Proyecciones Journal of Mathematics
Vol. 42, N^o 5, pp. 1191-1210, October 2023.
Universidad Católica del Norte
Antofagasta - Chile

doi 10.22199/issn.0717-6279-5317

Nonlocal partial fractional evolution equations with state dependent delay

Nardjis Lachachi-Merad
Djillali Liabes University, Algeria
Selma Baghli-Bendimerad
Djillali Liabes University, Algeria
Mouffak Benchohra
Djillali Liabes University, Algeria
and
Erdal Karapınar
Çankaya University, Turkey
Received : January 2022. Accepted : June 2023

Abstract

In this work, we propose sufficient conditions guaranteeing an existence result of mild solutions by using the nonlinear Leray-Schauder alternative in Banach spaces combined with the semigroup theory for the class of Caputo partial semilinear fractional evolution equations with finite state-dependent delay and nonlocal conditions.

Mathematics Subject Classification (2020): *34G20; 37G05; 37L05; 34K37; 74H20.*

Key words and phrases: *Functional evolution equations; Caputo fractional derivative; State dependent delay; Nonlocal conditions; Existence of solution; Semigroup theory; Nonlinear alternative.*

1. Introduction

In this paper, by using the Leray-Schauder alternative in a real Banach space $(E, |\cdot|)$ combined with the semigroup theory we establish an existence result of mild solutions for the following partial functional differential evolution equations with finite state-dependent delay involving the Caputo fractional derivative

$$(1.1)\quad {}^cD_0^\alpha y(t) = A(t)y(t) + f\left(t, y_{\rho(t,y_t)}\right), \quad \text{a. e. } t \in J := [0,b],\ 0 < \alpha < 1,$$

$$(1.2)\qquad y(t) + h_t(y) = \varphi(t), \quad t \in H := [-r, 0],$$

where $f : J \times C(H,E) \longrightarrow E$; $\rho : J \times C(H,E) \longrightarrow [-r,b]$; $\varphi \in C(H,E)$ and $h_t : C(H,E) \longrightarrow E$ are given functions ; ${}^cD_0^\alpha$ is the Caputo fractional derivative of order $\alpha \in (0,1)$ and $\{A(t)\}_{t\in J}$ is a family of operators from E into E which are linear, closed and not necessarily bounded.

For any continuous function y and any $t \in J$, we denote by y_t the element of E given by

$$y_t(\theta) = y(t+\theta) \quad \text{for } \theta \in H.$$

Functional differential equations of integer order arise in various areas of applied mathematics and other equations have received already much attention in recent years. The first appearance of a fractional derivative is in a letter written to De l'Hopital by Leibniz in 1695. Then, it has developed by Euler, Fourier, Liouville, Riemann, and so on. Recently, various phenomena in many fields of science and engineering are valuably modeled by differential equations of fractional order. Some numerous applications could be found in viscoelasticity, electromagnetism, control, electrochemistry, porous media, etc. see the works of Kilbas *et al.* [19], Miller and Ross [25], Podlubny [28, 29] and Samko *et al.* [30]. In recent years, there has been a significant development in fractional ordinary and partial differential equations by Benchohra and his collaborators [6], El Borai [14], El-Sayed [15] and Zhou *et al.* [31].

Firstly, Byszewski initiated the study of the nonlocal Cauchy partial functional evolution problem in [12]. Li *et al.* looked on controllability for nonlocal evolution inclusions in [20, 21], the nonlocal equations were studied by Liang *et al.* in [22], and Benchohra *et al.* considered several classes of problems with nonlocal conditions in [11].

Baghli *et al.* give existence, uniqueness and controllability results for mild solutions also in Fréchet spaces on semi-infinite interval of first class partial functional evolution equations and inclusions and neutral functional ones with delay in [2, 7, 8, 9]. Equations with delay which depend on the state have been proposed in modeling. Existence results were derived from semilinear functional differential equations in this case. Many results were derived recently for different functional differential equations with delay whose solution is depending on the delay and defined on a bounded interval as is proven in [1], and for an unbounded interval as developed by Baghli *et al.* in [4, 5, 10, 23]. Fractional non autonomous evolution equations in Fréchet spaces has been investigated by Mesri *et al.* in [24].

So in this paper, we give the existence of solutions for the Caputo's fractional semilinear differential equations with finite state-dependent delay in Banach spaces. Our results are based upon fixed point techniques combined with the semigroup theory. After preliminaries in Section 2, we give our main result about existence of mild solution of the problem (1.1)-(1.2) in Section 3. In Section 4, an example is given to illustrate the abstract theory.

2. Preliminaries

This section introduces notation, definitions, and fundamental facts that will be employed throughout this study.

Let $C(J;E)$ be the space of functions from J into E that are continuous with the norm $|.|$ and $B(E)$ be the space of linear bounded operators from E into E with the usual supremum norm

$$\|N\|_{B(E)} = \sup \{ \ |N(y)| \ : \ |y| = 1 \ \}.$$

A measurable function $y : J \to E$ is Bochner integrable if and only if $|y|$ is Lebesgue integrable. $L^1(J,E)$ denotes the Banach space of measurable functions $y : J \to E$ that are Bochner integrable normed by

$$\|y\|_{L^1} = \int_0^b |y(t)| \ dt.$$

Definition 2.1. *A function $f : J \times E \to E$ is said to be Carathéodory if it satisfies:*

(i) *for almost each $t \in J$, the function $f(t, \cdot) : E \to E$ is continuous;*

(ii) *for each $y \in E$, the function $f(\cdot, y) : J \to E$ is measurable;*

(iii) *for every positive integer k, there exists a function $\ell_k \in C(J; \mathbf{R}^+)$ such that*

$$|f(t, y)| \leq \ell_k(t)$$

for every $|y| \leq k$ and almost every $t \in J$.

The nonlocal condition $y(t) + h_t(y) = \varphi(t)$ for $t \in H$ is a physical application with better effect than the classical initial condition $y(0) = y_0$. $h_t(y)$, for example it can be given by

$$h_t(y) = \sum_{i=1}^{p} c_i \; y(t_i + t), \quad t \in H,$$

where $c_1, c_2, \ldots, c_p$ are given constants and $0 < t_1 < \ \ldots \ < t_p < b$.

In particular, at the initial time $t = 0$, we have

$$h_0(y) = \sum_{i=1}^{p} c_i \; y(t_i).$$

Assume that the function $\rho : J \times C(H; E) \to [-r, b]$ is continuous. Additionally, we introduce here the following hypothesis involving the set

$$\mathcal{R}(\rho^-) = \{\rho(s, \varphi) : (s, \varphi) \in J \times C(H; E), \; \rho(s, \varphi) \leq 0\}$$

(H_φ) The function $t \to \varphi_t$ from $\mathcal{R}(\rho^-)$ into $C(H; E)$ is continuous, and there exists a continuous and bounded function $\mathcal{L}^\varphi : \mathcal{R}(\rho^-) \to (0, +\infty)$ such that

$$\|\varphi_t\| \leq \mathcal{L}^\varphi(t) \|\varphi\|, \text{ for every } t \in \mathcal{R}(\rho^-).$$

Remark 2.2. *Continuous and bounded functions satisfy (H_φ) (see [1, 18]).*

Lemma 2.3. *([18], Lemma 2.4) If a function $y : H \cup J \to E$ is such that $y_0 = \varphi$, then*

$$\|y_s\| \leq \mathcal{L}^{\varphi}\|\varphi\| + \sup_{0\leq\theta\leq\hat{s}}\{|y(\theta)| \;\; s \in \mathcal{R}\left(\rho^{-}\right) \cup J\}, \;\; \hat{s} := \max(0; s),$$

where $\mathcal{L}^{\varphi} = \sup\limits_{t\in\mathcal{R}(\rho^{-})} \mathcal{L}^{\varphi}(t)$.

Proposition 2.4. *[4] If a function $y : [-r, b] \to E$ is such that $y|_H = \varphi$ and satisfies the condition (H_{φ}), then we have*

(2.1) $\quad \|y_{\rho(t,y_t)}\| \leq |y(t)| + \mathcal{L}^{\varphi}\|\varphi\|,$ *for every $t \in J$ and $\rho \in C(H, E)$.*

We give here fractional order derivative definitions.

Definition 2.5. *[19, 28] The fractional integral operator of Riemann-Liouville for the order $\alpha > 0$ of a function $f : \mathbf{R}^{+} \longrightarrow \mathbf{R}$ is defined as*

$$I_0^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds$$

where $\Gamma(z) = \int_0^{+\infty} t^{z-1}e^{-t}dt$ is the Euler's gamma function.

For all $\alpha > 0$, the integral $I^{\alpha}f$ exists when $f \in C\left(\mathbf{R}^{+}\right) \cap L^1_{loc}\left(\mathbf{R}^{+}\right)$. Notice that, when $f \in C\left(\mathbf{R}^{+}\right)$, then $I^{\alpha}f \in C\left(\mathbf{R}^{+}\right)$ and moreover $I^{\alpha}f(0) = 0$.

Definition 2.6. *[19, 28] The Caputo fractional derivative for order $\alpha > 0$ of a function $f : \mathbf{R}^{+} \longrightarrow \mathbf{R}$ is defined by*

$$\frac{d^{\alpha}f(t)}{dt^{\alpha}} = \frac{1}{\Gamma(m-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{m-\alpha-1} f(s)ds = \frac{d}{dt} I_0^{1-\alpha} f(t).$$

where $m = [\alpha] + 1$. Here $[\alpha]$ denotes the integer part of α.

In what follows, let us consider the family $\{A(t)\}_{t\in J}$ of linear, closed, bounded, and densely defined operators on a Banach space E with domain $D(A(t))$ which is independent of t. Additionally, we suppose that $A(t)$ verifies the following hypotheses (see [13], for more details):

(A_1) For any λ with $Re(\lambda) \geq 0$, the operator $[\lambda I - A(t)]$ exists, and there exists a bounded inverse $[\lambda I - A(t)]^{-1} \in B(E)$ such that

$$\left\|[\lambda I - A(t)]^{-1}\right\| \leq \frac{M}{|\lambda| + 1}$$

where M is a positive constant, independent of t and λ.

(A_2) For any $t, \tau, s \in I$, there exist constants $\gamma \in (0, 1]$ and $C > 0$ such that

$$\left\|[A(t) - A(\tau)]\, A^{-1}(s)\right\| \leq C|t - \tau|^{\gamma}$$

where the constant C are independent of t, τ and s.

Remark. From Henry [17], Temam [26] and Pazy [27], we know that (A_1) means that for each $s \in I$, the operator $A(s)$ generates an analytic semigroup $e^{-tA(s)}$ $(t > 0)$, and there exists a positive constant M, independent of both t and s, such that

$$\| - A(s)e^{tA(s)}\| \leq \frac{M}{t}, \text{ for } t > 0 \text{ and } s \in J.$$

Definition 2.7. *[13] Define the operators $\Psi(t, s)$, $\phi(t, s)$ and $U(t)$ by*

$$\Psi(t, s) = \alpha \int_0^{+\infty} \theta t^{\alpha-1} \xi_\alpha(\theta) e^{t^\alpha \theta A(s)} d\theta, \tag{2.2}$$

$$\phi(t, s) = \sum_{k=1}^{+\infty} \phi_k(t, s) \tag{2.3}$$

and

$$U(t) = A(t)A^{-1}(0) - \int_0^t \phi(t, s)A(s)A^{-1}(0)ds, \tag{2.4}$$

where ξ_α the function of probability density defined on $[0, +\infty)$ whose Laplace transform is given by

$$\int_0^{+\infty} \xi_\alpha(\theta)e^{\theta x} d\theta = \sum_{i=1}^{+\infty} \frac{(-x)^i}{\Gamma(1 + \alpha i)} \quad 0 < \alpha \leq 1,\ x > 0,$$

$$\phi_1(t, s) = [-A(t) + A(s)]\Psi(t - s, s),$$

and

$$\phi_{k+1}(t, s) = \int_s^t \phi_k(t, \tau)\phi_1(\tau, s)d\tau, \quad k = 1, 2, \ldots.$$

For more details about the definition and property of the probability density function and the property below, one can see [16].

Theorem 2.8. *(Leray-Schauder Nonlinear Alternative). Let X be a Banach space, C be a convex and closed subset of E, U an open subset of C, and $0 \in U$. Suppose that $N(Y) : \bar{U} \to C$ is continuous and compact map. Then either,*
$(LS1)$ N has a fixed point ; or
$(LS2)$ There exist $\lambda \in [0,1)$ and $x \in \partial U$ such that $x = \lambda N(x)$.

3. Existence of mild solution

We will use the following definition of mild solutions for the nonlocal problem (1.1)-(1.2).

Definition 3.1. *A continuous function $y(\cdot) : [-r,b] \to E$ is called a mild solution of the problem (1.1)-(1.2), if y satisfies for each $t \in J$ the following integral equation*

$$\begin{aligned} y(t) &= [\varphi(0) - h_0(y)] - \int_0^t \Psi(t-s,s)U(s)A(0)\,[\varphi(0) - h_0(y)]\,ds \\ &+ \int_0^t \Psi(t-s,s) f\left(s, y_{\rho(s,y_s)}\right) ds \\ &+ \int_0^t \int_0^s \Psi(t-s,s)\phi(s,\tau) f\left(\tau, y_{\rho(\tau,y_\tau)}\right) d\tau ds. \end{aligned} \tag{3.1}$$

In what follow, we give some properties concerning the operators Ψ, ϕ and U used later in our argument.

Lemma 3.2. *[13] The functions $\Psi(t-s,s)$ and $A(t)\Psi(t-s,s)$ are continuous in uniform topology, where $t \in J$, $0 \le s \le t - \epsilon$ for any $\epsilon > 0$ and*

$$\|\Psi(t-s,s)\| \le C(t-s)^{\alpha-1}, \tag{3.2}$$

where C is a positive constant, which is independent, of both t and s and α is a positive constant. Furthermore,

$$\|\phi(t,s)\| \le C(t-s)^{\gamma-1} \tag{3.3}$$

and

$$\|U(t)\| \leq C\left(1+t^{\gamma}\right). \tag{3.4}$$

where γ is a positive constant.

Set $\Theta_{\gamma}=\alpha^{-1}+b^{\gamma}\beta(\alpha,\gamma+1)$ and $\Upsilon=\alpha^{-1}+C\gamma^{-1}b^{\gamma}\beta(\alpha,\gamma+1)$ where $\beta(\alpha,\gamma)=\int_0^1 t^{\alpha-1}(1-t)^{\gamma-1}dt$ is the beta Euler's function.

In order to obtain the existence of mild solutions for problem (1.1)-(1.2), we suppose the following hypotheses

$(H1)$ The function f is Carathéodory.

$(H2)$ For all $R>0$, there exists $l_R \in L^{+\infty}(J;\mathbf{R}_+)$ such that

$$|f(t,u)| \leq f(t,0)+l_R(t)\|u\|$$

for almost ever $t \in J$ and each $u \in C(H,E)$ with $\|u\| \leq R$.
Also denote $f^* := ess\sup_{t\in J} f(t,0)$ and $l_R^* := \|l_R\|_{L^\infty}$.

$(H3)$ For all $R>0$, there exists a constant $\sigma>0$ such that

$$|h_t(u)| \leq \sigma$$

for all $t \in H$ and all $u \in C(H,E)$ with $\|u\| \leq R$.

Corollary 3.3. *From Proposition 2.4, if a function $y:[-r,b]\to E$ is such that $y(t)=\varphi(t)-h_t(y)$ satisfying the condition (H_φ), then we have*

$$\|y_{\rho(t,y_t)}\| \leq |y(t)|+\mathcal{L}_h^{\varphi}(\|\varphi\|+\sigma) \tag{3.5}$$

where $\mathcal{L}_h^{\varphi}=\sup_{t\in\mathcal{R}(\rho^-)}\mathcal{L}^{\varphi(t)-h_t(\cdot)}$.

Then we can give now our result.

Theorem 3.4. *Assume that (H_φ) and $(H1)-(H3)$ are satisfied, and $Cb^{\alpha}\Upsilon l_R^*<1$. Then the nonlocal fractional problem $(1.1)-(1.2)$ has at least one mild solution.*

Proof. Transform the problem (1.1) – (1.2) into a fixed point problem. Set $\Omega := C([-r,b];E)$ and consider the operator $N:\Omega\to\Omega$ defined by

$$\begin{aligned}(Ny)(t) &= [\varphi(0)-h_0(y)] - \int_0^t \Psi(t-s,s)U(s)A(0)\,[\varphi(0)-h_0(y)]\,ds\\ &+ \int_0^t \Psi(t-s,s)f\left(s,y_{\rho(s,y_s)}\right)ds\\ &+ \int_0^t\int_0^s \Psi(t-s,s)\phi(s,\tau)f\left(\tau,y_{\rho(\tau,y_\tau)}\right)d\tau ds.\end{aligned}$$

Clearly, all fixed points of the operator N are mild solutions of the nonlocal problem (1.1) – (1.2).

We prove that the operator N is continuous and is compact.

• N is a continuous operator. Let $(y_n)_{n\in\mathbf{N}}$ be a sequence such that $y_n\to y$. Then,

$$\begin{aligned}&|(Ny_n)(t)-(Ny)(t)| \le |-h_0(y_n)+h_0(y)|\\ &+\int_0^t |\Psi(t-s,s)U(s)A(0)\,[h_0(y_n)-h_0(y)]|\,ds\\ &+\int_0^t \left|\Psi(t-s,s)\left[f\left(s,y_{n\rho(s,y_{ns})}\right)-f\left(s,y_{\rho(s,y_s)}\right)\right]\right|ds\\ &+\int_0^t\int_0^s \left|\Psi(t-s,s)\phi(s,\tau)\left[f\left(\tau,y_{n\rho(\tau,y_{n\tau})}\right)-f\left(\tau,y_{\rho(\tau,y_\tau)}\right)\right]\right|d\tau ds.\end{aligned}$$

By Lemma 3.2 and $(H3)$, we have

$$\begin{aligned}&|(Ny_n)(t)-(Ny)(t)| \le \sigma\,\|y_n-y\|\\ &+\sigma C^2|A(0)|\int_0^t (t-s)^{\alpha-1}(1+s^\gamma)\,ds\ \ \|y_n-y\|\\ &+C\int_0^t (t-s)^{\alpha-1}ds\left\|f\left(\cdot,y_{n\rho(\cdot,y_\cdot)}\right)-f\left(\cdot,y_{\rho(\cdot,y_\cdot)}\right)\right\|\\ &+C^2\int_0^t (t-s)^{\alpha-1}\int_0^s (s-\tau)^{\gamma-1}d\tau ds\|f\left(\cdot,y_{n\rho(\cdot,y_{n\cdot})}\right)-f\left(\cdot,y_{\rho(\cdot,y_\cdot)}\right)\|.\end{aligned}$$

Since

$$\int_0^t (t-s)^{\alpha-1}ds = \alpha^{-1}t^\alpha,$$

$$\int_0^t (t-s)^{\alpha-1}(1+s^\gamma)ds = t^\alpha\left(\alpha^{-1}+t^\gamma\beta(\alpha,\gamma+1)\right)$$

and

$$\int_0^t (t-s)^{\alpha-1}\int_0^s (s-\tau)^{\gamma-1}d\tau ds = \gamma^{-1}t^{\alpha+\gamma}\beta(\alpha,\gamma+1),$$

we obtain

$$\begin{aligned}|\left(Ny_n\right)(t)-(Ny)(t)| &\leq \sigma\left[1+C^2|A(0)|t^\alpha\left(\alpha^{-1}+t^\gamma\beta(\alpha,\gamma+1)\right)\right]\|y_n-y\|\\ &+Ct^\alpha\left[\alpha^{-1}+C\gamma^{-1}t^\gamma\beta(\alpha,\gamma+1)\right]\left\|f\left(\cdot,y_{n\rho(\cdot,y_{n\cdot})}\right)-f\left(\cdot,y_{\rho(\cdot,y_\cdot)}\right)\right\|.\end{aligned}$$

Set

$$\Theta_\eta := \alpha^{-1}+b^\eta\beta(\alpha,\gamma+1)$$

and

$$\Upsilon := \alpha^{-1}+C\gamma^{-1}b^\gamma\beta(\alpha,\gamma+1),$$

to get for $t\leq b$

$$\begin{aligned}|\left(Ny_n\right)(t)-(Ny)(t)| &\leq \sigma\left[1+C^2|A(0)|b^\alpha\Theta_\gamma\right]\|y_n-y\|\\ &+ Cb^\alpha\Upsilon\left\|f\left(\cdot,y_{n\rho(\cdot,y_{n\cdot})}\right)-f\left(\cdot,y_{\rho(\cdot,y_\cdot)}\right)\right\|.\end{aligned}$$

Hence,

$$|\left(Ny_n\right)(t)-(Ny)(t)|\to 0 \text{as} n\to+\infty.$$

Then N is a continuous operator.

- N maps bounded sets into bounded sets in Ω.

We show that for each bounded set $B\subset\Omega$, $N(B)$ is a bounded set, i.e. there exists κ a positive constant such that $|y(t)|\leq\kappa$ implies that there exists ε a positive constant such that $|Ny(t)|\leq\varepsilon$ for each $t\in J$.

Using Lemma 3.2 and the hypothesis $(H2)$, we obtain

$$\begin{aligned}|(Ny)(t)| &\leq |\varphi(0)-h_0(y)|\\ &+\int_0^t |\Psi(t-s,s)U(s)A(0)[\varphi(0)-h_0(y)]|\,ds\end{aligned}$$

$$
\begin{aligned}
&+\int_0^t \left|\Psi(t-s,s)f\left(s,y_{\rho(s,y_s)}\right)\right| ds\\
&+\int_0^t\int_0^s \left|\Psi(t-s,s)\phi(s,\tau)f\left(\tau,y_{\rho(\tau,y_\tau)}\right)\right| d\tau ds\\
&\leq (\|\varphi\|+|h_0(y)|)\left[1+C^2|A(0)|\int_0^t (t-s)^{\alpha-1}\left(1+s^\gamma\right)ds\right]\\
&+C\int_0^t (t-s)^{\alpha-1}\left[f(s,0)+l_R(s)\|y_{\rho(s,y_s)}\|\right]ds\\
&+C^2\int_0^t\int_0^s (t-s)^{\alpha-1}(s-\tau)^{\gamma-1}\left[f(\tau,0)+l_R(\tau)\|y_{\rho(\tau,y_\tau)}\|\right]d\tau ds.
\end{aligned}
$$

By hypothesis $(H3)$ and Corollary 3.3, then we get for $t \leq b$

$$
\begin{aligned}
|(Ny)(t)| &\leq (\|\varphi\|+\sigma)\left[1+C^2|A(0)|t^\alpha\left(\alpha^{-1}+t^\gamma\beta(\alpha,\gamma+1)\right)\right]\\
&+ C\int_0^t (t-s)^{\alpha-1}\left[f^*+l_R^*(|y(s)|+\mathcal{L}_h^\varphi[\|\varphi\|+\sigma])\right]ds\\
&+ C^2\int_0^t\int_0^s (t-s)^{\alpha-1}(s-\tau)^{\gamma-1}\times\\
&\qquad\times\left[f^*+l_R^*(|y(\tau)|+\mathcal{L}_h^\varphi[\|\varphi\|+\sigma])\right]d\tau ds\\
&\leq (\|\varphi\|+\sigma)\left[1+C^2|A(0)|t^\alpha\left(\alpha^{-1}+b^\gamma\beta(\alpha,\gamma+1)\right)\right]\\
&+ C\left[f^*+l_R^*(\kappa+\mathcal{L}_h^\varphi[\|\varphi\|+\sigma])\right]t^\alpha\left[\alpha^{-1}+C\gamma^{-1}t^\gamma\beta(\alpha,\gamma+1)\right]\\
&\leq (\|\varphi\|+\sigma)\left[1+C^2|A(0)|b^\alpha\Theta_\gamma\right]\\
&+ C\left[f^*+l_R^*(\kappa+\mathcal{L}_h^\varphi[\|\varphi\|+\sigma])\right]b^\alpha\left[\alpha^{-1}+C\gamma^{-1}b^\gamma\beta(\alpha,\gamma+1)\right]\\
&\leq \left(\left[1+C^2|A(0)|b^\alpha\Theta_\gamma\right]+Cl_R^*b^\alpha\Upsilon\mathcal{L}_h^\varphi\right)(\|\varphi\|+\sigma)\\
&+ Cb^\alpha\Upsilon\left(l_R^*\kappa+f^*\right):=\varepsilon.
\end{aligned}
$$

Then there exists a positive constant ε such that $|Ny(t)| \leq \varepsilon$ for each $t \in J$; so the operator N maps bounded sets into bounded sets in Ω.

• $N(B)$ is equicontinuous. Let $t_1, t_2 \in J$ such that $t_1 < t_2$ and let $y \in B$. By the hypothesis $(H3)$, we get

$$
|Ny(t_2)-Ny(t_1)| \leq |A(0)||\varphi(0)-h_0(y)|\times
$$

$$\begin{aligned}&\times\left[\int_0^{t_1}\left|\Psi\left(t_1-s,s\right)-\Psi\left(t_2-s,s\right)\right||U(s)|ds\right.\\&\quad\left.+\int_{t_1}^{t_2}\left|\Psi\left(t_2-s,s\right)\right||U(s)|ds\right]\\&+\int_0^{t_1}\left|\Psi\left(t_2-s,s\right)-\Psi\left(t_1-s,s\right)\right|\left|f\left(s,y_{\rho(s,y_s)}\right)\right|ds\\&+\int_{t_1}^{t_2}\left|\Psi\left(t_2-s,s\right)\right|\left|f\left(s,y_{\rho(s,y_s)}\right)\right|ds\\&+\int_0^{t_1}\int_0^s\left|\Psi\left(t_2-s,s\right)-\Psi\left(t_1-s,s\right)\right||\phi(s,\tau)|\left|f\left(\tau,y_{\rho(\tau,y_\tau)}\right)\right|d\tau ds\\&+\int_{t_1}^{t_2}\int_0^s\left|\Psi\left(t_2-s,s\right)\right||\phi(s,\tau)|\left|f\left(\tau,y_{\rho(\tau,y_\tau)}\right)\right|d\tau ds.\end{aligned}$$

Set

$$\mathcal{I}(s)=|A(0)|\left(\|\varphi\|+\sigma\right)|U(s)|+\left|f\left(s,y_{\rho(s,y_s)}\right)\right|+\int_0^s\left|\phi(s,\tau)f\left(\tau,y_{\rho(\tau,y_\tau)}\right)\right|d\tau.$$

Hence

$$\begin{aligned}\left|Ny\left(t_2\right)-Ny\left(t_1\right)\right|&\le\int_0^{t_1}\left|\Psi\left(t_1-s,s\right)-\Psi\left(t_2-s,s\right)\right|\mathcal{I}(s)\ ds\\&+\int_{t_1}^{t_2}\left|\Psi\left(t_2-s,s\right)\right|\mathcal{I}(s)\ ds.\end{aligned}$$

Using the inequalities (3.3) and (3.4), the hypothesis $(H2)$ and Corollary 3.3, we get for $s\le b$ and $|y(s)|\le\kappa$

$$\begin{aligned}\mathcal{I}(s)&\le C|A(0)|\left(\|\varphi\|+\sigma\right)(1+s^\gamma)+\left[f(s,0)+l_R(s)\|y_{\rho(s,y_s)}\|\right]\\&+C\int_0^s(1+\tau^\gamma)\left[f(\tau,0)+l_R(\tau)\|y_{\rho(\tau,y_\tau)}\|\right]d\tau\\&\le C|A(0)|\left(\|\varphi\|+\sigma\right)(1+b^\gamma)+\left[f^*+l_R^*\left(|y(s)|+\mathcal{L}_h^\varphi[\|\varphi\|+\sigma]\right)\right]\\&+C\int_0^s(1+\tau^\gamma)\left[f^*+l_R^*\left(|y(\tau)|+\mathcal{L}_h^\varphi[\|\varphi\|+\sigma]\right)\right]d\tau\\&\le C|A(0)|\left(\|\varphi\|+\sigma\right)(1+b^\gamma)+\left[f^*+l_R^*\left(\kappa+\mathcal{L}_h^\varphi[\|\varphi\|+\sigma]\right)\right]\\&+C\left[f^*+l_R^*\left(\kappa+\mathcal{L}_h^\varphi[\|\varphi\|+\sigma]\right)\right]\int_0^s(1+\tau^\gamma)d\tau\\&\le C|A(0)|\left[(1+b^\gamma)+l_R^*\mathcal{L}_h^\varphi\left(1+Cb\left[1+\frac{b^\gamma}{\gamma+1}\right]\right)\right]\left(\|\varphi\|+\sigma\right)\\&+\left(1+Cb\left[1+\frac{b^\gamma}{\gamma+1}\right]\right)\left(f^*+l_R^*\kappa\right):=\varpi.\end{aligned}$$

Then

$$\begin{aligned}|Ny\,(t_2)-Ny\,(t_1)| &\leq \varpi\int_0^{t_1}|\Psi\,(t_1-s,s)-\Psi\,(t_2-s,s)|\,ds\\ &+ \varpi\int_{t_1}^{t_2}|\Psi\,(t_2-s,s)|\,ds.\end{aligned}$$

Then, $|Ny\,(t_2)-Ny\,(t_1)|$ tends to zero as $t_2-t_1\to 0$. Hence, $N(B)$ is equicontinuous.

• Estimates of solutions: Let y and $\lambda\in[0,1)$ such that $y=\lambda N(y)$. By Lemma 3.2 and the hypotheses $(H2)$ and $(H3)$, we have for each $t\in J$

$$\begin{aligned}|y(t)| &\leq \lambda|\varphi(0)-h_0(y)|+\lambda\int_0^t|\Psi(t-s,s)U(s)A(0)[\varphi(0)-h_0(y)]|\,ds\\ &+ \lambda\int_0^t\left|\Psi(t-s,s)f\left(s,y_{\rho(s,y_s)}\right)\right|ds\\ &+ \lambda\int_0^t\int_0^s\left|\Psi(t-s,s)\varphi(s,\tau)f\left(\tau,y_{\rho(\tau,y_\tau)}\right)\right|d\tau ds\\ &\leq (\|\varphi\|+\sigma)\left[1+C^2|A(0)|\int_0^t(t-s)^{\alpha-1}\left(1+s^\gamma\right)ds\right]\\ &+ C\int_0^t(t-s)^{\alpha-1}\left(f(s,0)+l_R(s)\|y_{\rho(s,y_s)}\|\right)ds\\ &+ C^2\int_0^t\int_0^s(t-s)^{\alpha-1}(s-\tau)^{\gamma-1}\left(f(\tau,0)+l_R(\tau)\|y_{\rho(\tau,y_\tau)}\|\right)d\tau ds.\end{aligned}$$

Using Corollary 3.3, we get

$$\begin{aligned}|y(t)| &\leq (\|\varphi\|+\sigma)\left[1+C^2|A(0)|t^\alpha\left(\alpha^{-1}+t^\gamma\beta(\alpha,\gamma+1)\right)\right]\\ &+ C\int_0^t(t-s)^{\alpha-1}\left[f^*+l_R^*\left(|y(s)|+\mathcal{L}_h^\varphi[\|\varphi\|+\sigma]\right)\right]ds\\ &+ C^2\int_0^t\int_0^s(t-s)^{\alpha-1}(s-\tau)^{\gamma-1}\times\\ &\quad\times\left[f^*+l_R^*\left(|y(\tau)|+\mathcal{L}_h^\varphi[\|\varphi\|+\sigma]\right)\right]d\tau ds.\end{aligned}$$

We consider the function

$$\mu(t):=\sup_{s\in[0,t]}|y(s)|.$$

Let $t^\star \in [-r,t]$ be such that $\mu(t^\star) = |y(t^\star)|$.
If $t^\star \in [-r,0]$, then $\mu(t^\star) = \|\varphi\| - \sigma$. If $t \in [0,b]$, by the previous inequality, we have

$$\begin{aligned}
\mu(t) &\le (\|\varphi\|+\sigma)\left[1 + C^2|A(0)|t^\alpha\left(\alpha^{-1} + t^\gamma\beta(\alpha,\gamma+1)\right)\right] \\
&+ C\int_0^t (t-s)^{\alpha-1}\left(f^* + l_R^*(\mu(s) + \mathcal{L}^\varphi\|\varphi\|)\right) ds \\
&+ C^2\int_0^t\int_0^s (t-s)^{\alpha-1}(s-\tau)^{\gamma-1}\times \\
&\quad \times\left[f^* + l_R^*\left(\mu(\tau) + \mathcal{L}_h^\varphi[\|\varphi\|+\sigma]\right)\right] d\tau ds \\
&\le (\|\varphi\|+\sigma)\left[1 + C^2|A(0)|b^\alpha\left(\alpha^{-1} + b^\gamma\beta(\alpha,\gamma+1)\right)\right] \\
&+ Ct^\alpha\left[\alpha^{-1} + C\gamma^{-1}t^\gamma\beta(\alpha,\gamma+1)\right]\left[f^* + l_R^*\left(\|\mu\|_\infty + \mathcal{L}_h^\varphi[\|\varphi\|+\sigma]\right)\right] \\
&\le (\|\varphi\|+\sigma)\left[1 + C^2|A(0)|b^\alpha\Theta_\gamma\right] \\
&+ Cb^\alpha\left[\alpha^{-1} + C\gamma^{-1}b^\gamma\beta(\alpha,\gamma+1)\right]\left[f^* + l_R^*\left(\|\mu\|_\infty + \mathcal{L}_h^\varphi[\|\varphi\|+\sigma]\right)\right] \\
&\le (\|\varphi\|+\sigma)\left[1 + C^2|A(0)|b^\alpha\Theta_\gamma\right] \\
&+ Cb^\alpha\Upsilon\left[f^* + l_R^*\left(\|\mu\|_\infty + \mathcal{L}_h^\varphi[\|\varphi\|+\sigma]\right)\right].
\end{aligned}$$

Consequently,

$$\|\mu\|_\infty \le \frac{Cb^\alpha\Upsilon f^* + (1 + C^2|A(0)|b^\alpha\Theta_\gamma + Cb^\alpha\Upsilon l_R^*\mathcal{L}_h^\varphi)\left(\|\varphi\|+\sigma\right)}{(1 - Cb^\alpha\Upsilon l_R^*)} := \widetilde{M}.$$

Hence we have for every $t \in H \cup J$,

$$\|y\|_\infty \le \max\{\ \|\varphi\| - \sigma\ \ \widetilde{M}\ \} := M^*.$$

Set

$$Z = \{\ y \in C([-r,b];E) : \sup\{|y(t)|\ \ : 0 \le t \le b\} < M^* + 1\ \}.$$

From the choice of Z there is no $y \in \partial Z$ such that $y = \lambda\ N(y)$ for some $\lambda \in (0,1)$. Then the statement $(LS2)$ in Theorem 2.8 does not hold here. Thus, the statement $(LS1)$ holds, so we can deduce that the operator N has at least one fixed-point $y^\star$, which is the mild solution of problem $(1.1) - (1.2)$. □

4. An Example

We consider the following problem

$$(4.1)\quad \begin{cases} \dfrac{\partial^\alpha u}{\partial t^\alpha}(t,\xi) = \kappa(t,\xi)\dfrac{\partial^2 u(t,\xi)}{\partial \xi^2} \\ + \displaystyle\int_{-r}^{0} a_1(s-t)u\left[s-\rho_1(t)\rho_2\left(\int_0^\pi a_2(\theta)|u(t,\theta)|^2 d\theta\right),\xi\right] ds \\ \hfill 0<\alpha<1,\ 0\le t\le b,\ \xi\in[0,\pi], \\ u(t,0) = \ u(t,\pi)\ = 0, \hfill 0\le t\le b, \\ u(t,\xi)+c_t(u) = \ u_0(t,\xi), \hfill -r\le t\le 0,\ \xi\in[0,\pi], \end{cases}$$

where $\kappa(t,\cdot)$ is a continuous function for $t\in[0,b]$ and $\kappa(\cdot,\xi)$ is uniformly Hölder continuous in $\xi\in[0,\pi]$; $a_1:[-r,0]\to\mathbf{R}$ and $a_2:[0,\pi]\to\mathbf{R}$, $\rho_1:[0,b]\to\mathbf{R}$, $\rho_2:\mathbf{R}\to\mathbf{R}$, $c_t: C([-r,0],\mathbf{R})\to\mathbf{R}$ and $u_0:[-r,0]\times[0,\pi]\to\mathbf{R}$ are continuous given functions.

Consider the space $E=L^2([0,\pi],\mathbf{R})$ and define A by

$$A(t)w=\kappa(t,w)w''$$

with domain

$$D(A)=H^2(0,\pi)\cap H_0^1(0,\pi).$$

Then $A(s)$ generates an analytic $e^{tA(s)}$ in E which satisfies the assumptions (A_1) and (A_2).

Theorem 4.1. *Assume that the functions* $a_1:[-r,0]\to\mathbf{R}$, $a_2:[0,\pi]\to\mathbf{R}$, $\rho_1:[0,b]\longrightarrow\mathbf{R}$, $\rho_2:\mathbf{R}\longrightarrow\mathbf{R}$, $c_t: C([-r,0],\mathbf{R})\to\mathbf{R}$ *and* $u_0:[-r,0]\times[0,\pi]\to\mathbf{R}$ *are continuous functions. Then there exists at least one mild solution of* (4.1) *on* $[-r,b]$.

Proof. We may deduce from the assertions that for $\xi\in[0,\pi]$, we have

$$y(t)(\xi)=u(t,\xi),$$

$$A^{-1}(0)=(\kappa(\cdot,0))^{-1},$$

$$f(t,\psi)(\xi)=\int_{-r}^{0} a_1(s)\psi(s,\xi)ds,$$

$$\rho(t,\psi)(\xi) = t - \rho_1(t)\rho_2\left(\int_0^{\pi} a_2(s)|\psi(0,\xi)|^2 ds\right),$$
$$h_t(\psi)(\xi) = c_t(\psi)(\xi)$$

and

$$\varphi(t)(\xi) = u_0(t,\xi)$$

are well defined functions, so we can transform the example (4.1) into the abstract system (1.1) $-$ (1.2). Hence, by Theorem 3.4, we may deduce the existence of at least one mild solution. From Remark 2.2, we can obtain the following.

Corollary 4.2. *Since the function $\varphi \in C(H,E)$ is continuous and bounded, there exists at least one mild solution of* (4.1).

□

5. Acknowledgement

This paper was supported by the General Directorate for Scientific Research and Technological Development (DGRSDT), University-Training Research Projects: C00L03UN220120210002 PRFU 2021 project. The authors thank also the anonymous reviewers for their precious remarks.

References

[1] N. Abada, R. P. Agarwal, M. Benchohra and H. Hammouche, "Existence results for nondensely defined impulsive semilinear functional differential equations with state-dependent delay", *Asian-European Journal of Mathematics,* vol. 1, no. 4, pp. 449-468, 2008. doi: 10.1142/S1793557108000382

[2] R. P. Agarwal, S. Baghli and M. Benchohra, "Controllability of solutions on semiinfinite interval for classes of semilinear functional and neutral functional evolution equations with infinite delay", *Applied Mathematics and Optimization,* vol.60, no. 2, pp. 253-274, 2009. doi: 10.1007/s00245-009-9073-1

[3] N. U. Ahmed, *Semigroup theory with applications to systems and control.* Pitman Res. Notes Math. Ser., 246. Longman Scientific & Technical. New York: Harlow John Wiley & Sons, Inc., 1991.

[4] D. Aoued and S. Baghli-Bendimerad, “Mild solution for perturbed evolution equations with infinite state-dependent delay”, *Electronic journal of qualitative theory of differential equations,* vol. 59, nos. 1-14, 2013.

[5] D. Aoued and S. Baghli-Bendimerad, ”Controllability of mild solutions for evolution equations with infinite state-dependent delay”, *European Journal of Pure and Applied Mathematics,* vol. 9, no. 4, pp. 383-401, 2016.

[6] A. Arara, M. Benchohra, N. Hamidi and J. J. Nieto, “Fractional order differential equations on a unbounded domain”, *Nonlinear Analysis*, vol. 72, pp. 580-586, 2010. doi: 10.1016/j.na.2009.06.106

[7] S. Baghli and M. Benchohra, “Uniqueness results for partial functional differential equations in Fréchet spaces”, *Fixed-Point Theory*, vol. 9, no. 2, pp. 395-406, 2008.

[8] S. Baghli and M. Benchohra, “Existence results for semilinear neutral functional differential equations involving evolution operators in Fréchet spaces”, *Georgian Mathematical Journal,* vol. 17, no. 3, pp. 423-436, 2010.

[9] S. Baghli and M. Benchohra, “Global uniqueness results for partial functional and neutral functional evolution equations with infinite delay”, *Differential Integral Equations,* vol. 23, nos. 1&2, pp. 31-50, 2010. doi: 10.57262/die/1356019385

[10] S. Baghli, M. Benchohra and J. J. Nieto, “Global uniqueness results for partial functional and neutral functional evolution equations with statedependent delay”, *Journal of Advanced Research,* vol. 2, no. 3, pp. 35-52, 2010.

[11] M. Benchohra, E. P. Gatsori, L. Górniewicz and S. K. Ntouyas, “Controllability results for evolution inclusions with non-local conditions”, *Zeitschrift für Analysis und ihre Anwendungen,* vol. 22, pp. 411-431, 2003.

[12] L. Byszewski, “Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem”, *Journal of Mathematical Analysis and Applications,* vol. 162, pp. 494-505, 1991. doi: 10.1016/0022-247X(91)90164-U

[13] M. M. El-Borai, “The fundamental solutions for fractional evolution equations of parabolic type”, *Journal of Applied Mathematics and Stochastic Analysis,* vol. 3, pp. 197-211, 2004. doi: 10.1155/S1048953304311020

[14] M. M. El-Borai, “On some fractional evolution equations with nonlocal conditions”, *International Journal of Pure and Applied Mathematics,* vol. 24, pp. 401-410, 2005.

[15] A. M. A. El-Sayed, “Fractional order evolution equations”, *J. Fract. Calc.*, vol. 7, pp. 89-100, 1995.

[16] R. Gorenflo and F. Mainardi, “Fractional calculus and stable probability distributions”, *Archives of Mechanics,* vol. 50, no. 3, pp. 377-388, 1998.

[17] D. Henry, *Geometric Theory of Semilinear Parabolic Equations.* Lecture Notes in Math., 840, New York: Springer, 1981.

[18] E. Hernandez, R. Sakthivel and S. Tanaka Aki, “Existence results for impulsive evolution differential equations with state-dependent delay”, *Electronic Journal of Differential Equations,* vol. 2008, no. 28, pp. 1-11, 2008.

[19] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations.* North-Holland Mathematics Studies, 204. Amsterdam: Elsevier Science B.V., 2006.

[20] G. Li, Sh. Song and Ch. Wu, “Controllability of evolution inclusions with nonlocal conditions”, *Journal of Systems Science and Complexity*, vol. 18, no. 1, pp. 35-42, 2005.

[21] G. Li and X. Xue, “Controllability of evolution inclusions with nonlocal conditions”, *Applied Mathematics and Computation,* vol. 141, pp. 375-384, 2003. doi: 10.1016/S0096-3003(02)00262-X

[22] J. Liang, J. H. Liu and T. J. Xiao, “Nonlocal impulsive problems for nonlinear differential equations in Banach spaces”, *Mathematical and Computer Modelling,* vol. 49, pp. 798-804, 2009. doi: 10.1016/j.mcm.2008.05.046

[23] A. Mebarki and S. Baghli-Bendimerad, “Neutral multi-valued integrodifferential evolution equations with infinite state-dependent delay”, *Turkish Journal of Mathematics,* vol. 44, no. 6, pp. 2312-2329, 2020. doi: 10.3906/mat-2007-66

[24] F. Mesri, J. E. Lazreg, M. Benchohra and J. Henderson, “Fractional non autonomous evolution equations in Fréchet spaces”, *Commun. Appl. Nonl. Anal.,* vol. 28, no. 2, pp. 35-45, 2021.

[25] K. S. Miller and B. Ross, *An introduction to the fractional calculus and differential equations.* New York: John Wiley, 1993.

[26] R. Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics,* 2nd ed. New York: Springer, 1997.

[27] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations.* New York: Springer, 1983.

[28] I. Podlubny, *Fractional Differential Equation.* San Diego: Academic Press, 1999

[29] I. Podlubny, “Geometric and physical interpretation of fractional integration and fractional differentiation”, *Fract. Calc. Appl. Anal.,* vol. 5, pp. 367-386, 2002. doi: 10.48550/arXiv.math/0110241

[30] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional integrals and derivatives. Theory and applications.* Yverdon: Gordon and Breach, 1993.

[31] Y. Zhou and F. Jiao, “Existence of mild solutions for fractional neutral evolution equations”, *Computers & Mathematics with Applications,* vol. 59, pp. 1063-1077, 2010. doi: 10.1016/j.camwa.2009.06.026

Nardjis Lachachi-Merad
Department of Mathematics,
Djillali Liabes University,
Sidi Bel-Abbès,
Algeria
e-mail: nardjis.lachachi@univ-sba.dz

Selma Baghli-Bendimerad
Department of Mathematics,
Djillali Liabes University,
Sidi Bel-Abbès,
Algeria
e-mail: selma.baghli@univ-sba.dz
selmabaghli@gmail.com

Mouffak Benchohra
Department of Mathematics,
Djillali Liabes University,
Sidi Bel-Abbès,
Algeria
e-mail: mouffak.benchahra@univ-sba.dz
e-mail: benchohra@yahoo.com

and

Erdal Karapınar
Department of Mathematics,
Çankaya University,
06790, Etimesgut, Ankara,
Turkey
e-mail: karapinar@mail.cmuh.org.tw
erdalkarapinar@yahoo.com
Corresponding author