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Paranormed Norlund N^t- difference sequence spaces and their α-, β- and γ-duals

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Abstract

Kizmaz [4] defined some difference spaces viz., $\ell_\infty(\Delta), c(\Delta)$ *and* $c_0(\Delta)$ *and studied by Et and Colak [1] thoroughly. In this paper, Norlund* N^t*- difference sequence spaces* $N^t(c_0, p, \Delta)$, $N^t(c, p, \Delta)$ *and* $N^t(\ell_\infty, p, \Delta)$ *contain the sequences whose* $N^t\Delta$*-transforms in* c_0, c *and* ℓ_∞ *are defined and the paranormed linear structures are developed on these spaces. It has been shown that the spaces* $N^t(c_0, p, \Delta)$, $N^t(c, p, \Delta)$ *&* $N^t(\ell_\infty, p, \Delta)$ *are linearly isomorphic and are of non-absolute type. Further, it is verified that* $N^t(c, p, \Delta)$, $N^t(c_0, p, \Delta)$ *and* $N^t(l_\infty, p, \Delta)$ *of non-absolute form are isomorphic to* $N^t(c_0, p)$, $N^t(c, p)$ *and* $N^t(\ell_\infty, p)$*, respectively. Topological properties such as the completeness and the isomorphism are also discussed. Some inclusion relations among these spaces are also verified. Finally, the* α-, β- *and* γ- *dual of these spaces are determined and constructed the Schauder-basis of* $N^t(c_0, p, \Delta)$ *and* $N^t(c, p, \Delta)$.

Keywords: *Paranormed sequence space,* N^t*-Difference sequence space, Norlund matrix, Schauder basis,* α-, β- *and* γ*-duals.*

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1. Introduction

Throughout the paper, w will denote the space of all sequence of complex numbers and ℓ_∞, c and c_0 are the spaces of all bounded, convergent and null sequences, respectively, cs, bs, ℓ_1 and ℓ_p for the sequence spaces of all convergent, bounded, absolutely and p-absolutely convergent series, respectively.

Definition 1.1. *Paranormed Space A linear space Y over $\mathbf{R}$ is a paranormed space if there is a sub-additive function $g : X \to \mathbf{R}$ such that $g(\theta) = 0$, $g(x) = g(-x)$ and $|\alpha_n - \alpha| \to 0$ and $g(x_n - x) \to 0 g(\alpha_n x_n - \alpha x) \to 0, \forall\, \alpha \in \mathbf{R}, x \in X$, where θ is the zero vector.*

Let λ_1 and λ_2 be any two sequence spaces and $A = (a_{nk})$ as any infinite matrix of $a_{nk} \in \mathbf{R}$, $n, k \in \mathbf{N}$. Then we say that A defines a matrix mapping from λ_1 into λ_2 as $A : \lambda_1 \to \lambda_2$ if $x = (x_n) \in \lambda_1 Ax = \{(Ax)_n\} \in \mu$, where

$$(Ax)_n = \sum_k a_{nk} x_k \quad (n \in \mathbf{N}) \tag{1.1}$$

Here$(\lambda_1 : \lambda_2)$, we denote the class of all matrices A such that the series in (1.1) converges for each $n \in \mathbf{N}$ and every $x \in \mu$, A sequence x is said to A summable to a if Ax converges to a, which is called as the A-limit of x.

Assume here and after that (p_n) is bounded sequence of strictly positive real numbers with $\sup p_n = H$ and $M = \max\{1, H\}$. Then the linear spaces ℓ_∞, $c(p)$ and $c_0(p)$ were defined by Maddox in [8, 11, 13]) as follows:

$$\ell_\infty = \left\{ x = (x_n) \in w : \sup_{n \in \mathbf{N}} |x_n|^{p_n} < \infty \right\}$$

$$c(p) = \left\{ x = (x_n) \in w : \lim_{n \to \infty} |x_n - L|^{p_n} = 0, \text{forsome}\, L > 0 \right\}$$

$$c_0(p) = \left\{ x = (x_n) \in w : \lim_{n \to \infty} |x_n|^{p_n} = 0 \right\}$$

These are complete sequence spaces in the paranormed

$$g(x) = \sup_{n \in \mathbf{N}} |x_n|^{\frac{p_n}{M}}, \quad \text{iff} \quad \inf_{k \in \mathbf{N}} p_k > 0 \tag{1.2}$$

For the sequence space μ and ν, the set $S(\mu,\ \nu)$ is defined as

$$S(\mu, \nu) = \{z \in w : xz \in \nu, \forall\, x \in \mu\} \tag{1.3}$$

The α-, β-, γ-duals of κ, which are respectively denoted by κ^α, κ^β and κ^γ are $\kappa^\alpha = S(\kappa, \ell_1), \kappa^\beta = S(\kappa, cs), \kappa^\gamma = S(\kappa, bs)$.

Definition 1.2 (Schauder Basis). *A sequence (b_n) is called Schauder basis of the paranormed sequence space (μ, g), if $x \in \mu$, $\exists(\beta_n)$ such that* $\lim_{k\to\infty} g(x - \sum_{n=0}^{k} \beta_n b_n) = 0.$

Peyerimhoff [12] and Mears [10] gave the concept of the Norlund Means. Let $T_n = \sum_{k=0}^{n} t_k, \forall n \in \mathbf{N}, t_k \geq 0, t_0 > 0$. Then the Norlund means for $t = (t_k)$ is a matrix $N^t = (a^t_{nk})$, where

$$a^t_{nk} = \begin{cases} \frac{t_{n-k}}{T_n}, & 0 \leq k \leq n \\ 0, & k > n \end{cases} \tag{1.4}$$

for all $n \in \mathbf{N}$.
Norlund matrix N^t is a Toeplitz matrix iff $t_n/T_n \to 0$, as $n \to \infty$. If $t = e = (1, 1, 1, ...)$, then the Norlund matrix N^t is reduced to the matrix C_1 of arithematic means. For $t_n = A_n^{r-1}$, the method N^t gives Cesaro method C_r with $r > -1$, where, for $n \in \mathbf{N}$:

$$A^r_n = \begin{cases} \frac{(r+1)(r+2).....(r+n)}{n!}, & n \in \mathbf{N} \\ 1, & n = 0. \end{cases} \tag{1.5}$$

For $t_0 = D_0 = 1$, define the determinant D_n, for $n \in \mathbf{N}$ as follows

$$D_n = \begin{vmatrix} t_1 & 1 & 0 & 0 & \cdots & 0 \\ t_2 & t_1 & 1 & 0 & \cdots & 0 \\ t_3 & t_2 & t_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ t_{n-1} & t_{n-2} & t_{n-3} & t_{n-4} & \cdots & 1 \\ t_n & t_{n-1} & t_{n-2} & t_{n-3} & \cdots & t_1 \end{vmatrix} \tag{1.6}$$

Let $V^t = (r^t_{nk})$ be the inverse of $N^t = (a^t_{nk})$, [10], then

$$r^t_{nk} = \begin{cases} (-1)^{n-k} D_{n-k} T_k, & 0 \leq k \leq n, \\ 0, & k > n \end{cases} \tag{1.7}$$

for all $n,\ k \in \mathbf{N}$. Also for all $k \in \mathbf{N}$, we have

$$D_k = \sum_{i=1}^{k-1} (-1)^{i-1} t_i\ D_{k-i} + (-1)^{k-1}\ t_k. \tag{1.8}$$

In this paper, the Norlund-difference sequence spaces $N^t(c_0, p, \Delta)$, $N^t(c, p, \Delta)$ and $N^t(\ell_\infty, p, \Delta)$ of the sequences whose $N^t\Delta$-transform are in c_0, c and ℓ_∞ respectively ate introduced and investigated some topological properties, inclusion relations between among these sequence spaces.

2. The Norlund sequence spaces $N^t(c_0, p, \Delta)$, $N^t(c, p, \Delta)$ and $N^t(l_\infty, p, \Delta)$

In this section, the paranormed spaces $N^t(c_0, p, \Delta)$, $N^t(c, p, \Delta)$ and $N^t(\ell_\infty, p, \Delta)$ are defined and the paranormed structures are developed on these spaces. It has been shown that these spaces are linearly isomorphic.

Kizmaz [4] defined some difference spaces viz., $\ell_\infty(\Delta), c(\Delta)$ and $c_0(\Delta)$ and studied by Et and Colak [1] thoroughly. Let $m(\geq 0) \in \mathbf{Z}$. Then for any given sequence space λ, we have $\lambda(\Delta^m) = \{z = (z_n) \in w : (\Delta^m x_n) \in \lambda\}$ for $\lambda = c_0, c$ and l_∞ where $\Delta^m x = (\Delta^m x_n) = (\Delta^{m-1} x_n - \Delta^{m-1} x_{n+1}$ and so that

$$\Delta^m x_n = \sum_{v=0}^{m} (-1)^v \binom{m}{v} x_{n+v}.$$

Yesilkayagil and Basar [15] defined the Norlund sequence space $N^t(p)$ as

$$N^t(p) = \left\{ z = (z_n) \in w : \sum_n \left| \frac{1}{T_k} \sum_{i=0}^{n} t_{k-i}\ x_i \right|^{p_n} < \infty \right\}$$

with $0 < p_n \leq H < \infty$.

We introduced the Δ-Norlund difference sequence spaces $N^t(c, p, \Delta), N^t(c_0, p, \Delta)$ and $N^t(\ell_\infty, p, \Delta)$, for $x \in w$, as follows (for $L > 0$):

$$N^t(c_0, p, \Delta) = \left\{ x : \lim_{n\to\infty} \left| \frac{1}{T_n} \sum_{i=0}^{n} t_{k-i}\ \Delta x_i \right|^{p_n} = 0 \right\}$$

$$N^t(c, p, \Delta) = \left\{ x : \lim_{n\to\infty} \left| \frac{1}{T_n} \sum_{i=0}^{n} t_{k-i} (\Delta x_i - L) \right|^{p_n} = 0 \right\}$$

$$N^t(\ell_\infty, p, \Delta) = \left\{ x : \sup_n \left| \frac{1}{T_n} \sum_{i=0}^{n} t_{k-i}\ \Delta x_i \right|^{p_n} < \infty \right\}$$

The sequence spaces can redefine the spaces $N^t(c_0, p, \Delta)$, $N^t(c, p, \Delta)$ and $N^t(\ell_\infty, p, \Delta)$ respectively as $N^t(c_0, p, \Delta) = (c_0(p))_{N^t}$, $N^t(c, p, \Delta) = (c(p))_{N^t}$ and $N^t(l_\infty, p, \Delta) = (\ell_\infty(p))_{N^t}$.

Define the sequence $y = (\Delta y_n)$ by the $N^t(\Delta)$-transform of sequence $x = (\Delta x_n)$, so we have

$$y = (y_n) = \frac{1}{T_n} \sum_{i=0}^{n} t_{n-i}\ \Delta x_i\ \forall n \in \mathbf{N} \tag{2.1}$$

Theorem 2.1. *$N^t(c,p,\Delta)$, $N^t(c_0,p,\Delta)$ and $N^t(\ell_\infty,p,\Delta)$ are the complete linear metric space paranormed by*

$$g_1(x) = \sup_n \left| \frac{1}{T_k} \sum_{i=0}^{n} t_{k-i}\ \Delta x_i \right|^{\frac{p_n}{M}}$$

with $0 < p_n \leq H < \infty$ with $M = \max\{1, H\}$.

Proof. The result is proved for $N^t(c_0,p,\Delta)$. And the supremum of every bounded sequence is finite, the result for the other spaces can be proved analogously.

Let $x, y \in N^t(c_0,p,\Delta)$, then

$$\begin{aligned} \sup_n \left| \frac{1}{T_n} \sum_{i=0}^{n} t_{k-i} \Delta(x_i + y_i) \right|^{\frac{p_n}{M}} \quad &\leq \quad \sup_n \left| \frac{1}{T_n} \sum_{i=0}^{n} t_{k-i} \Delta x_i \right|^{\frac{p_n}{M}} \\ &+ \quad \sup_n \left| \frac{1}{T_n} \sum_{i=0}^{n} t_{k-i}\ \Delta y_i \right|^{\frac{p_n}{M}} \end{aligned}$$

and for any $\alpha \in \mathbf{R}$, we have

$$\alpha^{p_k} \leq \{\max 1, \alpha^M\}, \tag{2.2}$$

Clearly, $g_1(\theta) = 0, g_1(x) = g_1(-x) \ \ \forall\ x \in N^t(c_0,p,\Delta)$. Therefore, inequalities (2.2) and (2.2) give sub-additivity of g_1 and $g_1(\alpha x) \leq \max(1, \alpha^M) g_1(x)$. Further, let $(x^{(n)}) \in N^t(c_0,p,\Delta)$, then $g_1(x^{(n)} - x) \to 0$ and let (α_n) be any sequence of scalars such that $\alpha_n \to \alpha$. Thus,

$$\begin{aligned} g_1(\alpha_n x^{(n)} - \alpha x) \quad &= \sup_n \left| \tfrac{1}{T_n} \textstyle\sum_{i=0}^{n} t_{k-i}\ \Delta(\alpha_n\ x_i^{(n)} - \alpha x_i) \right|^{\frac{p_n}{M}} \\ &\leq \alpha_n - \alpha^{\frac{p_n}{M}} g_1(x^n) + \alpha^{\frac{p_n}{M}} g_1(x^n - x) \\ &\to 0 \quad \text{as} \quad n \to \infty \end{aligned}$$

Hence g_1 is paranorm.

Now, let $\{x^j\}$ be any Cauchy sequence in $N^t(c_0,p,\Delta)$, with $x^j = \{x_0^{(j)}, x_1^{(j)}, x_2^{(j)}, ...\}$. For given $\epsilon > 0\ \exists\, n_0(\epsilon)$ such that $g_1(x^j - x^i) < \epsilon \ \ \forall\ j, i \geq n_0(\epsilon)$. Then, for $k \in \mathbf{N}$,

$$
\begin{aligned}
&(N^t(\Delta,p)x^j)_k - (N^t(\Delta,p)x^i)_k^{\frac{p_n}{M}} \\
&\leq \sup_n (N^t(\Delta,p)x^j)_k - (N^t(\Delta,p)x^i)_k^{\frac{p_n}{M}} \\
&\qquad < \frac{\epsilon}{2}, \forall j,i > n_0(\epsilon)
\end{aligned}
$$

which yields the Cauchy sequence of real numbers $\{(N^t(\Delta,p)x^0)_k, (N^t(\Delta,p)x^1)_k, \ldots\}$, for $k \in \mathbf{N}$. Hence, $(N^t(\Delta,p)x^j)_k \to (N^t(\Delta,p)x)_k$ as $j \to \infty$. For $(N^t(\Delta,p)x)_0, (N^t(\Delta,p)x)_1, \ldots$ infinitely many limits, there is a the sequence $\{(N^t(\Delta,p)x)_0, (N^t(\Delta,p)x)_1, (N^t(\Delta,p)x)_2, \ldots\}$. Using (2.3) as $i \to \infty$, we get

$$(N^t(\Delta,p)x^j)_k - (N^t(\Delta,p)x)_k < \frac{\epsilon}{2},\ j \geq n_0(\epsilon)$$

Since $x^j = (x_n^j) \in N^t(c_0,p,\Delta)$ for each $j \in \mathbf{N}$, there exists $n_0(\epsilon) \in \mathbf{N}$ such that
$(N^t(\Delta,p)x^j)_k^{\frac{p_n}{M}} < \frac{\epsilon}{2}$ for every $j \geq n_0(\epsilon)$ and $k \in \mathbf{N}$.
Taking a fixed $j \geq n_0(\epsilon)$, we obtain by (2.6) that

$$
\begin{aligned}
(N^t(\Delta,p)x)_k^{\frac{p_n}{M}} &\leq (N^t(\Delta,p)x^j)_k - (N^t(\Delta,p)x)_k^{\frac{p_n}{M}} \\
&+ (N^t(\Delta,p)x^j)_k^{\frac{p_n}{M}} < \epsilon
\end{aligned}
$$

for every $j \geq n_0(\epsilon)$. Therefore, $x \in N^t(c_0,p,\Delta)$. □

Remark 2.2. *For the spaces $N^t(c_0,p,\Delta)$, the property of absolute is not satisfied, i.e., $g_1(x) \neq g_1(x)$, so that $N^t(c_0,p,\Delta)$ is of non-absolute type, $x = (x_n)$.*

Theorem 2.3. *The spaces $N^t(c,p,\Delta)$, $N^t(c_0,p,\Delta)$ and $N^t(l_\infty,p,\Delta)$ of non-absolute type are paranorm or norm isomorphic to $N^t(c_0,p)$, $N^t(c,p)$ and $N^t(\ell_\infty,p)$ respectively, for $0 < p_n \leq H < \infty$.*

Proof. Define a linear transformation $T : N^t(c_0,p,\Delta) \to N^t(c_0,p)$ by $Tx = N^t(c_0,p,\Delta)x$. For $x = \theta$, whenever $Tx = \theta$ and hence T is injective. Suppose $y \in N^t(c_0,p)$ and define the sequence $x = (x_n) = (\Delta x_n)$ by $x = (x_n) = \sum_{j=0}^{n} (-1)^{n-j}\ D_{n-j}T_j\ \Delta y_j,\quad \forall\ n \in \mathbf{N}$. Thus, we have

$$\begin{aligned} g_1(x) &= \sup_n \left| \frac{1}{T_n} \sum_{i=0}^n t_{n-i} \; \Delta x_i \right|^{\frac{p_n}{M}} \\ &= \sup_n \left| \frac{1}{T_n} \sum_{i=0}^n t_{n-i} \sum_{j=0}^n (-1)^{n-j} \; D_{n-j} T_j \; \Delta y_j \right|^{\frac{p_n}{M}} \\ &= \sup_n |y_n|^{\frac{p_n}{M}} \; < \; \infty \end{aligned}$$

Thus, $x \in N^t(c_0, p, \Delta)$ and so T is onto and preserved under paranorm. Hence, $N^t(c_0, p, \Delta)$ and $N^t(c_0, p)$ are linearly isomorphic.

Analogously, it can be verifies that $N^t(c, p, \Delta) \cong N^t(c, p)$ and $N^t(\ell_\infty, p, \Delta) \cong N^t(l_\infty, p)$. □

Theorem 2.4. *Let $u^{(n)}(t) = \{u_k^{(n)}(t)\}$ be a sequence defined as*

$$u_k^{(n)}(t) \quad = \begin{cases} (-1)^{(k-n)} D_{k-n} T_n, & 0 \le n \le k \\ 0, & n > k \end{cases} .$$

Then

a) $\{u^{(n)}(t)\}_{n \in \mathbf{N}}$ is a basis for $N^t(c_0, p, \Delta)$ and every $x \in N^t(c_0, p, \Delta)$ has a unique representation as $x = \sum_n \alpha_n(t) u^{(n)}(t)$, where $\alpha_n(t) = (N^t(\Delta, p)x)_n, \forall \; n \in \mathbf{N}$ and $0 < p_n \le H < \infty$.

b) The set $\{e, u^{(n)}(t)\}$ is a basis of $N^t(c, p, \Delta)$ and every $x \in N^t(c, p, \Delta)$ has a unique representation as $x = \eta e + \sum_n [\alpha_n(t) - \eta] \; u^{(n)}(t)$, where $\eta = \lim\limits_{n \to \infty} (N^t(\Delta, p)x)_n$.

Proof.

a) Clearly, $\{u^{(n)}(t)\} \subset N^t(c_0, p, \Delta)$, also

(2.3) $\quad N^t u^{(n)}(t) = e^{(n)} \in l(c_0, \Delta), \quad \forall \; n \in \mathbf{N}, 0 < p_n \le H < \infty.$

Let $x \in (N^t(c_0, p, \Delta)$ be given. For every non-negative integer m, we take

$$x^{[m]} = \sum_{n=0}^{m} \alpha_n(t) u^{(n)}(t) \tag{2.4}$$

Then, by using N^t to (**??**) with (2.4), we have

$$N^t x^{[m]} = \; \sum_{n=0}^m \alpha_n(t) N^t u^{(n)}(t) = \; \sum_{n=0}^m (N^t x)_n \; e^{(n)}$$

Now $\forall i, m \in \mathbf{N}$, we obtain

$$\{N^t(x - x^{[m]})\}_i \quad = \begin{cases} 0\ , & 0 \le i \le m \\ (N^t x)_i\ , & i > m \end{cases}$$

For any given $\epsilon > 0$, there exists $m_0 \in \mathbf{N}$ such that

$$\left[\sum_{i=m}^{\infty} (N^t x)_i^{p_n}\right]^{\frac{1}{M}} < \frac{\epsilon}{2}, \quad \forall m \ge m_0.$$

Therefore,

$$\begin{aligned} g\left[N^t(x - x^{[m]})\right] \quad &= \left[\sum_{i=m}^{\infty} (N^t x)_i^{p_n}\right]^{\frac{1}{M}} \\ &\le \left[\sum_{i=m_0}^{\infty} (N^t x)_i^{p_n}\right]^{\frac{1}{M}} \\ &< \epsilon, \quad \forall m \ge m_0 \end{aligned}$$

Now, if possible assume that $x = \sum \mu_n(t) u^{(n)}(t)$. Then,

$$\begin{aligned} (N^t x)_k \quad &= \textstyle\sum_n \mu_n(t) \{N^t u^{(n)}(t)\}_k \\ &= \textstyle\sum_n \mu_n(t)\ e_k^{(n)} = \mu_k(t), \quad \forall\ k \in \mathbf{N} \end{aligned}$$

which is absurd.

b) Since $\{u^{(n)}(t)\} \subset N^t(c_0, p, \Delta)$ and $e \in c$ and the inclusion $\{e, u^{(n)}(t)\} \subset N^t(c, p, \Delta)$ is trivial. For $x \in N^t(c, p, \Delta)$, there exist unique η satisfying (2.4). So, $l \in N^t(c_0, p, \Delta)$ whenever $l = x - \eta e$. Hence, by part (a) that the representation of ℓ is unique.

□

3. The Inclusion Relations

Some inclusion relations between the sequence spaces $l_\infty(p), c(p), c_0(p)$ and $N^t(l_\infty, p, \Delta)$, $N^t(c, p, \Delta)$, $N^t(c_0, p, \Delta)$ have been defined and studied in this section.

Theorem 3.1. *The inclusion* $N^t(c_0, p, \Delta) \subset N^t(c, p, \Delta) \subset N^t(\ell_\infty, p, \Delta)$ *strictly hold.*

Proof. Let $y \in N^t(c_0, p)$, then $N^t\Delta x \in N^t(c_0, p)$. Since $N^t(c_0, p) \subset N^t(c, p)$, we obtain $N^t\Delta x \in N^t(c, p)$ and so that $x \in N^t(c, p, \Delta)$. Hence the inclusion $N^t(c_0, p, \Delta) \subset N^t(c, p, \Delta)$. Further, since $N^t\Delta x \in N^t(c, p)$ for every $x \in N^t(c, p, \Delta)$ and the inclusion $N^t(c_0, p) \subset N^t(c, p)$ is strict, for some $N^t\Delta x \in N^t(c, p)$. Thus, $x \notin N^t(c_0, p, \Delta)$.

By the similar discussion, it may easily be proved that the inclusion $N^t(c, p, \Delta) \subset N^t(\ell_\infty, p, \Delta)$ is strict. □

Theorem 3.2. *The inclusions $N^t(c, p) \subset N^t(c, p, \Delta)$, $N^t(c_0, p) \subset N^t(c_0, p, \Delta)$ and $N^t(\ell_\infty, p) \subset N^t(\ell_\infty, p, \Delta)$ hold for $1 \le p_n \le p_{n+1}$, $\forall n \in \mathbf{N}$.*

Proof. The inclusions are obvious for $p = e$, (see [9]). We are considering the case for $N^t(\ell_\infty, p) \subset N^t(\ell_\infty, p, \Delta)$. Let $x \in N^t(\ell_\infty, p)$ be given. Then $\Delta x^p \in \ell_\infty$, where $x^p = ({x_k}^{p_k})_{k=0}^{\infty}$. Choose fixed $m_0 \in \mathbf{N}$ such that $\Delta {x_k}^{p_k} < 1$ for all $k \ge m_0$. Then for any $n > m_0$ that

$$\Delta {x_k}^{p_n} = (\Delta {x_k}^{p_k})^{\frac{p_n}{p_k}} \le \Delta {x_k}^{p_k}, \quad m_0 \le k \le n \tag{3.1}$$

Since $p_k \le p_n$ for $k \le n$ and $n \in \mathbf{N}$. Further, since $p = (p_n)$ is bounded, then for $K > 0$, we have

$$\sup_n \left| \frac{1}{T_n} \sum_{i=0}^{m_0-1} t_{k-i}\ \Delta x_i \right|^{p_n} \le K \sup_n \frac{1}{T_n} \sum_{i=0}^{m_0-1} t_{k-i} \left|\Delta x_i\right|^{p_k} \tag{3.2}$$

Therefore, using (3.1) and (3.2) and by applying Holder inequality that

$$\begin{aligned}
&\left|N^t(l_\infty, p, \Delta)_k\right|^{p_n} \\
&\le \left[\sup_n \left(\tfrac{1}{T_n} \textstyle\sum_{i=0}^{n} t_{k-i}\right)\ \Delta x_i\right]^{p_n} \\
&\le \left[\sup_n \tfrac{1}{T_n} \textstyle\sum_{i=0}^{n} t_{k-i}\ \Delta {x_i}^{p_n}\right] \left[\sup_n \left(\tfrac{1}{T_n} \textstyle\sum_{i=0}^{n} t_{k-i}\right)\right]^{p_{n-1}} \\
&= \sup_n \tfrac{1}{T_n} \textstyle\sum_{i=0}^{n} t_{k-i}\ \Delta {x_i}^{p_n} \\
&= \sup_n \tfrac{1}{T_n} \left[\textstyle\sum_{i=0}^{n_0-1} t_{k-i}\ \Delta {x_i}^{p_n} + \textstyle\sum_{k=n_0}^{n} t_{k-i}\ \Delta {x_i}^{p_n}\right] \\
&\le \tfrac{K+1}{T_n} \textstyle\sum_{k=0}^{n} t_{k-i}\ \Delta {x_i}^{p_n} \\
&= (K+1)\ \left(N^t(\Delta x^p)\right)
\end{aligned}$$

Also we have $\Delta x^p \in l_\infty$, thus $N^t(\Delta x^p) \in l_\infty$, (see [9])
With this the above inequality leads to the fact that $N^t\Delta x \in N^t(l_\infty, p)$ and hence $x \in N^t(l_\infty, p, \Delta)$. Therefore, the inclusion $N^t(l_\infty, p) \subset N^t(l_\infty, p, \Delta)$ hold.
Similarly, it can be shown with some modifications that the inclusion $N^t(c, p) \subset N^t(c, p, \Delta)$ and $N^t(c_0, p) \subset N^t(c_0, p, \Delta)$ also hold. □

4. The α-, β- & γ-duals of the spaces $N^t(c_0, p, \Delta)$, $N^t(c, p, \Delta)$, $N^t(l_\infty, p, \Delta)$

In this section, we determine the α-, β- and γ- duals of the spaces $N^t(c_0, p, \Delta)$, $N^t(c, p, \Delta)$, $N^t(l_\infty, p, \Delta)$. We refer the following lemmas:

Lemma 4.1. (*see* [2], *Theorem* 5.1.0) *Let (a_{nk}) be an infinite matrix over the complex field. Then the following statements holds:*
(i) Let $1 < p_n \leq H < \infty, n \in \mathbf{N}$. Then $A = (a_{nk}) \in (\ell(p) : \ell_1)$ iff $R > 1 \in \mathbf{Z}$ such that

$$\sup_{N\in\mathcal{F}} \sum_k \left| \sum_{n\in\mathbf{N}} a_{nk} R^{-1} \right|^{p'_n} < \infty \tag{4.1}$$

(ii) Let $0 < p_n \leq 1$ for every $n \in \mathbf{N}$. Then $A = (a_{nk}) \in (\ell(p) : \ell_1)$ if and only if

$$\sup_{N\in\mathcal{F}} \sup_{k\in\mathbf{N}} \left| \sum_{n\in N} a_{nk} \right|^{p_n} < \infty \tag{4.2}$$

Lemma 4.2. (*see* [5], *Theorem* 1) *Let (a_{nk}) be an infinite matrix over the complex field. Then the following statements holds:*

Let $1 < p_n \leq H < \infty$, $\forall n \in \mathbf{N}$. Then $A = (a_{nk}) \in (\ell(p) : \ell_\infty)$ iff there exists $R > 1 \in \mathbf{Z}$ such that

$$\sup_{n\in\mathbf{N}} \sum_k \left| a_{nk} R^{-1} \right|^{p'_n} < \infty \tag{4.3}$$

Let $0 < p_n \leq 1$ $\forall n \in \mathbf{N}$. Then $A = (a_{nk}) \in (\ell(p) : \ell_\infty)$ iff

$$\sup_{n,k\in\mathbf{N}} |a_{nk}|^{p_n} < \infty \tag{4.4}$$

Lemma 4.3. (*see* [5], *Theorem* 1) *Let $0 < p_n \leq H < \infty$ for every $n \in \mathbf{N}$. Then $A = (a_{nk}) \in (\ell(p) : c)$ iff (4.3) and (4.4) hold, and there is $\beta_k \in \mathbf{C}$ such that $a_{nk} \to \beta_k$, for each $k \in \mathbf{N}$.*

Theorem 4.4. *Let $1 < p_n \leq H < \infty$ for every $n \in \mathbf{N}$. Then α-dual of the spaces $N^t(c_0, p, \Delta)$, $N^t(c, p, \Delta)$, $N^t(l_\infty, p, \Delta)$ is*

$$D_1 = \left\{ a \in w : \sup_{N\in\mathcal{F}} \sum_k \left| \sum_{n\in\mathbf{N}} (-1)^{n-k} D_{n-k} T_n \ \Delta a_n R^{-1} \right|^{p'_n} < \infty \right\}$$

Proof. For $a = (a_n) \in w$, consider the following equality

$$a_n x_n = \sum_{k=0}^{n} (-1)^{n-k} D_{n-k} T_n \, \Delta a_n y_k = (Cy)_n, \quad \forall \, n \in \mathbf{N} \tag{4.5}$$

where $C = (c_{nk})$ is defined by

$$C_{nk} = \begin{cases} (-1)^{n-k} D_{n-k} T_n \, \Delta a_n, & if \; 0 \le k \le n \\ 0 \,, & if \; k > n \end{cases},$$

for all $n, k \in \mathbf{N}$. Thus, by combining (4.5) with Part (i) of Lemma 4.1, we observe that $ax \in \ell_1$ whenever $x \in N^t(c_0, p, \Delta)$, $N^t(c, p, \Delta)$, or $N^t(\ell_\infty, p, \Delta)$ iff $Cy \in \ell_1$ when $y \in N^t(c_0, p)$, $N^t(c, p)$, $N^t(\ell_\infty, p)$. So that, a is in the α-dual of $N^t(c_0, p, \Delta)$, $N^t(c, p, \Delta)$, *and* $N^t(\ell_\infty, p, \Delta)$ iff $C \in N^t(c_0, l_1) = N^t(c, l_1) = N^t(l_\infty, l_1)$. Thus $a \in [N^t(c_0, p, \Delta)]^\alpha = [N^t(c, p, \Delta)]^\alpha = [N^t(l_\infty, p, \Delta)]^\alpha$ iff $\sup_{N \in \mathcal{F}} \sum_k \left| \sum_{n \in \mathbf{N}} C_{nk} R^{-1} \right|^{p'_n} < \infty$ which leads to the consequence that

$$[N^t(c_0, p, \Delta)]^\alpha = [N^t(c, p, \Delta)]^\alpha = [N^t(l_\infty, p, \Delta)]^\alpha = D_1 \tag{4.6}$$

□

Theorem 4.5. *Define the sets D_2, D_3, D_4 and D_5 as follows:*

$$\begin{aligned} D_2 &= \left\{ a \in w : \sup_{n \in \mathbf{N}} \sum_{k \in \mathbf{N}} \left| \sum_{j=k}^{n} (-1)^{j-k} Y R^{-1} \right|^{p'_n} < \infty \right\} \\ D_3 &= cs \\ D_4 &= \left\{ a \in w : \sup_{N \in \mathcal{F}} \sum_{k \in \mathbf{N}} \left| \sum_{n \in N} (-1)^{n-k} P \Delta a_n \right|^{p_n} < \infty \right\} \\ \textit{and} \;\; D_5 &= \left\{ a \in w : \sup_{n \in \mathbf{N}} \sum_{k \in \mathbf{N}} \left| \sum_{j=k}^{n} (-1)^{j-k} P \Delta a_j \right|^{p_n} < \infty \right\} \end{aligned}$$

where $Y = \Delta a_j D_{j-k} T_n$, $P = D_{j-k} T_n$, $0 < p_n \le 1, \forall \, n \in \mathbf{N}$.

Then, $[N^t(c_0, p, \Delta)]^\beta = D_2 \cap D_3$, $[N^t(c, p, \Delta)]^\beta = D_2 \cap D_3 \cap D_4$, $[N^t(l_\infty, p, \Delta)]^\beta = D_3 \cap D_5$.

Proof. Now we give the $\beta-$dual of the sequence space $N^t(c_0, p, \Delta)$. Consider the equality

$$\sum_{k=0}^{n} a_k x_k = \sum_{k=0}^{n-1} \sum_{j=k}^{n} (-1)^{j-k} P \Delta a_j y_k + T_n \Delta a_n y_n = (Ey)_n \tag{4.7}$$

where, $P = D_{j-k}T_n$, $\forall n \in \mathbf{N}, E = (e_{nk})$ is defined by

$$e_{nk} = \begin{cases} \sum_{j=k}^{n} (-1)^{j-k} D_{j-k}T_n \ \Delta a_j, & if \ 0 \leq k < n \\ T_n \ \Delta a_n, & if \ k = n \\ 0 \ , & if \ k > n \end{cases}$$

for all $n, k \in \mathbf{N}$. Then, from equation (4.6) we obtain, $ax = (a_n x_n) \in cs$ whenever $x = (x_n) \in N^t(c_0, p, \Delta)$ iff $Ey \in c$ whenever $y = (y_n) \in N^t(c_0, p)$. This means that $a = (a_n) \in [N^t(c_0, \Delta, p)]^\beta$ if and only if $E \in N^t(c_0, p)$. Therefore by using Lemma (4.2) with Part (i) we have

$$\sup_{n\in\mathbf{N}} \sum_k \left| \sum_{j=k}^{n} (-1)^{j-k} D_{j-k}T_n \ \Delta a_j R^{-1} \right|^{p'_n} < \infty$$

and by Lemma (4.3), $\lim_n a_{nk}$ exists for all $k \in \mathbf{N}$. Hence we conclude that $[N^t(c_0, p, \Delta)]^\beta = D_2 \cap D_3$.

Analogously, the $\beta-$ dual of the sequence spaces $N^t(c, p, \Delta)$ and $N^t(\ell_\infty, p, \Delta)$ can be obtained. □

Theorem 4.6. *The γ- dual of $N^t(c, p, \Delta)$, $N^t(c_0, p, \Delta)$ and $N^t(l_\infty, p, \Delta)$ is D_2.*

Proof. This result is easily obtained by proceeding as in the proof of Theorem (4.5) with Lemma (4.3). □

5. Conclusions

The Norlund N^t-difference sequence spaces $N^t(c_0, p, \Delta)$, $N^t(c, p, \Delta)$ and $N^t(l_\infty, p, \Delta)$ has been studies thoroughly and the their α-, β- and γ-duals have been obtained with the help of some particular subsets containing sequences viz., D_1, D_2, D_3, D_4 and D_5.

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