $f(\eta)(y) \leq \mu(y)$, for all $y \in Y$. Hence, $f(\eta) \leq \mu$. By hypothesis, $p_n \underset{n\to+\infty}{\longrightarrow} p$, i.e. there is $N \in \mathbf{N}$, for all $n > N$, $p_n \in \eta$. Furthermore, we have $f(\eta)(y) = \sup_{x\in f^{-1}(y)} \eta(x)$. Then, for all $n > N$, $f(\eta)(f(x_n)) = \sup_{z\in f^{-1}\circ f(x_n)} \eta(z)$. Therefore, for all $n > N$, $f(\eta)(f(x_n)) > 0$. Hence, for all $n > N$, $f(p_n) \in f(\eta) \leq \mu$, i.e. $f(p_n) \in \mu$, for all $n > N$. Thus, $f(p_n) \underset{n\to+\infty}{\longrightarrow} f(p)$ in (X, τ). □

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