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Some open questions in real algebraic geometry

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Abstract

Many interesting problems arise on the borderline between real algebraic geometry and topology. We focus on 12 open questions. Some of them come from regulous geometry, which emerged as a subfield of real algebraic geometry less than 15 years ago.

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In what follows we discuss 12 open questions concerning 7 topics in real algebraic geometry. Emphasis is placed on problems relating to the comparison of algebraic and topological categories.

Throughout this note the term *real algebraic variety* designates a ringed space with structure sheaf of $\mathbb{R}$-algebras of $\mathbb{R}$-valued functions, which is isomorphic to a Zariski locally closed subset of real projective n-space $\mathbb{P}^n(\mathbb{R})$, for some n, endowed with the Zariski topology and the sheaf of regular functions. This is compatible with [7], which contains a detailed exposition of real algebraic geometry. Recall that each real algebraic variety in the sense used here is actually affine, that is, isomorphic to an algebraic subset of $\mathbb{R}^n$, for some n, see [7, Proposition 3.2.10 and Theorem 3.4.4]. Morphisms of real algebraic varieties are called *regular maps*. Each real algebraic variety carries also the Euclidean topology determined by the usual metric on $\mathbb{R}$. Unless explicitly stated otherwise, all topological notions relating to real algebraic varieties refer to the Euclidean topology.

Given two real algebraic varieties X and Y, we denote by $\mathcal{R}(X, Y)$ the set of all regular maps from X to Y. Regular functions and regular maps can be described in a straightforward explicit way.

Let X be an algebraic subset of $\mathbb{R}^n$ and let $U \subset X$ be a Zariski open subset. By [7, Proposition 3.2.3], a function $\varphi\colon U \to \mathbb{R}$ is regular if and only if there exist two polynomial functions $P, Q\colon \mathbb{R}^n \to \mathbb{R}$ with

$$U \subset \{x \in \mathbb{R}^n : Q(x) \neq 0\} \quad \text{and} \quad f(x) = \frac{P(x)}{Q(x)} \quad \text{for all } x \in U.$$

A map $f = (f_1, \ldots, f_p)\colon U \to Y \subset \mathbb{R}^p$, where Y is an algebraic subset of $\mathbb{R}^p$, is regular if and only if the components f_i are regular functions.

As a matter of convention, all $\mathcal{C}^\infty$ manifolds will be Hausdorff and second countable. The space $\mathcal{C}^k(M, N)$ of $\mathcal{C}^k$ maps between $\mathcal{C}^\infty$ manifolds, where k is either a nonnegative integer or $k = \infty$, is endowed with the $\mathcal{C}^k$ topology (see [13, pp. 34, 36] where it is called the weak $\mathcal{C}^k$ topology; the $\mathcal{C}^0$-topology is just the compact-open topology). If X, Y are nonsingular real algebraic varieties, then $\mathcal{R}(X, Y) \subset \mathcal{C}^\infty(X, Y)$.

1 Maps between spheres

As usual, we denote by $\mathbb{S}^n$ the unit n-sphere,

$$\mathbb{S}^n := \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} : x_0^2 + \cdots + x_n^2 = 1\}.$$

Question 1. Let (n, p) be a pair of positive integers. Can every $\mathcal{C}^\infty$ map from $\mathbb{S}^n$ to $\mathbb{S}^p$ be approximated by regular maps in the $\mathcal{C}^\infty$ topology? In other words, is the set $\mathcal{R}(\mathbb{S}^n, \mathbb{S}^p)$ of regular maps dense in the space $\mathcal{C}^\infty(\mathbb{S}^n, \mathbb{S}^p)$ of $\mathcal{C}^\infty$ maps?

This problem has been studied since at least the 1980's. New methods have been introduced in the recent paper by Bochnak and the author [11]. According to [11, Corollary 1.5], a $\mathcal{C}^\infty$ map $f\colon \mathbb{S}^n \to \mathbb{S}^p$ can be approximated by regular maps in the $\mathcal{C}^\infty$ topology if and only if f is homotopic to a regular map. As an application, we obtained the positive answer to Question 1 in the following five cases [11, Theorem 5.6]:

(i) $p = 1, 2$ or 4;

(ii) $n - p \leq 3$;

(iii) $4 \leq n - p \leq 5$ with possible exception for the pairs: $(9, 5), (7, 3), (11, 6), (10, 5), (8, 3)$;

(iv) the homotopy group $\pi_n(\mathbb{S}^p)$ is finite cyclic of odd order, and p is odd with $n \leq 2p - 2$;

(v) $n = p + 13$, where p is odd and $p \geq 15$.

2 Maps with values in odd-dimensional spheres

Question 2. Let X be a compact connected nonsingular real algebraic variety of odd dimension n. Can every $\mathcal{C}^\infty$ map from X to $\mathbb{S}^n$ be approximated by regular maps in the $\mathcal{C}^\infty$ topology?

The answer is known to be positive for $n = 1$ [8, Corollary 1.5]. For all other odd n, Question 2 remains open. By [11, Theorem 1.8], if X is orientable as a $\mathcal{C}^\infty$ manifold, then either

(i) the set $\mathcal{R}(X, \mathbb{S}^n)$ is dense in the space $\mathcal{C}^\infty(X, \mathbb{S}^n)$, or

(ii) the closure of $\mathcal{R}(X, \mathbb{S}^n)$ in the space $\mathcal{C}^\infty(X, \mathbb{S}^n)$ coincides with the set

$$\{f \in \mathcal{C}^\infty(X, \mathbb{S}^n) : \deg(f) \in 2\mathbb{Z}\},$$

where $\deg(f)$ is the topological degree of f.

The behavior of regular maps into even-dimensional spheres is entirely different. For example, every regular map $\mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^2$ is null homotopic, that is, has topological degree 0, see [9].

3 Maps from real algebraic curves to real algebraic varieties

Question 3. Which pairs (X, Y) of nonsingular real algebraic varieties have the property that every $\mathcal{C}^\infty$ map from X to Y can be approximated by regular maps in the $\mathcal{C}^\infty$ topology?

Approximation as in Question 3 is known to be possible if X is a compact nonsingular real algebraic curve and either

(i) Y is a rational nonsingular real variety (that is, Y is nonsingular and birationally equivalent to $\mathbb{R}^p$, where $p = \dim Y$), or

(ii) Y is a homogeneous real algebraic G-variety for a Zariski closed subgroup
$G \subset \mathrm{GL}_m(\mathbb{R})$, for some m (that is, G acts transitively on Y, the action $G \times Y \to Y$, $(a, y) \mapsto a \cdot y$ being a regular map).

The case (i) was settled by Bochnak and the author [10, Theorem 1.1], while the case (ii) is contained in the recent paper of Benoist and Wittenberg [4, Theorem A].

A real algebraic variety Y is said to be *rationally connected* if for any two points y_0, y_1 in Y there exists a regular map $f : \mathbb{P}^1(\mathbb{R}) \to Y$ such that y_0, y_1 belong to $f(\mathbb{P}^1(\mathbb{R}))$. Note that $\mathbb{P}^1(\mathbb{R})$ is biregularly isomorphic to $\mathbb{S}^1$.

Question 4. Let X be a compact nonsingular real algebraic curve and let Y be a rationally connected nonsingular real algebraic variety. Can every $\mathcal{C}^\infty$ map from X to Y be approximated by regular maps in the $\mathcal{C}^\infty$ topology?

The following question is also undecided.

Question 5. Let X be a compact nonsingular real algebraic curve and let

$$Y \coloneqq \{(x, y, z) \in \mathbb{R}^3 : x^4 + y^4 + z^4 = 1\}.$$

Can every $\mathcal{C}^\infty$ map from X to Y be approximated by regular maps in the $\mathcal{C}^\infty$ topology?

4 Regulous maps

Regulous geometry has emerged recently as a subfield of real algebraic geometry, see the survey paper [22]. It deals with objects described by real rational functions that can be extended to continuous ones.

Let X, Y be two nonsingular real algebraic varieties. A map $f\colon X \to Y$ is said to be *regulous* if it is continuous on X and there exists a Zariski open dense subset U of X such that the restriction $f|_U\colon U \to Y$ is a regular map. Let $X(f)$ denote the union of all such U. The complement $P(f) \coloneqq X \setminus X(f)$ of $X(f)$ is the smallest Zariski closed subset of X for which the restriction $f|_{X\setminus P(f)}\colon X \setminus P(f) \to Y$ is a regular map. If $f(P(f)) \neq Y$, we say that f is a *nice* regulous map. In the literature regulous maps are also called *continuous rational maps* [15–17, 19]. The concise name “regulous” was coined by Fichou, Huisman, Mangolte and Monnier [14]. Since the publication of [16] in 2009 several mathematicians have devoted their attention to regulous maps, see the surveys [18, 22] and the references therein.

A map $f\colon X \to Y$ is said to be k-*regulous*, where k is a nonnegative integer or $k = \infty$, if it is both regulous and of class $\mathcal{C}^k$. Thus, less formally, a k-regulous map is a $\mathcal{C}^k$ map that admits a rational representation. Obviously, “0-regulous” is the same as “regulous”. As observed in [16, Proposition 2.1], ∞-regulous maps coincide with regular maps, and these are usually studied separately. A standard example of a k-regulous function, with k a nonnegative integer, is $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \frac{x^{3+k}}{x^2+y^2} \quad \text{for } (x,y) \neq (0,0) \quad \text{and} \quad f(0,0) = 0.$$

Clearly, f is not of class $\mathcal{C}^{k+1}$.

Assume that X is compact. Denote by $H_d^{\mathrm{alg}}(X;\mathbb{Z}/2)$ the subgroup of the homology group $H_d(X;\mathbb{Z}/2)$ generated by the homology classes represented by d-dimensional Zariski closed subsets of X, $0 \leq d \leq \dim X$ [7]. Set

$$H^c_{\mathrm{alg}}(X;\mathbb{Z}/2) \coloneqq D_X^{-1}(H_d^{\mathrm{alg}}(X;\mathbb{Z}/2)),$$

where $c + d = \dim X$ and $D_X\colon H^c(X;\mathbb{Z}/2) \to H_d(X;\mathbb{Z}/2)$ is the Poincaré duality isomorphism. By [16, Proposition 1.3], if both X, Y are compact, and $f\colon X \to Y$ is a regulous map, then

$$f^*(H^c_{\mathrm{alg}}(Y;\mathbb{Z}/2)) \subset H^c_{\mathrm{alg}}(X;\mathbb{Z}/2),$$

where $f^*\colon H^c(Y;\mathbb{Z}/2) \to H^c(X;\mathbb{Z}/2)$ is the induced homomorphism.

Clearly, $H^c_{\mathrm{alg}}(\mathbb{S}^p;\mathbb{Z}/2) = H^p(\mathbb{S}^p;\mathbb{Z}/2)$. Therefore, if $f\colon X \to \mathbb{S}^p$ is a regulous map, then

$$f^*\colon (H^p(\mathbb{S}^p;\mathbb{Z}/2)) \subset H^p_{\mathrm{alg}}(X;\mathbb{Z}/2).$$

Question 6. Let k, p be integers with $k \geq 0$, $p \geq 1$. Let X be a compact nonsingular real algebraic variety and let $f\colon X \to \mathbb{S}^p$ be a $\mathcal{C}^\infty$ map such that

$$f^*(H^p(\mathbb{S}^p;\mathbb{Z}/2)) \subset H^p_{\mathrm{alg}}(X;\mathbb{Z}/2).$$

Can f be approximated by k-regulous maps in the $\mathcal{C}^k$ topology?

The answer is known to be positive if $k = 0$ and $p \in \{1, n-1, n\}$, where $n = \dim X$, see [17].

Question 7. Let k, p be integers with $k \geq 0$, $p \geq 1$. Can every k-regulous map from X to $\mathbb{S}^p$ be approximated by nice k-regulous maps in the $\mathcal{C}^k$ topology?

It may seem that Question 7 is rather technical, but in view of [16, 17] the positive answer would have several interesting consequences.

Let k be a nonnegative integer, X a nonsingular real algebraic variety, and Z a nonsingular Zariski closed subvariety of X. By Kollár and Nowak [15, Proposition 8], if $F\colon X \to \mathbb{R}$ is a k-regulous function, then the restriction $F|_Z$ is also a k-regulous function.

Question 8. Let $f\colon Z \to \mathbb{R}$ be a k-regulous function. Does there exist a k-regulous function $F\colon X \to \mathbb{R}$ such that $F|_Z = f$?

According to [15, Theorem 10], the answer is positive for $k = 0$ (even in a more general setting).

5 Systems of linear equations with polynomial coefficients

Let m, n, p be positive integers. Consider a system of linear equations

$$\begin{cases} f_{11} \cdot y_1 + \cdots + f_{1p} \cdot y_p = g_1 \\ \qquad\qquad \vdots \\ f_{m1} \cdot y_1 + \cdots + f_{mp} \cdot y_p = g_m, \end{cases} \tag{1}$$

where the f_{ij}, g_i are polynomial real-valued functions on $\mathbb{R}^n$, and the y_j are unknowns.

Question 9. Which systems (1) that have a continuous solution $y_1 = \varphi_1, \ldots, y_p = \varphi_p$, where the $\varphi_j \colon \mathbb{R}^n \to \mathbb{R}$ are continuous functions, have also a regulous solution $y_1 = \psi_1, \ldots, y_p = \psi_p$, where the $\psi_j \colon \mathbb{R}^n \to \mathbb{R}$ are regulous functions?

The case $n = 1$ is an exercise. By [20], if $n = 2$ and $m = 1$, then (1) has a regulous solution precisely when it has a continuous solution. According to [15, Example 6], for $n = 3$, $m = 1$, $p = 2$, the equation

$$(x_1^3 x_2) \cdot y_1 + (x_1^3 - (1 + x_3^2) x_2^3) \cdot y_2 = x_1^4$$

has a continuous solution

$$y_1 = (1 + x_3^2)^{1/3}, \quad y_2 = \frac{x_1^3}{x_1^2 + (1 + x_3^2)^{1/3} x_1 x_2 + (1 + x_3^2)^{2/3} x_2^2},$$

but does not have a regulous solution.

Problems related to Question 9 are discussed in [1, 5, 12, 15].

6 Stratified-algebraic vector bundles

Let $\mathbb{F}$ stand for $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ (the quaternions). All $\mathbb{F}$-vector spaces will be left $\mathbb{F}$-vector spaces. When convenient, $\mathbb{F}$ will be identified with $\mathbb{R}^{d(\mathbb{F})}$, where $d(\mathbb{F}) = \dim_{\mathbb{R}} \mathbb{F}$.

Let X be a real algebraic variety. For any nonnegative integer n, let $\varepsilon_X^n(\mathbb{F})$ denote the standard trivial $\mathbb{F}$-vector bundle on X with total space $X \times \mathbb{F}^n$, where $X \times \mathbb{F}^n$ is regarded as a real algebraic variety. An *algebraic* $\mathbb{F}$*-vector bundle* on X is an algebraic $\mathbb{F}$-vector subbundle of $\varepsilon_X^n(\mathbb{F})$ for some n (see [7, Chapters 12 and 13] for basic properties of algebraic $\mathbb{F}$-vector bundles).

By a *stratification* of X we mean a finite collection $\mathcal{S}$ of pairwise disjoint Zariski locally closed subvarieties whose union is X. Each subvariety in $\mathcal{S}$ is called a *stratum.* A *stratified-algebraic* $\mathbb{F}$*-vector bundle* on X is a topological $\mathbb{F}$-vector subbundle ξ of $\varepsilon_X^n(\mathbb{F})$, for some n, such that for some stratification $\mathcal{S}$ of X, the restriction $\xi|_S$ of ξ to each stratum $S \in \mathcal{S}$ is an algebraic $\mathbb{F}$-vector subbundle of $\xi_S^n(\mathbb{F})$ (see also [24] for a different but equivalent description).

A topological $\mathbb{F}$-vector bundle ξ on X is said to *admit an algebraic structure* if it is isomorphic to an algebraic $\mathbb{F}$-vector bundle on X. Similarly, ξ is said to *admit a stratified-algebraic structure* if it is isomorphic to a stratified-algebraic $\mathbb{F}$-vector bundle on X. These two types of $\mathbb{F}$-vector bundles have been extensively investigated in [6, 7] (see also the references therein) and [21, 23, 24], respectively. In general, their behaviors are quite different, see [21, Example 1.11].

Denote by $K_{\mathbb{F}}(X)$ the Grothendieck group of topological $\mathbb{F}$-vector bundles on X. Since X has the homotopy type of a compact polyhedron, it follows that the abelian group $K_{\mathbb{F}}(X)$ is finitely generated. Let $K_{\mathbb{F}\text{-str}}(X)$ be the subgroup of $K_{\mathbb{F}}(X)$ generated by the classes of all $\mathbb{F}$-vector bundles admitting a stratified-algebraic structure.

If the variety X is compact, then the group $K_{\mathbb{F}\text{-str}}(X)$ contains complete information on $\mathbb{F}$-vector bundles admitting a stratified-algebraic structure: A topological $\mathbb{F}$-vector bundle ξ on X admits a stratified-algebraic structure if and only if its class in $K_{\mathbb{F}}(X)$ belongs to $K_{\mathbb{F}\text{-str}}(X)$ (in other words, ξ admits a stratified-algebraic structure if and only if there is a stratified-algebraic $\mathbb{F}$-vector bundle η on X such that the direct sum $\xi \oplus \eta$ admits a stratified-algebraic structure), see [21, Corollary 3.14].

For any topological $\mathbb{F}$-vector bundle ξ on X, we regard $\operatorname{rank}\xi$ (the rank of ξ) as a function $\operatorname{rank}\xi\colon X \to \mathbb{Z}$, which assigns to every point x in X the dimension of the fiber of ξ over x. Clearly, $\operatorname{rank}\xi$ is constant on each connected component of X. We say that ξ has *property* (rk) if for every integer d , the set $\{x \in X : (\operatorname{rank}\xi)(x) = d\}$ is algebraically constructible. Recall that a subset of X is said to be *algebraically constructible* if it belongs to the Boolean algebra generated by the Zariski closed subsets of X. It readily follows that each stratified-algebraic $\mathbb{F}$-vector bundle on X has property (rk). Thus, property (rk) is a necessary condition for ξ to admit a stratified-algebraic structure.

Let us illustrate the role of property (rk). The real algebraic curve

$$C = \{(x, y) \in \mathbb{R}^2 : x^2(x^2 - 1)(x^2 - 4) + y^2 = 0\}$$

is irreducible with singular locus $\{(0, 0)\}$. It has three connected components, the singleton $\{(0, 0)\}$ and two ovals. Clearly, every algebraic $\mathbb{F}$-vector bundle on C has constant rank, while the rank function of a topological $\mathbb{F}$-vector bundle on C may take three distinct values. On the other hand, the rank function of a stratified-algebraic $\mathbb{F}$-vector bundle on C need not be constant, but must be constant on $C \setminus \{(0,0)\}$.

Returning to the general case, denote by $K_{\mathbb{F}}^{(\mathrm{rk})}(X)$ the subgroup of $K_{\mathbb{F}}(X)$ generated by the classes of all topological $\mathbb{F}$-vector bundles having property (rk). By construction,

$$K_{\mathbb{F}\text{-str}}(X) \subset K_{\mathbb{F}}^{(\mathrm{rk})}(X).$$

Since the group $K_{\mathbb{F}}(X)$ is finitely generated, so is the quotient group

$$\Gamma_{\mathbb{F}}(X) \coloneqq K_{\mathbb{F}}^{(\mathrm{rk})}(X)/K_{\mathbb{F}\text{-str}}(X).$$

Thus the group $\Gamma_{\mathbb{F}}(X)$ is finite if and only if $r\Gamma_{\mathbb{F}}(X) = 0$ for some positive integer r.

For any $\mathbb{F}$-vector bundle ξ on X and any positive integer r, we denote by

$$\xi(r) = \xi \oplus \cdots \oplus \xi$$

the r-fold direct sum.

As observed in [23, Proposition 1.2], if X is compact, then for a positive integer r the following conditions are equivalent:

(a) The group $\Gamma_{\mathbb{F}}(X)$ is finite and $r\Gamma_{\mathbb{F}}(X) = 0$.

(b) For each topological $\mathbb{F}$-vector bundle ξ on X having property (rk), the $\mathbb{F}$-vector bundle $\xi(r)$ admits a stratified-algebraic structure.

(c) For each topological $\mathbb{F}$-vector bundle η on X having constant rank, the $\mathbb{F}$-vector bundle $\eta(r)$ admits a stratified-algebraic structure.

It is known that $\Gamma_{\mathbb{F}}(X) \neq 0$ in general, see [21, Example 1.11].

Question 10. Let X be a compact real algebraic variety. Is the group $\Gamma_{\mathbb{F}}(X)$ finite?

By [23, Theorem 1.8], the answer is positive if $\dim X \leq 8$.

7 Approximation of $\mathcal{C}^\infty$ manifolds

Let X be a nonsingular real algebraic variety and let M be a compact $\mathcal{C}^\infty$ submanifold (without boundary) of X. We say that M admits an *algebraic* (resp. a *weak algebraic*) approximation in X if for every neighborhood $\mathcal{U} \subset \mathcal{C}^\infty(M, X)$ of the inclusion map $M \hookrightarrow X$ there exists a $\mathcal{C}^\infty$ embedding $e\colon M \to X$ in $\mathcal{U}$ such that $e(M)$ is a nonsingular Zariski closed (resp. Zariski locally closed) subset of X. A bordism class in the mth unoriented bordism group $\mathrm{MO}_m(X)$ of X is said to be *algebraic* if it can be represented by a regular map from a compact nonsingular m-dimensional real algebraic variety to X. It readily follows that if M admits a weak algebraic approximation in X, then the bordism class of the inclusion map $M \hookrightarrow X$ is algebraic.

Question 11. Let X be a nonsingular real algebraic variety and let M be a compact $\mathcal{C}^\infty$ submanifold of X. Assuming that the bordism class of the inclusion map $M \hookrightarrow X$ is algebraic, does M admit a weak algebraic approximation in X?

It is conjectured in [19] that the answer is always positive. This, if true, would have very interesting consequences in regulous geometry. Recently, Benoist [3, Theorems 0.6 and 0.7] obtained the following remarkable result: If the bordism class of $M \hookrightarrow X$ is algebraic and $2\dim M + 1 \leq \dim X$, then M admits an algebraic approximation in X. Furthermore, the assumption $2\dim M + 1 \leq \dim X$ cannot be weakened in general.

By [2, Lemma 2.7.1], if X is one of the real algebraic varieties $\mathbb{R}^n$, $\mathbb{S}^n$, $\mathbb{P}^n(\mathbb{R})$ or $\mathbb{P}^k(\mathbb{R}) \times \mathbb{P}^l(\mathbb{R})$, then each unoriented bordism class of X is algebraic.

Question 12. Let X stand for $\mathbb{R}^n$, $\mathbb{S}^n$ or $\mathbb{P}^n(\mathbb{R})$. Does every compact $\mathcal{C}^\infty$ submanifold of X admit an algebraic approximation in X?

One should stress that the varieties $\mathbb{R}^n$, $\mathbb{S}^n$ and $\mathbb{P}^n(\mathbb{R})$ play a very particular role in Question 12. Indeed, by [3, Theorem 0.8], if $l = 2^{d+1} - 1$ for some positive integer d, then there exists a compact 2^d-dimensional $\mathcal{C}^\infty$ submanifold of $X \coloneqq \mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^l(\mathbb{R})$ which does not admit an algebraic approximation in X.

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