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Open global shadow graph and it's zero forcing number

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Abstract

Zero forcing number of a graph is the minimum cardinality of the zero forcing set. A zero forcing set is a set of black vertices of minimum cardinality that can colour the entire graph black using the colour change rule: each vertex of G is coloured either white or black, and vertex v is a black vertex and can force a white neighbour only if it has one white neighbour. In this paper we identify a class of graph where the zero forcing number is equal to the minimum rank of the graph and call it as a new class of graph that is open global shadow graph". Some of the basic properties of open global shadow graph are studied. The zero forcing number of open global shadow graph of a graph with upper and lower bound is obtained. Hence giving the upper and lower bound for the minimum rank of the graph.

Keywords: *Zero forcing set, Zero forcing number, Open global shadow graph.*

2010 Mathematics Subject Classification: *05C07, 05C38, 05C50, 05C69.*

1. Introduction

Throughout this article we consider only simple un-directed finite graphs without loops and multiple edges. A graph consist of a vertex set $V(G)$ and an edge set $E(G)$. Two vertices u and v of a graph G are adjacent if there is an edge uv joining them, and we denote this as $v \sim u$. The open neighborhood of a vertex v denoted by $N(v)$, consist of all the vertices which are adjacent to v. The closed neighborhood of v denoted by $N[v]$, consist of the vertex v and every vertex adjacent to v. Degree of a vertex v is the number of edges incident with the vertex v. The minimum and maximum degree of a graph is represented respectively as δ and Δ.

The zero forcing set $S \subseteq V(G)$ is a set consisting of black colored vertices, which are colored black based on the color changing rule. The color changing rule states that a black color vertex can force at most one white neighbour provided it is the only white neighbour of it. The minimum cardinality of S gives the zero forcing number of a graph G. The concept of zero forcing number was introduced by AIM Minimum rank- special graphs work group [2] to bound the minimum rank of the graph. Independently the concept of zero forcing was introduced in 2007 [5] to understand the controlability of quantum system. The major advantage of introducing the concept of zero forcing over other tools to bound the minimum rank is that its definition is purely combinatorial. The problem of finding the zero forcing number of a graph is an NP hard problem [1].

The shadow graph is defined as, let G be a simple connected graph and G' be a copy of G such that each vertex $u \in V(G)$ is made adjacent to $N(u')$. Where u' is the corresponding vertex of u. Motivated by the definition of shadow graph we introduce a new class of graph called the open global shadow graph. Let G be a graph and G' be a copy of G such that $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $V(G') = \{v'_1, v'_2, \ldots, v'_n\}$. The open global shadow graph denoted by $GS(G)$ is obtained by taking two copies of G say, G and G' and joining the vertex v_i to each of the vertex in $V(G') \setminus N[v'_i]$, where $1 \leq i \leq n$. In this paper we refer to the vertices v_i and v'_i as the corresponding vertices.

2. Some results on the open global shadow graph of a graph

In this section few characteristic properties of open global shadow graph are discussed. The total number of edges, regularity, the domination number and the connected domination number of the open global shadow graph are found.

Observation 2.1. *For a graph G of order n, $|V(GS(G))| = 2n$.*

Theorem 2.2. *For a non-trivial graph G, $|E(GS(G))| = n(n-1)$.*

Proof. For n number of available vertices in a graph G the maximum possible edges are $\frac{n(n-1)}{2}$. Which is that of a complete graph. It can be seen from the definition of open global shadow graph that if G has t edges $0 \le t \le \frac{n(n-1)}{2}$, G' also has t edges. Therefore, there are $\frac{n(n-1)}{2} - t$ missing edges in G and G' as compared to K_n. Now according to the definition of open global shadow graph, if v_i and v_j are not adjacent in G then v_i will be adjacent to v'_j and v_j will be adjacent to v'_i in $GS[G]$. Clearly, total number of edges $= t + t + 2(\frac{n(n-1)}{2} - t) = n(n-1) = 2|V(K_n)|$. □

From now on whenever we write a 'missing edge', we mean that from the total possible edge set for n vertices in the graph G (that is $\frac{n(n-1)}{2}$), an edge is removed.

If G is the complete graph K_n, then none of the edges are missing. Hence in $GS(K_n)$ we will get G and a copy of G that is G'. Therefore the open global shadow graph of the complete graph K_n is $K_n \cup K_n$. Suppose that there is an edge missing between v and u for $\{v, u\} \subseteq V(G)$. Then the open global shadow graph will have G and its copy G' and the edge connecting v to u' and u to v', where $\{v', u'\} \subseteq V(G')$. From this the following observation can be drawn.

Observation 2.3. *For each of the missing edges in G there are 2 edges added in between G and G' in $GS(G)$.*

Theorem 2.4. *Let G be a connected graph. Then $GS(G)$ is disconnected if and only if G is a complete graph.*

Proof. Without loss of generality, assume that G is a connected graph. If G is a complete graph, then clearly no edges are missing so open global shadow graph is isomorphic to $2K_n$. Since $2K_n$ is disconnected, it is proved.

Conversely assume that G is not a complete graph. Then there will be at least one missing edge in G. Also since G is connected in the open global shadow graph of G there will be at least two edge between (From observation 2.3) G and G'. Clearly $GS(G)$ is connected, a contradiction to our assumption that G is not a complete graph. □

In a simple connected graph G, if by removing an edge the graph G get disconnected then such an edge is called cut edge. By removing a vertex from a simple connected graph G, if the graph splits into two or more components then such a vertex is called the cut vertex. Next theorem shows that the open global shadow graph has no cut edge and cut vertex.

Theorem 2.5. *Let G be a connected graph of order $n \geq 3$. Then*
i) $c[GS(G) - v] = c[GS(G)]$, where v is any vertex in $GS(G)$ and $c[GS(G)]$ is the number of connected components of $GS(G)$.
ii) $c[GS(G) - e] = c[GS(G)]$, where e is any edge in $GS(G)$.

Proof. Assume that G is a connected graph. Now we divide the proof into two cases:

Case 1: To prove that $GS(G)$ is a graph without cut edge

Sub case 1.1: Assume that G is not a complete graph K_n. Since G is connected and G is not K_n, there will be at least 2 vertices in G (similarly in G') that are not adjacent to each other. According Observation 2.3 for each of this missing edges in G there will be two extra edges added in $GS(G)$. Hence there cannot be a cut edge in this case.

Sub case 1.2: Assume that G is the complete graph K_n. Clearly when $G \cong K_n$ the open global shadow graph $GS(G) \cong 2K_n$.

Case 2: To show that $GS(G)$ doesn't have a cut vertex.

Sub case 2.1: Suppose G is a graph with no cut vertex. This implies that G' has no cut vertex. Clearly from observation 2.3 the graph $GS(G)$ doesn't have cut vertex.

Sub case 2.2: Suppose G is a graph with a cut vertex. Let v be a cut vertex in G and K and H be the components of the graph $G - v$. Let K' and H' be the graphs corresponding to K and H respectively in $GS(G)$. Clearly in $GS(G)$ all the vertices in the component H will be adjacent to all the vertices in K' similarly all the vertices in the component K will be adjacent to all the vertices in H'. Therefore we cannot find any cut vertex. □

Theorem 2.6. *The open global shadow graph is $n - 1$ regular.*

Proof. Let G be a graph with n vertices. For any vertex say $v_i \in V(G)$ where $1 \leq i \leq n$, $V(G) \setminus N(v_i)$ are the non neighbors of v_i in G, according to the definition of open global shadow graph $GS(G)$, vertices that are corresponding to $V(G) \setminus N(v_i)$ in G' are made adjacent to v_i making the degree of v_i to be $n-1$. Same is true when we consider any vertex say v_i' in G'. Hence we can say that open global shadow graph is a $n-1$ regular graph. □

Observation 2.7. *If G is an n order disconnected graph with 2 components with each component forming a clique, then $GS(G)$ is isomorphic to $2K_n$.*

A dominating set for a graph G is a subset D of $V(G)$ such that every vertex not in D is adjacent to at least one vertex of D.
Minimum cardinality of the dominating set is known as the domination number of the graph G and denoted by $\gamma(G)$.
When the subgraph induced by D is connected then it forms a connected dominating set and the minimum cardinality of the connected dominating set is known as connected domination number and is denoted by $\gamma_c(G)$ [10].
It can be observed that any of the two corresponding vertices of the open global shadow graph are enough to dominate the entire graph. therefore we have the following:

Proposition 2.8. *For a connected graph G, $\gamma(GS(G)) = 2$.*

Proposition 2.9. *For a connected graph GK_n, $\gamma_c(GS(G)) \leq 4$.*

Proof. From Proposition 2.8 it is clear that v_i and v_i', where $1 \leq i \leq n$ are enough to dominate the entire open global shadow graph. However v_i and v_i' are disconnected and there exists no vertex in $GS(G)$ such that it is adjacent to both v_i and v_i'. Let v_k be adjacent to v_i then clearly v_k is not adjacent to v_i'. Either exist at least one vertex v_t which is adjacent to v_k and v_i' in this case the connected dominating set will have $\{v_i, v_k, v_t, v_i'\}$ or vertex v_t which is adjacent to v_k' and v_i in this case the connected dominating set will have $\{v_k, v_i, v_t', v_k'\}$. Hence $\gamma_c(GS(G)) \leq 4$. □

Theorem 2.10 ([6]). *If G is a simple connected t-regular graph with at most $2t+2$ vertices, then G is Hamiltonian.*

Theorem 2.11. *If $GS(G)$ is connected then $GS(G)$ is a Hamiltonian graph.*

Observation 2.12. *The graph $GS(G)$ is a $n-1$ regular graph with $2n$ number of vertices. From theorem 2.10 we know that, if G is a connected r-regular graph with at most $2r+2$ vertices then G is Hamiltonian. Hence if $GS(G)$ is connected, then it is Hamiltonian.*

Observation 2.13. *The open global shadow graph is Hamiltonian or its components are Hamiltonian.*

3. Basic results on the zero forcing number of open global shadow graph

In this section, the bound for zero forcing number of open global shadow graphs are studied. Few characterization of the open global shadow graphs are given. Another objective of this section is to characterize simple graph G for which $Z(GS(G)) = 1, Z(GS(G)) = 2$ and $Z(GS(G)) = 3$. The edge arrows of each figure in this section indicate the direction of forcing. That is if a vertex say v_1 forces another vertex say v_2 black, then the edge arrow from v_1 to v_2 indicates the direction of forcing from the vertex v_1 to the vertex v_2.

Theorem 3.1 ([4]). *The Zero forcing number of graph G is bounded by the minimum degree $\delta(G)$ as $Z(G) \geq \delta$.*

The following observation is a consequence of Theorem 3.1 and Theorem 2.6.

Observation 3.2. *For a graph G of order n, $Z(GS(G)) \geq n-1$.*

Theorem 3.3 ([3])). *Let G be a graph with minimum degree $\delta \geq 1$, then $Z(G) \leq n\frac{\Delta}{\Delta+1}$.*

Theorem 3.4. *Let G be a graph of order n. Then $n-1 \leq Z(GS(G)) \leq 2n-2$.*

Proof. Lower bound follows from Observation 3.2. For a simple graph G of order $n \geq 2$, $\delta(GS(G)) \geq 1$. From Theorem 3.3 we know that,

$$Z(G) \leq n\frac{\Delta}{\Delta+1}$$

$$Z(GS(G)) \leq \frac{2n(n-1)}{(n-1)+1} = 2(n-1)$$
$$Z(GS(G)) \leq 2n - 2$$

.

□

From Observation 3.2 we can say that if $Z(GS(G)) = 1$, then the order of the graph G is 2. When the order of the graph G is 2, the only possible simple graphs are, K_2 and $\overline{K_2}$. Open global shadow graph of the graphs, K_2 and $\overline{K_2}$ are depicted in the figure 1. It can be observed that none of the graphs in figure 1 has zero forcing number 1. This implies that $Z(GS(G)) \geq 2$. As a consequence we have the following observation.

Observation 3.5. *There exists no simple graph G of order n with $Z(GS(G)) = 1$.*

Observation 3.6. *For an isolated vertex graph K_1. The graph $GS(K_1) \cong 2K_1$ and the zero forcing number $Z(GS(K_1)) = 2$.*

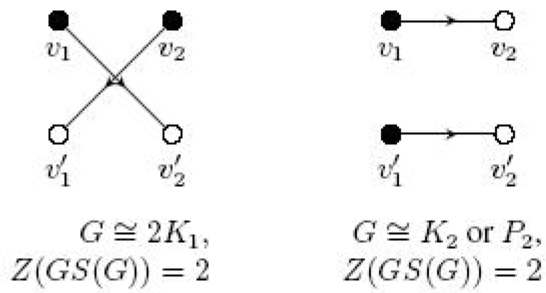


Figure 1: All the possible open global shadow graph of graph when n=2 with their zero forcing number.

Now we characterize the simple graph G for which $Z(GS(G)) = 2$

Theorem 3.7. *For a graph G, $Z(GS(G)) = 2$ if and only if G is isomorphic to one of the following graphs: $P_2, 2K_1, 3K_1$ and P_3.*

Proof. According to Observation 3.2, if $Z(GS(G)) = 2$, then $n \leq 3$. All the open global shadow graph with $n = 2, 3$ is shown in figure 1 and figure 2 respectively along with the zero forcing number. It is clear from both the figures that $Z(GS(G)) = 2$ if and only if G belonging to any of the graph $P_2, 2K_1, 3K_1$ and P_3. □

Observation 3.8. *It can be seen that the zero forcing number in figure 1 and figure 2 are obtained by considering the following conditions. $Z(G) \geq \delta$ ([4]) and $Z(C_n) = 2$ (any two consecutive vertices of the cycle forms a zero forcing set). If G is a disconnected graph with two connected components H_1 and H_2, then $Z(G) = Z(H_1) + Z(H_2)$.*

It is important to note that there exists no simple open global shadow graph for which the zero forcing number is 3.

Theorem 3.9. *There exists no simple graph for which $Z(GS(G)) = 3$.*

Proof. From Observation 3.2 $Z(GS(G)) = 3$ implies that $n \leq 4$. Clearly for $n \geq 5$, the zero forcing number of open global shadow graph cannot be equal to 3. On the other hand, from Observation 3.8 , Figure 1 and Figure 2 shows that the zero forcing number of open global shadow graph for $n = 1, 2$ and 3 are not equal to 3. Now we are left to show that for $n = 4$ there exists no open global shadow graph $GS(G)$ such that $Z(GS(G)) = 3$.

On contrary let us assume that for $n = 4$ there exist at least one graph for which $Z(GS(G)) = 3$.

Case 1: Assume that $GS(G)$ is disconnected. Then the possible simple graph G of order 4 are the complete graph K_4, the complement of the cycle C_4 that is $\overline{C_4}$ and the Co-claw graph that is $K_3 \cup k_1$. It can be seen that in all the 3 graphs, G is either a complete graph or has two components as clique. Hence $GS(G)$ is $2K_4$. With 3 black vertices at most one white vertex can be forced and the forcing process stops in $GS(G)$.

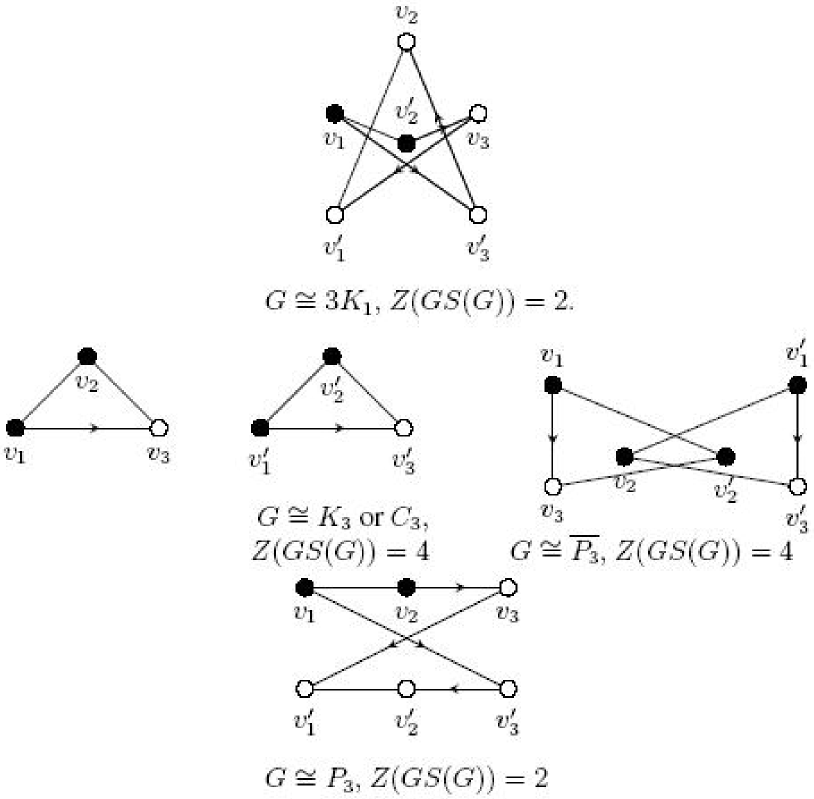


Figure 2: The figure illustrates all the possible open global shadow graph of a simple graph G when the order $n = 3$. In the first figure a minimum zero forcing set for the open global shadow graph of the graph $G \cong 3K_1$ is marked by the black vertices v_1 and v_2'. The forcing sequence v_1 forces v_3' and v_2' forces v_3 are indicated using the arrows. Again the vertex v_3' forces v_2 and v_3 forces v_1' also indicated using arrows.

Case 2: Assume that $GS(G)$ is connected.
Sub Case 2.1: Consider the graph G to be $\overline{K_4}$. Let $GS(G)$ be its open global shadow graph. In the graph $GS(G)$, it can be observed that $\forall\, v \in V(G)$, all 3 of its neighbours will be in $V(G')$ and vice-versa. Clearly by taking v and two of its neighbour black, it is not possible to force the entire graph $GS(G)$ to be black.

Sub Case 2.2: Consider the graph G to be the Co-diamond. Label the two vertices with degree zero as v_1 and v_2 and two vertices of degree one as v_3 and v_4. The forcing of $GS(G)$ can be split into two cases as the forcing pattern is same when we take either one of the two zero degree vertex or one of the two one degree vertex as initial black vertex.

Sub Case 2.2.1: When v_1 is considered black, then two of its neighbours in $V(G')$ say v_2', v_3' are considered black so that v_1 can force v_4' black. v_2' will have 2 white neighbours v_3 and v_4 hence v_2' doesn't force any white neighbour black. Further v_3' or v_4' can force v_2 black as it is the only white neighbour of v_3' or v_4'. In the next step v_2 has v_1' as its only white neighbour. However v_1' has two white neighbours v_3 and v_4. Hence 3 black vertices are not enough for the forcing process.
Case is similar when we consider v_2 as the initially colored black vertex.

Sub Case 2.2.2: When v_4 is taken to be black, two of its neighbours v_3 and v_2' are taken black. v_4 forces v_1' black. Now v_3 cannot force any vertex black as all its neighbours are black. v_2' can force v_1 black and v_1' can force v_2 black. Both v_2 and v_1 has 2 white vertices v_3' and v_4'. Hence the forcing stops.
Case is similar when we consider v_3 as the initially colored black vertices.

Sub Case 2.3: Consider the graph G to be the diamond graph. Label the two vertices with degree 3 as v_2 and v_4 and two vertices with degree 2 as v_1 and v_3 such that $v_1 v_3$. The forcing of $GS(G)$ can be split into two cases as the forcing pattern is same when we take either one of the two 2 degree vertex or one of the two 3 degree vertex as initial black vertex.

Sub Case 2.3.1: When v_1 is taken initially black. Clearly two of its neighbour should be black say, v_2 and v_4, then v_1 forces v_3' and v_4 or v_2 forces v_3 black. v_3 can force v_1' black after which the forcing process stops as both v_1' and v_3' have two white vertices.

Sub Case 2.3.2: When v_2 is taken black along with two of its neighbours say v_1 and v_3. v_2 can force v_4 also v_1 forces v_3' and v_3 forces v_1' and the process stops as both v_1' and v_3' have two white neighbours v_2' and v_4'.

Sub Case 2.4: Consider the graph G to be the co-paw label the one isolated vertex as v_1 , two degree one vertices as v_2 and v_4 and one degree two vertex as v_3 ($v_3 \sim v_2$ and $v_3 \sim v_4$). The forcing of $GS(G)$ can be split into three cases based on the degree of the vertices.

Sub Case 2.4.1: When v_1 along with two of its neighbours v'_2 and v'_3. v_1 can force v'_4. v'_2 and v'_4 respectively can force v_4 and v_2. The process of forcing stops as both v_2 and v_4 have two white vertices v_3 and v'_1.

Sub Case 2.4.2: When v_2 and two of its neighbours v_3 and v'_1 are black then, v_2 can force v'_4 black. Then v_3 can force v_4 black. v_4 can force v'_2 black. Both v'_2 and v'_4 are adjacent to two white vertices v'_3 and v_1. Hence the process stops.
Case is similar when we consider v_4 as the initially colored black vertices.

Sub Case 2.4.3: When v_3 and two of its neighbours v_2 and v_4 are black. Then v_3 forces v'_1 black. v_2 and v_4 forces v'_4 and v'_2 respectively. Now v'_4 and v'_2 has two white neighbours v'_3 and v_1. Hence the process stops.

Sub Case 2.5 Consider the graph G to be the paw, label the one vertex with degree one as v_1 , vertex with degree 3 as v_2 and two vertices with degree 2 as v_3 and v_4.(v_3v_1 and v_4v_1) The forcing of $GS(G)$ can be split into three cases based on the degree of the vertices.

Sub case 2.5.1: v_1 and two of its neighbours v_2 and v'_3 are taken black then v_1 forces v'_4 black. v'_3 or v'_4 forces v'_2 black v'_2 further forces v'_1 black. then process stops as both v'_1 and v_2 are adjacent to two white neighbour v_3 and v_4.

Sub Case 2.5.2: v_2 and two of its neighbours v_3 and v_4 are taken initially black then v_2 forces v_1 and v_3 or v_4 forces v'_1 black. v'_1 can force v'_2 black. However v'_2 and v_1 are adjacent to two white vertices.

Sub Case 2.5.3: v_3 and two of its neighbours v_2 and v_4 are black then,v_3 can force v'_1 black and v_2 can force v_1 black. v'_1 can force v'_2 black. However v_1 and v'_2 both are adjacent to two white neighbour v'_3 and v'_4.
Case is similar when we consider v_4 as the initially colored black vertices.

Sub Case 2.6: Consider the graph G to the cycle C_4. Clearly all the vertices are of degree 2. The forcing of $GS(G)$ is same irrespective of the vertex that we choose. Let v_1 and two of its neighbours v_2 and v_4 be colored initially black, then v_1 can force v_3' black. Now the forcing process stops as v_2, v_4 and v_3' have 2 white neighbours (v_3 and v_4', v_3 and v_2', v_2' and v_4' respectively).

Sub Case 2.7: Consider the graph G to be the claw. Label the 3 vertices with degree one as v_1, v_3, v_4 and one vertex with degree three as v_2. The forcing of $GS(G)$ can be split into two cases as the forcing pattern is same when we take either one of the three 1 degree vertex as initial black vertex.

Sub Case 2.7.1: When v_1 along with two of its neighbour v_2, v_3' are taken black. v_1 can force v_4'. $v_2, v_3' and v_4'$ have two white neighbours hence the process stops.
Case is similar when we consider v_4 or v_3 as the initially colored black vertices.

Sub Case 2.7.2: When v_2 and two of its neighbours sayv_1, v_3 as black.Then v_2 can force v_4 black but further forcing is not possible as each of these black vertices are connected to two white neighbours.

Sub Case 2.8: Consider the graph G to be the path P_4. Label the two vertices with degree one as v_1, v_4 and two vertices with degree two as v_2, v_3. The forcing of $GS(G)$ can be split into two cases based on the degree of the graph.

Sub Case 2.8.1: v_1 and two of its neighbours v_2 and v_3' are considered black. v_1 can force v_4' black. Then v_2 can force v_3 black and v_3' can force v_2' black. The forcing process stops as v_3 and v_2' have two white neighbours v_4 and v_1'.
Case is similar when we consider v_4 as the initially colored black vertices.

Sub Case 2.8.2: v_2 and two of its neighbours v_1 and v_3 are taken black. Now v_2 can force v_4' black. v_4' can force v_3' black and v_3' can force v_2' black. Then the forcing process stops as both v_2' and v_3 have two white neighbours v_1' and v_4.
Case is similar when we consider v_3 as the initially colored black vertices.

From all the above cases we can conclude that for $n = 4$ there exists no open global shadow graph $GS(G)$ such that $Z(GS(G)) = 3$. Table 3.1 shows all the possible graph G when $n = 4$ and its $Z(GS(G))$. It can be observed that none of the graph G takes $Z(GS(G))$ to be 3. □

Graph G	Z(G)	Z(GS(G))
$\overline{K_4}$	4	4
K_4	3	6
Co-diamond	3	4
Diamond	2	4
Co-paw	2	4
paw	2	4
$\overline{C_4}$	2	6
C_4	2	4
Claw	2	4
Co-claw	3	6
P_4	1	4

Table 3.1: Value of $Z(GS(G))$ for all possible G when $n = 4$

It is an open problem to characterize graphs for which $Z(GS(G)) = 4$. In figure 3 we provide few examples of graphs for which $Z(GS(G)) = 4$.

Theorem 3.10. *For a connected graph G of order n, $Z(GS(G)) = 2n - 2$ if and only if $G \cong K_n$.*

Proof. If $G \cong K_n$, then the open global shadow graph $GS(G) \cong 2K_n$. We know that $Z(G) = n - 1$ where n is the number of vertices in G. Now we have $2K_n$, for each K_n we need $n-1$ black vertices. Hence on the whole $Z(G) = 2(n - 1) = 2n - 2$.
To prove the converse part, let us assume the contrary that $Z(GS(G)) = 2n - 2$ and $G \neq K_n$. Since G is connected and $G \neq K_n$, there will be at least two vertices in G that are not adjacent to each other. Let v_i and v_j be the two vertices that are not adjacent to each other in G. In the open global shadow graph of G, v_i will be adjacent to v'_j and v_j will be adjacent to v'_i. In $GS(G)$ let us take $V(GS(G)) \setminus \{v_k, v'_i, v'_j\}$ (where v_k is one of the neighbour ofv'_j such that $k \neq i$)in the zero forcing set S. Clearly, v'_j is the only white neighbour of v_i. Hence v_i forces v'_j. Also v_k is a white vertex adjacent to v'_j, so v'_j can force v_k as it is the only white neighbour. Similarly, v'_i is the only white neighbour of v_j in $GS(G)$. Hence v_j can force

v_i' black, thereby forcing the entire graph black. It can be concluded that $Z(GS(G)) \leq |S| = n - 3$. Hence a contradiction. □

Observation 3.11. *Let $GS(G)$ be disconnected. Then* $Z(GS(G)) = 2n - 2$.

Lemma 3.12. *Let G be a null graph. Then $GS(G)$ is a triangle free graph.*

Proof. Let us assume that $GS(G)$ is not triangle free graph. Clearly there is at least one K_3 in the graph $GS(G)$. Consider the K_3 present in $GS(G)$, where at least two vertices are from the same component G or G'. However any two vertices in G or G' cannot be adjacent as G is a null graph. Hence the contradiction. □

Theorem 3.13 ([7]). *If a graph G is triangle free, then $Z(G) \geq 2\delta - 2$.*

Theorem 3.14. *For a null graph G of order $n > 2$, $Z(GS(G)) = 2n - 4$.*

Proof. It is known that $GS(G)$ is a triangle free graph from Lemma 3.12. Theorem 3.13 shows that if a graph is triangle free then $Z(G) \geq 2\delta - 2$. $GS(G)$ is an $n-1$ regular graph, Theorem 2.6. Hence $Z(GS(G)) \geq 2(n-1) - 2 = 2n - 4$.
Now we are left to show that $Z(GS(G)) \leq 2n - 4$. According to the definition of $GS(G)$ each vertex in G will be adjacent to all the $n-1$ vertices of G' except for the corresponding vertex. Let S be the zero forcing set, by taking v_i and $n-2$ of its neighbour black v_i can force the only white neighbour black. To further carryout the forcing process we need to take $n-3$ neighbours of v_j', $1 \leq j \leq n$ & $j \neq i$ black there by v_j forcing its only white neighbour. At this stage we are left with one white vertex in G and one in G'. Both these vertices can be forced by the black vertices in $GS(G)$. Thereby forcing the entire graph black. Hence $|S| = 1+n-2+n-3 = 2n-4$ implies $Z(GS(G)) \leq 2n - 4$. □

Theorem 3.15. *If G is $n-2$ regular, then $Z(GS(G)) \leq n$.*

Proof. If G is an $n-2$ regular graph, then according to the handshaking lemma n must be even. Meaning there are $\frac{n}{2}$ missing edges in G. Due to which there will be n edges between G and G' in $GS(G)$. In $GS(G)$ each vertex in G will have exactly one unique neighbour in G'. Same is true with vertices of G'. By taking all the n vertices of G or G' as black we can force the entire graph $GS(G)$ black. Hence $Z(GS(G)) \leq n$. □

Theorem 3.16. *For a path P_n, $Z(GS(P_n)) \leq n + 1$.*

Proof. Let G be a path P_n. In the $GS(G)$, The vertices v_1 and v_n are adjacent to $n-2$ vertices of G', the remaining $n-2$ vertices in G has $n-3$ adjacent vertices in G'. Similarly v'_1 and v'_n are adjacent to $n-2$ vertices of G, the remaining $n-2$ vertices in G' has $n-3$ adjacent vertices in G. By taking all the vertices of G or G' to be black. $GS(G)$ looks like a path graph. We need one more vertex to force the entire graph $GS(G)$ black. Hence $Z(GS(P_n)) \leq n+1$. □

Now we consider the hamiltonicity of $GS(G)$

Theorem 3.17 ([2]). *If G is a Hamiltonian graph, then $Z(G) = M(G)$ where $M(G)$ is the maximum nullity of graph G.*

Corollary 3.18. *For a graph G, $Z(GS(G)) = M(GS(G))$.*

Proof. From Observation 2.12 it can be seen that open global shadow graph is Hamiltonian. From Theorem 3.17, it can be concluded that $Z(GS(G)) = M(GS(G))$. □

4. Relation between $Z(GS(G))$ and $2Z(G)$

In this section we try to characterize graph classes for which $Z(GS(G)) = 2Z(G)$. We start with the complete graph K_n.

Theorem 4.1. *If $G \cong K_n$, then $Z(GS(K_n)) = 2Z(K_n)$.*

Proof. From Theorem 3.10 it is known that if $G \cong K_n$, then $Z(GS(G)) = 2n-2$. Also [9] we know that $Z(G) = n-1$ when $G \cong K_n$. Clearly $Z(GS(G)) = 2Z(G)$, when $G \cong K_n$. □

Let us recall the following theorem from [8] to prove the next theorem.

Theorem 4.2 ([8]). *If G is a complete bipartite graph $K_{p,q}$, then the zero forcing number $Z(K_{p,q}) = p+q-2$, where $p+q = n$..*

Next we consider the complete bipartite graph $K_{p,q}$.

Theorem 4.3. *If $G \cong K_{p,q}$ and $|V(G)| = n$, then $Z(GS(K_{p,q})) = 2Z(K_{p,q})$.*

Proof. Zero forcing number of the complete bipartite graph is $Z(K_{p,q}) = p + q - 2$ from Theorem 4.2.
We need to prove that Zero forcing number of open global shadow graph of complete bipartite graph $Z(GS(K_{p,q}))$ is $2n - 4$.
From Theorem 3.13 we know that if a graph G is triangle free graph then the zero forcing number of graph $Z(G) \geq 2\delta - 2$.
Claim 1: Open global shadow graph of complete bipartite graph is triangle free. Since $K_{p,q}$ is triangle free, it can be seen that there is no induced C_3 in G or G'. In graph $GS(G)$ only possibility for it to have C_3 is when any two vertices $u, v \in V(G)$ such that $u \sim v$ and there exist a vertex $w \in V(G)$ such that uw and vw. So that $u \sim w'$ and $v \sim w'$. Forming an induced $C_3 : u \sim v \sim w' \sim u$. But in a complete bipartite graph G if $u, v \in V(G)$ and $u \sim v$, then u and v will be in two different partite set. In other words we can never find a vertex $w \in V(G)$ such that uw and vw. Hence open global shadow graph of complete a bipartite graph is a triangle free graph. Hence if G is a complete bipartite graph,

$$Z(GS(G)) \geq 2(n-1) - 2 = 2n - 4. \tag{4.1}$$

Let $K_{p,q}$ be a complete bipartite graph with p number of vertices say $u_1, u_2, \ldots u_p$ in 1st partite set P and q number of vertices say $v_1, v_2, \ldots, v_q$ in the second partite set Q. Now consider the open global shadow graph of $K_{p,q}$ and let $K'_{p,q}$ be the copy of $K_{p,q}$ with p number of vertices say $u'_1, u'_2, \ldots, u'_p$ in the set P' (P' is the partite set in $K'_{p,q}$ corresponding to the partite set P in $K_{p,q}$) and q number of vertices say $v'_1, v'_2, \ldots, v'_q$ in the set Q'(Q' is the partite set in $K'_{p,q}$ corresponding to the partite set Q in $K_{p,q}$). With out loss of generality, in the graph $GS(G)$ if $u_i \in P$ is taken black, then all the vertices in $\{P'\} \setminus u'_i$ ($p - 1$ number of vertices) and q vertices in Q are adjacent to u_i. By taking $N(u_i) \setminus v$ (Where $v \in Q \cup P' \setminus u_i$) to be black v can be forced black by u_i. Now all the q vertices in Q are colored black. If v_j is the vertex in Q, v_j is adjacent to all the vertices in P and $\{Q'\} \setminus v'_j$. By taking $N(v_j) \setminus u$ (Where $u \in Q' \setminus v_j \cup P$) to be black u can be forced black by v_j. Now any of the vertex other than u_i in P and any of the vertex other than v_j in Q can force u'_i and v'_j black respectively. So the cardinality of zero forcing set is given as $|S| = 1+p-1+q-1+q-1+p-2 = 2p + 2q - 4 = 2(p + q) - 4 = 2n - 4$. Therefore the bound for the zero forcing number will be

$$Z(GS(G)) \leq 2n - 4. \tag{4.2}$$

From equation 1 and 2, $Z(GS(G)) = 2n - 4$. $\square$

Corollary 4.4. *If G is a star graph on $n+1$ vertices, then $Z(GS(K_{1,n})) = 2Z(K_{1,n})$.*

Proof. Star graph is a bipartite graph $K_{p,q}$ where $p = 1$ and $q = n$. Clearly from the above Theorem 4.3 $Z(GS(G)) = 2(n+1) - 4 = 2n - 2$ and $Z(GS(G)) = 2Z(G)$. □

Observation 4.5. *If $GS(G) \cong 2K_n$, then $Z(GS(G))$ need not be twice $Z(G)$.*

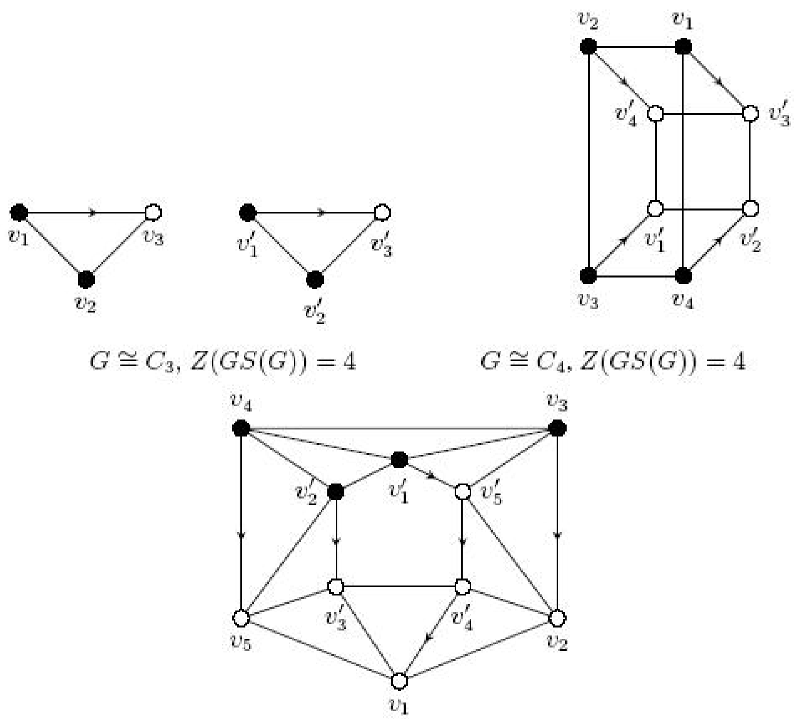
Let G be an n vertex graph with $GS(G) \cong 2K_n$ and $Z(GS(G)) = 2Z(K_n)$. Let us assume that $Z(GS(G)) = 2Z(G)$. We know that if $GS(G) \cong 2K_n$, then $Z(GS(G)) = 2n - 2$.

If $GS(G)$ is obtained by taking $G \cong K_n$, then $Z(GS(G)) = 2Z(G) = 2Z(K_n)$. However if $GS(G) \cong 2K_n$ doesn't always implies that $G \cong K_n$. According to Observation 2.7 suppose if G is a graph with two cliques (K_p and K_q, where $p + q = n$), then we know that $GS(G) \cong 2K_n$, but $Z(G) = n - 2 \neq n - 1$.

Theorem 4.6. *Let G be the graph C_n. Then $Z(GS(C_n)) = 2Z(C_n)$, if and only if, the cycle is either C_3, C_4 or C_5.*

Proof. Clearly when G is one of the graph C_3, C_4 or C_5 then $Z(GS(G)) = 4$ (refer figure 3). It is enough if we can show that n cannot take values greater than 6. When G is a cycle $Z(G) = 2$ (two of the consecutive vertices forms a zero forcing set). From the Observation 3.2 if $Z(GS(G)) = 2Z(G)$ implies $n6$. Clearly $3 \leq n \leq 5$ that is G is either C_3, C_4 or C_5. □

Observation 4.7. *In figure 3 the zero forcing number of C_3 and C_5 is obtained from $Z(G) \geq \delta$. The zero forcing number of C_4 can be observed in table 3.1.*



$G \cong C_5$, $Z(GS(G)) = 4$

Figure 3: All the possible open global shadow graphs of C_n, $3 \leq n \leq 5$.

Theorem 4.8. *If* $Z(G) < \frac{n-1}{2}$, *then* $Z(GS(G)) > 2Z(G)$.

Proof. Let G be a simple graph such that $Z(G) < \frac{n-1}{2}$. From Observation 3.2, we know that
$Z(GS(G)) \geq n - 1$.
$Z(GS(G)) \geq n - 1 = 2\frac{n-1}{2} > 2Z(G)$.
$Z(GS(G)) > 2Z(G)$. □

Converse of this theorem need not be true. If $Z(GS(G)) > 2Z(G)$, then this doesn't mean that $Z(G) < \frac{n-1}{2}$.

5. Conclusion and Scope

In this article we have discussed about the zero forcing number of the open global global shadow graph. It is an open problem to characterize connected graphs G for which $Z(GS(G)) = 2Z(G)$. It is an open problem to Characterize $Z(GS(G)) = 4$.

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References

[1] A. Aazami, *Hardness results and approximation algorithms for some problems on graphs,* UWSpace, 2008. [On line]. Available: https://bit.ly/3X7NwI1

[2] AIM Minimum rank- special graphs work group, "Zero forcing sets and the minimum rank of graphs", *Linear Algebra and its Applications,* vol. 428, no. 7, pp. 1628-1648, 2008. doi: 10.1016/j.laa.2007.10.009

[3] D. Amos, Y. Caro, R. Davila, and R. Pepper, "Upper bounds on the k-forcing number of a graph", *Discrete Applied Mathematics, vol. 181, pp. 1-10, 2015.* doi: 10.1016/j.dam.2014.08.029

[4] A. Berman, S. Friedland, L. Hogben, U. G. Rothblum, and B. Shader, "An upper bound for the minimum rank of a graph", *Linear Algebra and its Applications,* vol. 429, no. 7, pp. 1629-1638, 2008. doi: 10.1016/j.laa.2008.04.038

[5] D. Burgarth and V. Giovannetti, "Full control by locally induced relaxation", *Physical review letters,* vol. 99, no. 10, p. 100501, 2007. doi: 10.1103/PhysRevLett.99.100501

[6] D. W. Cranston and O. Suil, "Hamiltonicity in connected regular graphs", *Information Processing Letters,* vol. 113, no. 22-24, pp. 858-860, 2013. doi: 10.1016/j.ipl.2013.08.005

[7] M. Gentner, L. D. Penso, D. Rautenbach, and U. S. Souza, "Extremal values and bounds for the zero forcing number", *Discrete applied mathematics,* vol. 214, pp. 196-200, 2016. doi: 10.1016/j.dam.2016.06.004

[8] Leah.S.Mays, *The Zero Forcing Number Of Bipartite Graphs.* 2013. [On line]. Available: https://bit.ly/3F403Tz

[9] D. D. Row, "A technique for computing the zero forcing number of a graph with a cut- vertex", *Linear Algebra and its Applications,* vol. 436, no. 12, pp. 4423-4432, 2012. doi: 10.1016/j.laa.2011.05.012

[10] E. Sampathkumar and H. Walikar, "The connected domination number of a graph", *Journal of Mathematical Physics Science,* vol. 13, no. 6, pp. 607-613, 1979

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