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On the generalized Ornstein-Uhlenbeck operators perturbed by regular potentials and inverse square potentials in weighted L^2-spaces

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Abstract

Generation of a quasi-contractive semigroup by generalized Ornstein-Uhlenbeck operators

$$\mathcal{L} = -\Delta + \nabla\Phi \cdot \nabla - G \cdot \nabla + V + c|x|^{-2}$$

in the weighted space $L^2(\mathbf{R}^N, e^{-\Phi(x)}dx)$ *is proven, where* $\Phi \in C^2(\mathbf{R}^N, \mathbf{R})$, $G \in C^1(\mathbf{R}^N, \mathbf{R}^N)$, $0 \leq V \in C^1(\mathbf{R}^N)$ *and* $c > 0$. *The proofs are carried out by an application of* L^2*-weighted Hardy inequality and bilinear form techniques.*

Subjclass: *47D60, 47D06, 35K15.*

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1. Introduction

In recent years there has been an increasing interest in generalized Ornstein-Uhlenbeck operators, see [1], [2], [3], [4], [5], [6], [7], [8], [9], [12] and the references therein. Let us recall first some known results for generalized Ornstein-Uhlenbeck operators. It is known, see [4], under an appreciate conditions on Φ and G, that

$$A_{\Phi,G} = \Delta - \nabla\Phi \cdot \nabla u + G \cdot \nabla,$$

with domain $W^{2,p}(\mathbf{R}^N, d\mu)$ generates a positive quasi-contractive analytic C_0-semigroup in $L^p(\mathbf{R}^N, d\mu)$, where $\Phi \in C^2(\mathbf{R}^N, \mathbf{R})$, $G \in C^1(\mathbf{R}^N, \mathbf{R}^N)$, $d\mu = e^{-\Phi(x)}dx$ and $1 < p < \infty$. This result in [4] has been extended later partially to the case where $A_{\Phi,G}$ is perturbed by a potential $V \in C^1(\mathbf{R}^N)$ in [3] and [12]. Precisely, the authors established, under suitable assumptions on Φ, G and V, that $A_{\Phi,G} - V$ endowed with domain

$$W^{2,p}_V(\mathbf{R}^N, d\mu) = \Big\{u \in W^{2,p}_\mu(\mathbf{R}^N) : \ Vu \in L^p_\mu(\mathbf{R}^N)\Big\}$$

generates a quasi-contractive analytic C_0-semigroup on $L^p_\mu(\mathbf{R}^N)$ for $1 < p < \infty$. Afterwards, the operator $A_{\Phi,G}$ perturbed by a nonnegative singular potential νV in the space $L^p(\mathbf{R}^N, d\mu)$, $1 < p < \infty$, has been investigated in [6]. More specifically, by using perturbation techniques, it is proven that $A_{\Phi,G} - \nu V$ with a suitable domain generates a quasi-contractive and positive analytic C_0-semigroup in $L^p(\mathbf{R}^N, d\mu)$.

The aim of this present paper is to study the operator $A_{\Phi,G}$ perturbed with regular potential $0 \le V \in C^1(\mathbf{R}^N)$ and inverse square potential $c|x|^{-2}$ with $c > 0$ in the weighted space $L^2_\mu(\mathbf{R}^N)$.

We look for conditions on Φ, G, V and c ensuring that

$$A_{\Phi,G,V,c} = \Delta - \nabla\Phi \cdot \nabla u + G \cdot \nabla - V + c|x|^{-2}$$

with a suitable domain generates a positivity preserving C_0-semigroup in $L^2(\mathbf{R}^N, d\mu)$.
The proofs are based on an L^2-weighted Hardy's inequality and bilinear form techniques.

Now, we introduce the following conditions on Φ, G and V :

(A1) The function $\Phi \in C^2(\mathbf{R}^N, \mathbf{R})$.

(A2) The function $G \in C^1(\mathbf{R}^N, \mathbf{R}^N)$ satisfies $|G| \leq \kappa\Big(|\nabla\Phi|^2 + V + \alpha_0\Big)^{\frac{1}{2}}$ for some constants $\kappa \geq 0$ and α_0.

(A3) For every $\xi > 0$, there is a constant $C_\xi > 0$ such that $|D^2\Phi| \leq \xi|\nabla\Phi|^2 + C_\xi$.

(A4) There is constant $\beta \in \mathbf{R}$ such that $G \cdot \nabla\Phi - div\, G - V \leq \beta$.

(A5) There is constant $\kappa_1 > 0$ and $\alpha_1 \geq 0$ such that

$$|\nabla V| \leq \kappa_1 V^{\frac{3}{2}} + \alpha_1.$$

We point out that under the assumptions $(A1)-(A5)$ and the following conditions

$$G \cdot \nabla\Phi - div\, G - \theta V \leq \beta,$$

for some $\theta \in \mathbf{R}$ and

$$\frac{\theta}{p} + (p-1)\kappa_1\Big(\frac{\kappa}{p} + \frac{\kappa_1}{4}\Big) < 1,$$

Sobajima-Yokota proved in [12, Theorem 1.1] that the operator $A_{\Phi,G,V}$ with domain

$$W_V^{2,p}(\mathbf{R}^N, d\mu) = \Big\{u \in W_\mu^{2,p}(\mathbf{R}^N) : \; Vu \in L_\mu^p(\mathbf{R}^N)\Big\}$$

generates an analytic semigroup on $L_\mu^p(\mathbf{R}^N)$ for $1 < p < \infty$.
The paper is structured as follows, in Section 2, we prove an L^2-weighted Hardy inequality. Subsequently, we use them to investigate the perturbation of $A_{\Phi,G,V}$ with a singular potential $c|x|^{-2}, \quad c > 0$.

Notation Throughout this paper, we use the following notation: In the N-dimensional Euclidean space $\mathbf{R}^N$, $N \geq 3$, the Euclidean scalar product is denoted by $x \cdot y$ and $|x|$ is the corresponding norm. $C_c^\infty(\mathbf{R}^N)$ means the space of C^∞-functions with compact support.

The weighted space $L_\mu^2(\mathbf{R}^N) = L^2(\mathbf{R}^N, d\mu)$, where $d\mu = e^{-\Phi(x)}dx$. In addition, we denote by $H_\mu^1(\mathbf{R}^N)$ the set of all functions $f \in L_\mu^2(\mathbf{R}^N)$ having distributional derivative ∇f in $\Big(L_\mu^2(\mathbf{R}^N)\Big)^N$. Besides, we denote the following weighted Sobolev space

$$H^1_V(\mathbf{R}^N,\mu)=\Big\{u\in H^1_\mu(\mathbf{R}^N), Vu\in L^2_\mu(\mathbf{R}^N)\Big\}.$$

Finally, by D_k, ∇, D^2, and Δ we designate, respectively, the (distributional) partial derivatives $\frac{\partial}{\partial x_k}$, the gradient, Hessian matrix and Laplace operator. If u is smooth enough, we set

$$|\nabla u(x)|^2=\sum_{k=1}^N |D_k u(x)|^2,\quad |D^2u(x)|^2=\sum_{k,j=1}^N |D_kD_ju(x)|^2.$$

2. Hardy inequality

In this section, we establish an L^2-weighted Hardy inequality, which will be useful later on.

Theorem 2.1. *Assume $N\geq 3$ and $(A3)$ hold. Then, for any $u\in C_c^\infty(\mathbf{R}^N)$, one has*

$$\left(\frac{N-2}{2}\right)^2\int_{\mathbf{R}^N}\frac{|u|^2}{|x|^2}d\mu\leq (4+\delta)\int_{\mathbf{R}^N}|\nabla u|^2d\mu+c_\delta\int_{\mathbf{R}^N}|u|^2d\mu \tag{2.1}$$

for any $\delta>0$ with a corresponding constants $c_\delta>0$.

Proof. Let $u\in C_c^\infty(\mathbf{R}^N)$. We have

$$u(x)\exp(-\frac{\Phi(x)}{2})=-\int_1^\infty \frac{d}{dt}\Big(u(tx)\exp(-\frac{\Phi(tx)}{2})\Big)dt.$$

Thus, by a change of variables, we infer that

$$\begin{aligned}\Big\|\tfrac{u}{|x|}\Big\|_{L^2_\mu} &\leq \Big(\textstyle\int_1^\infty t^{-\frac{N}{2}}dt\Big)\Big\|\nabla u-\tfrac12 u\nabla\Phi\Big\|_{L^2_\mu}\\ &\leq \tfrac{2}{N-2}\Big\|\nabla u-\tfrac12 u\nabla\Phi\Big\|_{L^2_\mu}.\end{aligned}$$

Furthermore, by applying the Hölder, Young and Jensen inequalities, we get

$$\begin{aligned}&\Big(\tfrac{N-2}{2}\Big)^2\textstyle\int_{\mathbf{R}^N}\frac{u^2}{|x|^2}d\mu\\ &\leq \textstyle\int_{\mathbf{R}^N}|\nabla u|^2u^2d\mu+\frac14\int_{\mathbf{R}^N}|\nabla\Phi|^2u^2d\mu-\int_{\mathbf{R}^N}u\nabla\Phi\cdot\nabla u d\mu\\ &\leq \textstyle\int_{\mathbf{R}^N}|\nabla u|^2u^2d\mu+\frac14\int_{\mathbf{R}^N}|\nabla\Phi|^2u^2d\mu+\Big(\int_{\mathbf{R}^N}|\nabla u|^2d\mu\Big)^{\frac12}\Big(\int_{\mathbf{R}^N}|\nabla\Phi|^2u^2d\mu\Big)^{\frac12}\\ &\leq (\tfrac14+\tfrac{\kappa}{2})\textstyle\int_{\mathbf{R}^N}|\nabla\Phi|^2u^2d\mu+(1+\frac{1}{2\kappa})\int_{\mathbf{R}^N}|\nabla u|^2d\mu,\end{aligned}$$

for every $\kappa > 0$.
In addition, combining integration by parts, $(A3)$ and Young's inequalities, we deduce that

$$\begin{aligned}\int_{\mathbf{R}^N} |\nabla\Phi|^2 u^2 d\mu \; &= \int_{\mathbf{R}^N} \Delta\Phi u^2 d\mu + 2\int_{\mathbf{R}^N} u\nabla\Phi\cdot\nabla u d\mu \\ &\le (N\xi + \tfrac{1}{2})\int_{\mathbf{R}^N} |\nabla\Phi|^2 u^2 d\mu + NC_\xi \int_{\mathbf{R}^N} u^2 d\mu + 2\int_{\mathbf{R}^N} |\nabla u|^2 d\mu.\end{aligned}$$

Hence, we have

$$\int_{\mathbf{R}^N} |\nabla\Phi|^2 u^2 d\mu \;\le \frac{2NC_\xi}{1-2N\xi}\int_{\mathbf{R}^N} u^2 d\mu + \frac{4}{1-2N\xi}\int_{\mathbf{R}^N} |\nabla u|^2 d\mu,$$

for every $\xi \in (0, \frac{1}{2N})$. Then, collecting all the terms, we conclude that

$$\begin{aligned}\left(\frac{N-2}{2}\right)^2 \int_{\mathbf{R}^N} \frac{u^2}{|x|^2} d\mu \; &\le \left[(\tfrac{1}{4}+\tfrac{\kappa}{2})\frac{4}{1-2N\xi} + 1 + \frac{1}{2\kappa}\right]\int_{\mathbf{R}^N} |\nabla u|^2 d\mu \\ &\quad + (\tfrac{1}{4}+\tfrac{\kappa}{2})\frac{2NC_\xi}{1-2N\xi}\int_{\mathbf{R}^N} u^2 d\mu.\end{aligned}$$

Therefore, taking the minimum with respect to κ, that is, choosing $\kappa = \frac{1}{2}$ and ξ small, we get (2.1). $\square$

3. Generation result via the bilinear form technique

This section devoted to study the generation of the quasi-contractive semigroup by $A_{\Phi,G,V,c}$.
First of all, we introduce the bilinear form

$$b(u,v) = \int_{\mathbf{R}^N} \nabla u\cdot\nabla\overline{v}\, d\mu - \int_{\mathbf{R}^N} \overline{v}\, G\cdot\nabla u\, d\mu + \int_{\mathbf{R}^N} V u\overline{v}\, d\mu - c\int_{\mathbf{R}^N} \frac{u\overline{v}}{|x|^2}\, d\mu$$

for $u, v \in D(b) = H^1_V(\mathbf{R}^N, \mu)$, where $N \ge 3$ and $c > 0$.

Proposition 3.1. *Suppose that (A1)-(A4) hold, and let $N \ge 3$. Set $\gamma = \frac{1}{4}\left(\frac{N-2}{2}\right)^2$.*
Then, b is closed and quasi-accretive for all $c \in (0, \gamma)$.

Proof. We fix $c \in (0,\gamma)$, where $\gamma = \frac{1}{4}\left(\frac{N-2}{2}\right)^2$. Let δ be small such that $K = c(4+\delta)\left(\frac{N-2}{2}\right)^{-2} < 1$. We observe that

$$\begin{aligned}b(u,u) \; &\ge K\Big[\int_{\mathbf{R}^N} |\nabla u|^2 d\mu - \frac{1}{(4+\delta)}\left(\frac{N-2}{2}\right)^2 \int_{\mathbf{R}^N} |x|^{-2}|u|^2 d\mu \\ &\quad - \frac{1}{K}\int_{\mathbf{R}^N} \overline{u}G\cdot\nabla u|u|^2 d\mu + \frac{1}{K}\int_{\mathbf{R}^N} V|u|^2 d\mu\Big] + (1-K)\int_{\mathbf{R}^N} |\nabla u|^2 d\mu.\end{aligned}$$

By virtue of integration by parts we have

$$\begin{aligned} b(u,u) &\geq K\Bigg[\int_{\mathbf{R}^N}|\nabla u|^2 d\mu - \frac{1}{(4+\delta)}\left(\frac{N-2}{2}\right)^2\int_{\mathbf{R}^N}|x|^{-2}|u|^2 d\mu \\ &+\frac{1}{2K}\int_{\mathbf{R}^N}\Big(div\, G - G\cdot\nabla\Phi + 2V\Big)|u|^2 d\mu\Bigg] + (1-K)\int_{\mathbf{R}^N}|\nabla u|^2 d\mu. \end{aligned}$$

By applying (A4) and the weighted Hardy inequality (2.1), we obtain

$$b(u,u) \geq -K\left(\frac{\beta}{2K}+\frac{c_\delta}{4+\delta}\right)\int_{\mathbf{R}^N}|u|^2 d\mu + (1-K)\int_{\mathbf{R}^N}|\nabla u|^2 d\mu.$$

Hence, we see that

$$\begin{aligned} &b(u,u) + \left[K\left(\frac{\beta}{2K}+\frac{c_\delta}{4+\delta}\right)+1-K\right]\|u\|^2_{L^2_\mu} + (1-K)\|Vu\|^2_{L^2_\mu} \\ &\geq (1-K)\|u\|^2_{H^1_V(\mathbf{R}^N,\mu)}. \end{aligned}$$

Thus, we have

$$b(u,u) + \|u\|^2_{L^2_\mu} + |Vu\|^2_{L^2_\mu} \geq \frac{1-K}{\left(K\left(\frac{\beta}{2K}+\frac{c_\delta}{4+\delta}\right)+1\right)}\|u\|^2_{H^1_V(\mathbf{R}^N,\mu)}.$$

Therefore this proves that the norm $\|.\|_b$ associated to the bilinear form is equivalent to $\|.\|_{H^1_V(\mathbf{R}^N,\mu)}$.

In the sequel, we want to establish that b is quasi-accretive. Indeed, thanks to integration by parts we have

$$\begin{aligned} b(u,u) &\geq \int_{\mathbf{R}^N}|\nabla u|^2 d\mu - c\int_{\mathbf{R}^N}|x|^{-2}|u|^2 d\mu \\ &+\frac{1}{2}\int_{\mathbf{R}^N}\Big(div\ G - G\cdot\nabla\Phi + 2V\Big)|u|^2 d\mu. \end{aligned}$$

By means of the inequality (2.1) and the assumption (A4), we deduce that

$$b(u,u) \geq (\gamma - c)\int_{\mathbf{R}^N}|x|^{-2}|u|^2 d\mu - \left(\frac{\beta}{2}+\frac{c_\delta}{4+\delta}\right)\int_{\mathbf{R}^N}|u|^2 d\mu.$$

Since $c<\gamma$, it follows that

$$b(u,u) \geq -\left(\frac{\beta}{2}+\frac{c_\delta}{4+\delta}\right)\int_{\mathbf{R}^N}|u|^2 d\mu. \qquad \square$$

Whence, for $c<\gamma$, one can associate with the form b the operator $\mathcal{L}$ on $L^2_\mu(\mathbf{R}^N)$, defined by $D(\mathcal{L}) = \Big\{u \in D(b)$: there exists $v \in L^2_\mu(\mathbf{R}^N)$ such that $b(u,\psi) = \int_{\mathbf{R}^N} v\psi d\mu$ for all $\varphi \in D(b)\Big\}$ and $\mathcal{L}u = v$.

We are now in position to state and establish the main results in this paper:

Theorem 3.2. *Assume that (A1)-(A4) are satisfied, and let $N \geq 3$. Set $\gamma = \frac{1}{4}\left(\frac{N-2}{2}\right)^2$.*
The operator $-\mathcal{L} = A_{\Phi,G,V,c}$ generates a quasi-contractive and positive

semigroup on $L^2_\mu(\mathbf{R}^N)$ for all $c \in (0, \gamma)$.
In particular, the semigroup $\{T(t)\}_{t\geq 0}$ is symmetric if $G = 0$.

Proof. By virtue of [10, Theorem 1.51] and [10, Theorem 1.52], we conclude via Proposition 3.1 that $-\mathcal{L}$ generates a quasi-contractive semigroup $\{T(t)\}_{t\geq 0}$ on $L^2_\mu(\mathbf{R}^N)$. It remains therefore only to prove the semigroup $\{T(t)\}_{t\geq 0}$ is symmetric if $G = 0$. Indeed, it is clear that the form b is symmetric if $G = 0$. Hence, the associated operator $A_{\Phi,G,V,c}$ is also symmetric. Moreover, through Proposition 3.1, $A_{\Phi,G,V,c}$ is quasi-dissipative since the form b is quasi-accretive. Therefore, we have

$$\langle u, (1 - \lambda A_{\Phi,G,V,c})^{-1} v\rangle = \langle (1 - \lambda A_{\Phi,G,V,c})^{-1} u, v\rangle$$

Setting $\lambda = \frac{t}{n}$ for $t > 0$ and $n \in \mathbf{N}$, we have

$$\langle u, (1 - \frac{t}{n} A_{\Phi,G,V,c})^{-n} v\rangle = \langle (1 - \frac{t}{n} A_{\Phi,G,V,c})^{-n} u, v\rangle$$

Letting n to infinity, we infer that the semigroup $T(t)$ is symmetric. $\square$

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