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A new proof of Fillmore's theorem for integer matrices

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Abstract

Fillmore's theorem is a matrix completion problem that states that if A is a nonscalar matrix over a field $\mathbf{F}$ and $\gamma_1, \ldots, \gamma_n \in \mathbf{F}$ so that $\gamma_1 + \ldots + \gamma_n = tr(A)$ then there is a matrix similar to A with diagonal $(\gamma_1, \ldots, \gamma_n)$. Borobia [1] extended Fillmore's Theorem to the matrices over the ring of integers and Soto, Julio and Collao [3] studied it with the nonnegativity hypothesis. In this paper we prove the same result by modifying the initial proof of Fillmore, a subsequent new algorithm is proposed and some new information about the final matrix will be given.

Keywords: *Inverse problem, Smilarity, Diagonal, Integer matrix, Fillmore*

AMS subject classifications: *15A18, 15A29.*

1. Introduction

Fillmore stated a theorem in 1969, commonly known as Fillmore's theorem. Some years after, Zhan [4] gave another proof to this theorem including some modifications since the original one has some inaccuracy.

Theorem 1.1. *(Fillmore's theorem)[2] Let A be a nonscalar matrix of order n over a field $\mathbf{F}$ and let $\{\gamma_1, \ldots, \gamma_n\} \in \mathbf{F}$ such that $\sum_{i=1}^{n} \gamma_i = tr(A)$. Then there is a matrix similar to A with diagonal $(\gamma_1, \ldots, \gamma_n)$.*

Proof. The proof is by induction on the size of A and it is based on the next result.

Lemma 1.1. *[1] Let A be a nonscalar matrix of order $n \geq 3$ over a field $\mathbf{F}$ and let $\gamma \in \mathbf{F}$. Then there is a nonsingular matrix, P, such that*

$P^{-1}AP = \begin{pmatrix} \gamma & * \\ * & A_1 \end{pmatrix}$, *where A_1 is a nonscalar matrix of order $n-1$.*

Moreover, $(P^{-1}AP)_{23} = 1$.

The proof of this lemma uses an argument based on a change of basis of the matrix A. The new basis used is $\{x, Ax - \gamma x, x_3, \ldots, x_n\}$, where x and Ax are linearly independent. If A is not diagonal, we can take x as one of the standard vectors. If it is diagonal, x can be sum of two standard basis vectors. The matrix P is the identity except on its entry $p_{13} = 1 - \alpha_{23}$, where α_{23} is the entrance (2,3) of A in the new basis.

By applying this lemma repeatedly to the similar matrix so that the pair $\{x, Ax - \gamma x\}$ is linearly independent, we get to proof the theorem. $\square$

Here, we can highlight the importance that the change of basis matrix has in Fillmore's proof. Taking this into account, if we take the change of basis matrix in a certain way and we introduce a small variation at the beginning, then the similar matrix that results from the theorem has all its entries in $\mathbf{Z}$ and we can extend Fillmore's proof to integer matrices.

2. Fillmore's theorem for integer matrices

Fillmore's theorem have been studied in different moments. In this paper, we are going to present a new proof of Fillmore's theorem to conclude that the final matrix is over $\mathbf{Z}$ and give some more information about the final matrix. Borobia [1] also gave an algorithm that extended the theorem to integer matrices but, in this case, we are not going to give an alternative algorithm as he did but introduce a small variation at the beginning of Fillmore's proof and then use the same argument that Fillmore.

This problem was also studied by Soto, Julio and Collao [3], considering the nonnegativity hypothesis, that was not considered in the results of Fillmore and Borobia and giving an alternative algorithm. In this paper, we will follow Fillmore's initial proof (except for a small variation at the beginning) and we will give some new information about the final matrix.

Tan [5] extended Fillmore's theorem to factorial rings, although it is not proved that the matrix P^{-1} is also over the initial factorial ring, but over it's field of fractions. If Fillmore's theorem can be extended to an integral domain is a problem that has not been solved yet (see Remark 2.3 in [5]).

As it is said, the change of basis matrix is important in Fillmore's proof. If the initial matrix, A, is over $\mathbf{Z}$ and the change of basis matrix, B, is such that $\det(B) = \pm 1$, then the matrix A in the new basis is going to be also over $\mathbf{Z}$. This way, since we have by construction of P that $\det(P) = \pm 1$, then the similar matrix $P^{-1}AP$ will be over $\mathbf{Z}$.

The main point we want to prove is that, if $a_{12} = \pm 1$, then we can obtain an integer matrix after applying Fillmore's argument. After this, we'll prove it for matrices where $a_{12} \notin \{\pm 1\}$. This way, whether for Borobia it is necesary to have an off-diagonal entry equal to 1, to continue Fillmore's argument, we'll need it to be in position (1,2) or (2,1). We'll also admit value -1 in these positions.

Without any loss of generality, we can suppose that $x = e_2$.

As we need the matrix in the new basis $(B^{-1}AB)$ to be over $\mathbf{Z}$, $\det(B) = \pm 1$ so that the inverse matrix, B^{-1}, is also an integer matrix. Since $\det(B) = -a_{12}$, we can conclude that $a_{12} = \pm 1$.

Theorem 2.1. *(Fillmore's theorem for integer matrices) Let A be a nonscalar and nondiagonal matrix of order $n \geq 3$ over $\mathbf{Z}$ and let $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbf{Z}^n$. Then, there exists a non singular matrix over $\mathbf{Z}$, C, similar to A, with diagonal γ and with $C_{nj} = 1 \; \forall j \neq n$.*

Proof. We will start proving the first part of the theorem and then, we will proof the new information about the matrix C, that is not present in the results of Fillmore, Borobia or Soto.

After knowing that if $a_{12} \in \{\pm 1\}$, the matrix A is over $\mathbf{Z}$, we'll give the new algorithm and proof the theorem distinguishing different cases:

- *Case 1*: A is a non diagonal matrix over $\mathbf{Z}$ with $a_{12} \in \{\pm 1\}$ or $a_{21} \in \{\pm 1\}$.

- *Case 2*: A is a non diagonal matrix over $\mathbf{Z}$ with $a_{12} \notin \{0, \pm 1\}$ and $a_{21} \notin \{\pm 1\}$.
- *Case 3*: A is a non diagonal matrix over $\mathbf{Z}$ with $a_{12} = 0$.
- *Case 4*: A is a diagonal matrix.

Case 1: Since we want to follow Fillmore's theorem argument, the change of basis is given by the matrix $\{x, Ax - \gamma x, x_3, \ldots, x_n\}$. If we take x a vector of the standard basis, e_i, then, $Ax - \gamma x$ will be the column i of A except for the element $(Ax - \gamma x)_i$ that will be $a_{ii} - \gamma_i$.

As before, without any loss of generality, we can suppose that $x = e_2$. This way, the change of basis matrix
will have $\det(B) = \mp 1$, so we can conclude that B^{-1} is an integer matrix. Therefore, the matrix A in the new basis, $B^{-1}AB$, will also be an integer matrix.

If we take the matrix P as it appears in the initial proof, we have that P^{-1} is over $\mathbf{Z}$ because $\det(P) = 1$, so $P^{-1}AP$ is an integer matrix. Following the initial proof, we can conclude that $(P^{-1}AP)_{11} = \gamma_1$ and $(P^{-1}AP)_{23} = 1$.

Since after the first iteration we have $(P^{-1}AP)_{23} = 1$, we are under the initial conditions to continue the iterations and the proof of the theorem will follow by induction over the different submatrices.

Observation 2.1. *If we have $a_{21} \in \{\pm 1\}$ rather than $a_{12} \in \{\pm 1\}$, then we can use the same argument but, in this case the first iteration must be done with the standard basis vector $x = e_1$ instead of e_2. After, as we have in position (2,3) the value 1, we can continue using the equivalent standard basis vector to $x = e_2$ in the rest of the iterations submatrices.*

Observation 2.2. *The algorithm of this case is exactly the same as Fillmore's one, but by choosing the change of basis matrix in this way, we have the theorem proved for integer matrices. Now, we'll study the rest of cases and try to modify them so that we reduce them to this one and we don't change Fillmore's main argument.*

Case 2: As we said before, the small variation we introduce is made so that $a_{12} \in \{\pm 1\}$. This way, we are in the previous case. For Borobia, the important thing is that the matrix has an off-diagonal element with value 1 but for us, we need it to be in position (1,2) or (2,1). To do so, we'll use the lemma that Borobia uses in [1].

Lemma 2.1. *([1]) Let $A = (a_{ij})_{i,j=1}^n$ be a non scalar matrix over* $\mathbf{Z}$*. Then, there exists an integer matrix similar to A with a non diagonal element equal to 1.*

Observation 2.3. *Although the lemma doesn't specify the position (1,2) of the non diagonal element equal to 1, after applying the algorithm proposed, we end with a 1 in position (1,2).*

Case 3: Now we are going to proof the theorem when $a_{12} = 0$. This case is motivated because the previous lemma assumes that $a_{12} \neq 0$. If we apply a transformation over A taking a permutation matrix, Q, so that we have a nonzero element in position (1,2), then we'll be under case 1 or case 2. We can do this because A is a non diagonal matrix.

Case 4: Now, we are going to proof Fillmore's theorem for integers matrices when the initial matrix, A, is a diagonal matrix. If we follow Fillmore's argument, and we want $\det(B) \in \{\pm 1\}$ so we continue being under $\mathbf{Z}$, we'll have

$$\det(B) = \pm 1 \Longleftrightarrow a_{11} - a_{22} = \pm 1 \text{ or if } \ a_{11} - \gamma_1 \text{ and } a_{22} - \gamma_2$$

$$\text{are multiples of } a_{11} - a_{22},$$

but this would add more conditions to the initial problem, and it's not the point we want to proof.

In Borobia's argument (see [1]), if the matrix is diagonal, he first apply the next lemma so that the matrix is no longer diagonal.

Lemma 2.2. *[1] Let A be a nonscalar diagonal matrix over a field* $\mathbf{F}$*, let s such that $a_{11} \neq a_{ss}$, and let B be equal to the identity except on its entry $b_{1s} = \frac{1}{a_{11} - a_{ss}}$. Then $B^{-1}AB$ is equal to A except on its entry (1,s) that is equal to 1.*

Applying this lemma and taking $s = 2$ (in case $a_{11} \neq a_{22}$), then we are in case 1, where the value 1 will be in position (1,2). If not, we can take $s \neq 2$ and after applying the lemma we'll be in case 3.

Now that we have proved the theorem, we are going to give some new information about the final matrix, C. We are going to proof that, if C is the final matrix, then $C_{nj} = 1 \ \forall j \neq n$.

Let $C = P^{-1}\tilde{A}P$, where $\tilde{A}$ is A in the new basis and P as in Fillmore's proof, i.e., the identity matrix with $P_{13} = 1 - \tilde{a}_{23}$. The proof is then made by induction over the different iterations. After the first iteration, we have

$$c_{21} = (P^{-1}\tilde{A}P)_{21} = \sum_{k=1}^{n}(P^{-1}\tilde{A})_{2k}P_{k1} = (P^{-1}\tilde{A})_{21} = \sum_{k=1}^{n}(P^{-1})_{2k}\tilde{A}_{k1} = \tilde{a}_{21}$$

Since $x = e_2$ the basis is $B = \{e_2, Ae_2 - \gamma_2, x_3, \ldots, x_n\}$, and we have

$$\begin{aligned} \tilde{a}_{21} &= (B^{-1}AB)_{21} = \textstyle\sum_{k=1}^{n}(B^{-1}A)_{2k}B_{k1} = (B^{-1}A)_{22} \\ &= \textstyle\sum_{k=1}^{n}(B^{-1}_{2k})A_{k2} = a_{12}a_{12} = 1 \end{aligned}$$

as $a_{12} \in \{\pm 1\}$.

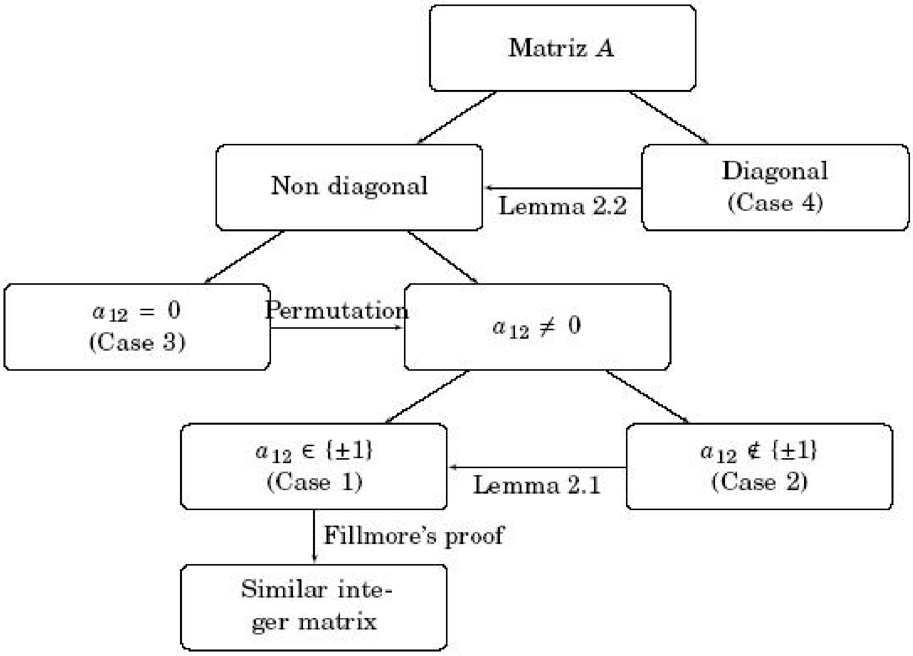
Now, we'll apply the induction hypothesis so that if, after k steps we have $a_{k+1,j} = 1\ \forall j = 1, \ldots, k$, and $a_{k+1,k+2} = 1$, then $a_{k+2,j} = 1\ \forall j = 1, \ldots, k+1$. Since here $j \neq k+3$, and taking into account that now, we have $x = e_{k+2}$ we can conclude that

$$\begin{aligned} c_{k+2,j} &= (P^{-1}\tilde{A}P)_{k+2,j} = \textstyle\sum_{i=1}^{n}(P^{-1}\tilde{A})_{k+2,i}P_{ij} = (P^{-1}\tilde{A})_{k+2,j} \\ &= \textstyle\sum_{i=1}^{n}(P^{-1})_{k+2,i}\tilde{A}_{ij} = \\ &= \tilde{a}_{k+2,j} = (B^{-1}AB)_{k+2,j} = \textstyle\sum_{i=1}^{n}(B^{-1}A)_{k+2,i}B_{ij} = \end{aligned}$$

$$= \begin{cases} j \leq k \\ \qquad (B^{-1}A)_{k+2,j} = \sum_{i=1}^{n}(B^{-1})_{k+2,i}A_{ij} = \\ \qquad\qquad = a_{k+1,k+2}a_{k+1,j} = 1 \\ j = k+1 \\ \qquad (B^{-1}A)_{k+2,k+2} = \sum_{i=1}^{n}(B^{-1})_{k+2,i}A_{i,k+2} = \\ \qquad\qquad = a_{k+1,k+2}a_{k+1,k+2} = 1 \end{cases}$$

□

3. Flowchart



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References

[1] A. Borobia, "Fillmore's Theorem for integer matrices", *Linear Algebra and its Applications*, vol. 531, pp. 281-284, 2017. doi: 10.1016/j.laa.2017.06.011

[2] P. A. Fillmore, "On similarity and the diagonal of a matrix", *The American Mathematical Monthly*, vol. 76, no. 2, pp. 167-169, 1969. doi: 10.2307/2317264

[3] R. L. Soto, A. I. Julio, M. A. Collao, “Brauer’s theorem and nonnegative matrices with prescribed diagonal entries”, *Electronic Journal of Linear Algebra,* vol. 35, pp 53-64, 2019. doi: 10.13001/1081-3810.3886

[4] X. Zhan, *Matrix Theory*. Providence, RI: American Mathematical Society, 2013.

[5] Y.-J. Tan, “Fillmore’s theorem for matrices over factorial rings”, *Linear and Multilinear Algebra,* vol. 68, no. 3, pp. 563-567, 2018. doi: 10.1080/03081087.2018.1509045

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