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Some open questions about line arrangements in the projective plane

To the memory of Luis Dissett and his love of teaching geometry

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Abstract

Despite that the study of line arrangements in the projective plane is old and elemental, there is still a long list of intriguing open questions and applications to modern mathematics. Our goal is to discuss part of that list, focusing on the connection with Chern invariants and pointing towards configurations of rational curves.

Keywords: Line arrangements, Chern numbers

MSC numbers: 14N20,52C30,51E30.

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A classical and beautiful reference on the topic of configurations of lines is "Geometry and the Imagination" [21, Chapter III], which traces back several results, and in particular surveys point-line configurations in detail. On page 95, it is stated that *"... there was a time when the study of configurations was considered the most important branch of all geometry."* citing the book "Geometrische Konfigurationen" by F. Levi. It is perhaps surprising that such an old and elemental subject is still relevant today in ongoing research, especially in algebraic geometry. To mention just two important results: the construction of special algebraic surfaces (together with combinatorial consequences for line arrangements) by Hirzebruch [22], and the Murphy's law in algebraic geometry due to Vakil [59], which is based on the Mnëv Universality Theorem (see e.g. [35]). Beyond line arrangements, the use of configurations of special curves has produced results on the geography of surfaces of general type (see e.g. [41], [57], [46]), and the construction of exotic blow-ups of the complex projective plane at few points, which even have complex structures (see e.g. [34], [39], [40]), among other applications. (See e.g. [4] for another important line of research.) We will not focus on any applications, the purpose of this note is to present a short guide to various open problems about arrangements of lines in the projective plane, and to hint connections with current research.

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1 Arrangements of lines

Our main reference will be [15]. Let k be an arbitrary field. The most relevant fields for us will be $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, and the algebraic closure $\overline{\mathbb{F}}_p$ of the field of p elements $\mathbb{F}_p$. The projective plane over k will be denoted by $\mathbb{P}^2_k$.

A set

$$L = \{[x, y, z] \in \mathbb{P}^2_k : ax + by + cz = 0\},$$

for some $a, b, c \in k$ not all zero, will be called a *line*. A *line arrangement* is a finite collection of d lines $\mathcal{A} = \{L_1, \cdots, L_d\}$. For $m \geq 2$, an *m-point* is a point in $\mathcal{A}$ which belongs to exactly m lines in $\mathcal{A}$. We denote the number of m-points by t_m. Any arrangement of d lines satisfies

$$\binom{d}{2} = \sum_{m \geq 2} \binom{m}{2} t_m,$$

which is a purely combinatorial fact. It is proved by counting pairs of lines in two different ways, using only that two distinct lines intersect at one point.

A line arrangement is said to be in *general position* if the arrangement satisfies $t_m = 0$ for every $m > 2$, and so it only has double points. An arrangement of d lines with $t_d = 1$ is called *trivial*. An arrangement of d lines is called *quasi-trivial* if $t_{d-1} = 1$.

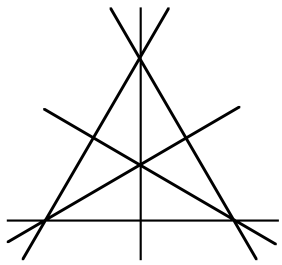

Figure 1: The complete quadrilateral.

Many interesting line arrangements come from drawings in a blackboard, this is when $k = \mathbb{R}$ (cf. [22, 1.1]). Nontrivial line arrangements in the real projective plane partition $\mathbb{P}^2_{\mathbb{R}}$ into polygons. When all of these polygons are triangles, the arrangement is called *simplicial* (cf. [17], [7]). A long-standing open problem is to classify simplicial arrangements. Any quasi-trivial arrangement is simplicial. A more interesting example is the *complete quadrilateral*, meaning a quadrilateral and its two diagonals, which is defined by the set of zeros of the polynomial

$$xyz(x-y)(x-z)(y-z).$$

It has 6 lines, and $t_2 = 3$, $t_3 = 4$, $t_m = 0$ else. That arrangement can be thought of as coming from an equilateral triangle together with its 3 symmetry lines, as in Figure 1. This can be generalized as follows: Any regular polygon of n sides defines a simplicial arrangement (see [15, Section 2]) of $2n$ lines with $t_2 = n$, $t_3 = n(n-1)/2$, $t_n = 1$, $t_m = 0$ else. They partition $\mathbb{P}^2_{\mathbb{R}}$ into n^2+n triangles. For example, the simplicial arrangement in Figure 2 defines 30 triangles. There are many simplicial arrangements which do not come from quasi-trivial or regular polygons, see e.g. [7] for a classification up to 27 lines.

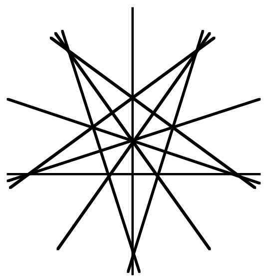

Figure 2: The simplicial arrangement from a pentagon.

Only finitely many of the regular polygon arrangements can be defined over $\mathbb{Q}$. For $n > 6$ the coefficients of the lines cannot be all in $\mathbb{Q}$ [6].

Question 1. Are there infinitely many (non quasi-trivial) simplicial arrangements defined over $k = \mathbb{Q}$?

An answer to this question would produce a new infinite family of simplicial arrangements. On the other hand, it could solve Question 7 in relation to a density property on its combinatorics, which is true for arrangements defined over $\mathbb{R}$.

For any given nontrivial line arrangement over $\mathbb{R}$, one has that

$$t_2 \geq 3 + \sum_{m \geq 4} (m-3) t_m, \tag{1}$$

and equality holds if and only if the arrangement is simplicial. One can prove both statements through the computation of the Euler characteristic of $\mathbb{P}^2_{\mathbb{R}}$ via the partition into polygons defined by the arrangement. At this moment, the reader is invited to prove it. To start you need to count the number of points, edges, and polygons in the partition using only the t_m numbers. An immediate consequence is that the number of double points is bigger than or equal to 3, which can be seen as a way to strongly solve the classical problem of Sylvester (cf. [1, Chapter 9]): *"Prove that it is not possible to arrange any finite number of real points so that a right line through every two of them shall pass through a third, unless they all lie in the same right line"*. To have an statement about lines and 2-points, one just *dualizes* the points into lines, and so, for example, the collinearity condition for points becomes a trivial arrangement for lines. We recall that any arrangement of lines could be seen as an arrangement of points, where each line $\{ax + by + cz = 0\}$ is represented by the point $[a, b, c]$, and vice-versa. (There is an analog inequality in [49] for pseudo-line arrangements, meaning a collection of smooth closed curves in $\mathbb{P}^2_{\mathbb{R}}$ which behave as lines but are not necessarily lines.)

Still in $\text{char}(k) = 0$, we could now consider complex arrangements. Then we are talking about configurations of Riemann spheres in the 4-manifold $\mathbb{P}^2_{\mathbb{C}}$. It turns out that we have fewer constraints on the geometry. For example, we do not need to have double points.

Let us consider the arrangement of $3n$ lines defined by

$$(x^n - y^n)(x^n - z^n)(y^n - z^n) = 0$$

in $\mathbb{P}^2_{\mathbb{C}}$. For $n \geq 4$ we have $t_3 = n^2$, $t_n = 3$, $t_m = 0$ else. When $n = 3$, we have an arrangement of 9 lines with $t_3 = 12$ and no other m-points. It is called *dual Hesse arrangement*. The *Hesse arrangement* can be defined as the arrangement of 12 lines given by dualizing the twelve 3-points. It has $t_2 = 12$, $t_4 = 9$, $t_m = 0$ else. It is unique up to projective equivalence (i.e. up to changing coordinates by a linear transformation of $\mathbb{P}^2_{\mathbb{C}}$). The

Hesse arrangement could also be realized as the 12 lines which join the 9 inflection points of any given nonsingular plane cubic. Finally, we note that $n = 2$ defines the complete quadrilateral, and $n = 1$ gives a triangle, both of which are field independent arrangements.

For other interesting complex arrangements, we refer to [22, 1.2], where we can find Klein's and Wiman's arrangements for example.

It was proved by Hirzebruch in [22] (improved in the remarks added to the proof via a result of Sakai, and then by Sommese in [50, (5.3)Theorem]; see also [23]) that any nontrivial and nonquasi-trivial arrangement of complex lines satisfies

$$2t_2 + t_3 \geq 3 + d + \sum_{m \geq 5} (m-4) t_m, \tag{2}$$

and equality holds if and only if the arrangement is the dual Hesse arrangement. All known proofs of it (and similar inequalities) invoke some version of the Bogomolov-Miyaoka-Yau (BMY) inequality, which is a deep result in the theory of complex algebraic surfaces. (In [33, Section 11] one can find other applications.) The idea of Hirzebruch was to construct a surface of general type X as a Kummer covering of $\mathbb{P}^2_{\mathbb{C}}$ of degree 3^{d-1} branch along an arbitrary arrangement of d lines, and then apply the BMY inequality to X, which can be improved by adding the information of some rational and elliptic curves inside of X. See [23] and the book [2] for variations of that idea, which also produce nontrivial constraints on the t_m numbers.

Thinking on the inequality (1) for real arrangements, and taking into consideration related questions about the nature of the Bogomolov-Miyaoka-Yau inequality (see e.g. [30]):

Question 2. Is there a topologically based proof of the Hirzebruch-Sakai inequality?

Warning: this could be a hard question. Is there a cell decomposition that may be of help as it was for arrangements over $\mathbb{R}$? On the opossite side, one may think that incidences from line arrangements defined over fields of positive characteristic may help to produce counterexamples, but [47] shows topological difficulties in that plan of attack. Below we will elaborate about the geometry in positive characteristic.

The Hirzebruch-Sakai inequality (2) shows that complex arrangements cannot have $t_2 = t_3 = 0$. On the other extreme, arrangements with only double points are the general ones. How about only triple points? The only known complex arrangement with only triple points is the dual Hesse arrangement. Combinatorially speaking, one can write down incidences of potential examples with an unbounded amount of lines, but none of them has been proved to be realizable over $\mathbb{C}$. (In characteristic 2, as we will see below, the Fano arrangement consists of 7 lines and 7 triple points.)

Question 3. Is the dual Hesse arrangement the only (nontrivial) complex line arrangement with only triple points?

See [14] for various aspects about triple points in arrangements. Speaking on special incidences at points in an arrangement of lines, we should write a bit about the world of point-line configurations (cf. [21, Chapter III], [12], [18]).

A (r_a, d_b)-*configuration* for us is an arrangement of d lines which contains r points, such that each line passes through b points and each point is in exactly a lines. These points are included in the set of all singular points of the arrangement, but they are not necessarily all of them. One observes that

$$r \cdot a = d \cdot b.$$

When $a = b$ (and so $r = d$), we call it an r_a-configuration. It is an easy exercise to find all r_2-configurations. The r_3-configurations are richer and diverse in general. For $r = 7$ we have essentially one possible combinatorics, and its realization is the Fano plane configuration defined by the 7 lines in $\mathbb{P}^2_{\mathbb{F}_2}$. In this case, there is no more room for extra singular points. Moreover, the following fact due to de Bruijn and Erdös [8] implies a characterization of all r_a-configurations where the number of singular points of the arrangement is exactly r. Consider any nontrivial arrangement of d lines. Then we have

$$\sum_{m \geq 2} t_m \geq d, \tag{3}$$

and equality holds if and only if the arrangement is quasi-trivial or it is defined by all the lines in $\mathbb{P}^2_{\mathbb{F}_q}$ for some q. (See [15, Theorem 2.7] for a proof, where the complicated part is to classify equality.) These very special configurations are called *finite projective plane arrangements.* They consists of q^2+q+1 lines with $t_{q+1} = q^2+q+1$, $t_m = 0$ else, and so they are all r_{q+1}-configurations. They are only possible in positive characteristics.

In general, it seems that r_a-configurations have not been classified. Not even their combinatorics, which is not unique (see e.g. the possible incidences for 9_3-configurations in [21, Chapter III]).

Let us finish with another big family of special arrangements of lines: (a,b)-nets. Motivation to study them comes from topological invariants of complements of line arrangements, in particular components of the resonance varieties that they define (see [60], [42]). Some references for (a,b)-nets are [60], [52], [42], [56], [37] , [31], [32]. A (a,b)-net is an arrangement $\mathcal{A}$ of ab lines which is the union of a arrangements $\mathcal{A}_i$ which do not share lines, such that there is a set of b^2 points $\mathcal{X}$ satisfying (1) The intersection point of any line in $\mathcal{A}_i$ with any line in $\mathcal{A}_j$ belongs to $\mathcal{X}$ for $i \neq j$, and (2) Through every point in $\mathcal{X}$ there passes exactly one line from each $\mathcal{A}_i$. One can check that each $\mathcal{A}_i$ contains b lines, and that a (a,b)-net is in particular a (b^2_a, ab_b)-configuration.

For example, one can check that all $(3,3)$-nets belong to the following family (up to change of coordinates):

$\mathcal{A}_1 : L_1 = (y)$, $L_2 = (\frac{1}{c}x + y + z)$, $L_3 = (\frac{b}{b-1}x + z)$,

$\mathcal{A}_2 : L_4 = (x)$, $L_5 = (x + cy + z)$, $L_6 = (by + z)$,

$\mathcal{A}_3 : L_7 = (x + c(1-b)y)$, $L_8 = (x + y + z)$, $L_9 = (z)$,

where b, c are parameters (see [56, Section 4]). In Figure 3 we evaluate this family for suitable real numbers b, c.

The incidence of the b^2 a-points in $\mathcal{X}$ is determined by $(a-2)$ $b \times b$ Latin squares which form an orthogonal set (cf. [56]). Any (a,b)-net defines a pencil of curves of degree b in $\mathbb{P}^2$ (see [56, Section 3] for details). A very simple argument on the topological Euler characteristic of the corresponding fibration shows that the only possible values for (a,b) over $\mathbb{C}$ are: $(3, b \geq 2)$, $(4, b \geq 3)$, and $(5, b \geq 6)$. An alternative and elemental argument [32] shows that actually $(5,b)$-nets are impossible over $\mathbb{C}$, and that $(4,b)$-nets cannot exist for b congruent to 2 mod 3.

For $(3,b)$-nets we do have classification when $b < 7$ (see [56]) in characteristic zero. Here we need just one Latin square. For example, the set of 6×6 Latin squares below represent all the combinatorial possibilities (up to reorganize lines and points), but only nine of them (not the bold ones) are realizable as $(3,6)$-nets in $\mathbb{P}^2_{\mathbb{C}}$. See [56, Section 3] for details.

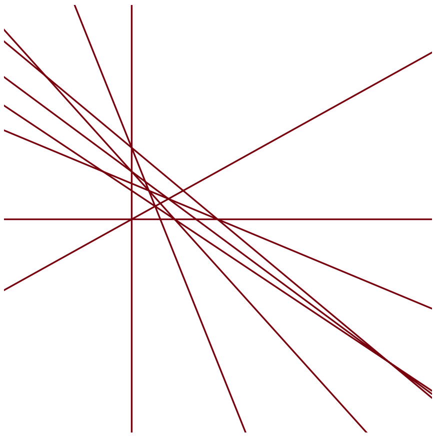

Figure 3: A real $(3,3)$-net.

1	2	3	4	5	6
2	3	4	5	6	1
3	4	5	6	1	2
4	5	6	1	2	3
5	6	1	2	3	4
6	1	2	3	4	5

1	2	3	4	5	6
2	1	5	6	3	4
3	6	1	5	4	2
4	5	6	1	2	3
5	4	2	3	6	1
6	3	4	2	1	5

1	2	3	4	5	6
2	3	1	5	6	4
3	1	2	6	4	5
4	6	5	2	1	3
5	4	6	3	2	1
6	5	4	1	3	2

1	**2**	**3**	**4**	**5**	**6**
2	**1**	**4**	**3**	**6**	**5**
3	**4**	**5**	**6**	**1**	**2**
4	**3**	**6**	**5**	**2**	**1**
5	**6**	**1**	**2**	**4**	**3**
6	**5**	**2**	**1**	**3**	**4**

1	**2**	**3**	**4**	**5**	**6**
2	**1**	**4**	**3**	**6**	**5**
3	**4**	**5**	**6**	**1**	**2**
4	**3**	**6**	**5**	**2**	**1**
5	**6**	**2**	**1**	**4**	**3**
6	**5**	**1**	**2**	**3**	**4**

1	2	3	4	5	6
2	1	4	5	6	3
3	6	2	1	4	5
4	5	6	2	3	1
5	3	1	6	2	4
6	4	5	3	1	2

1	2	3	4	5	6
2	1	4	3	6	5
3	5	1	6	4	2
4	6	5	1	2	3
5	3	6	2	1	4
6	4	2	5	3	1

1	2	3	4	5	6
2	1	6	5	3	4
3	6	1	2	4	5
4	5	2	1	6	3
5	3	4	6	1	2
6	4	5	3	2	1

1	2	3	4	5	6
2	3	1	6	4	5
3	1	2	5	6	4
4	6	5	1	2	3
5	4	6	2	3	1
6	5	4	3	1	2

1	2	3	4	5	6
2	1	6	5	4	3
3	5	1	2	6	4
4	6	2	1	3	5
5	3	4	6	2	1
6	4	5	3	1	2

1	2	3	4	5	6
2	1	4	5	6	3
3	4	2	6	1	5
4	5	6	2	3	1
5	6	1	3	2	4
6	3	5	1	4	2

1	**2**	**3**	**4**	**5**	**6**
2	**1**	**5**	**6**	**4**	**3**
3	**5**	**4**	**2**	**6**	**1**
4	**6**	**2**	**3**	**1**	**5**
5	**4**	**6**	**1**	**3**	**2**
6	**3**	**1**	**5**	**2**	**4**

Typically there are some free parameters in their construction, as in the case of $(3,3)$-nets above, so we obtain families for each Latin square. On the other hand, the question on which multiplication tables of groups can be realized as $(3,b)$-nets was completely answered in [31].

Question 4. Is it possible to characterize Latin squares which realize $(3,b)$-nets?

For $(4,b)$-nets over $\mathbb{C}$ we know only one example: The Hesse arrangement. One can prove that this is the unique $(4,3)$-net.

Question 5. Is the Hesse arrangement the only $(4,b)$-net over $\mathbb{C}$?

Very recently appeared this pre-print [3] which claims to solve that question in the positive. The argument involves the computation of the topological signature of a particular covering branched along a hypothetical $(4,b)$-net. There is an algebraic way and a topological way to compute it, and they cannot agree, unless the $(4,b)$-net is the Hesse arrangement.

2 Chern numbers

To any given arrangement of d lines $\mathcal{A}$ we can assign the so-called logarithmic *Chern numbers*:

$$\bar{c}_1^2(\mathcal{A}) = 9 - 5d + \sum_{m\geq 2}(3m-4)t_m \qquad \bar{c}_2(\mathcal{A}) = 3 - 2d + \sum_{m\geq 2}(m-1)t_m.$$

We usually write just $\bar{c}_1^2, \bar{c}_2$. These numbers have been studied in [26], [22], [50], [57], [15] to mention some places. One could see these integers as invariants which stay constant under deformations of line arrangements which do not change their point-line incidences. They produce a label for the parameter space which classifies them.

The origin of these numbers comes from the following. Think about the projective plane $\mathbb{P}_k^2$ as an example of a nonsingular projective surface over k. Let $\sigma\colon X \to \mathbb{P}_k^2$ be the blow-up of all the m-points of $\mathcal{A}$ with $m > 2$. Let D be the reduced total transform under σ of the arrangement, and so it contains all strict transforms of the lines and all exceptional divisors of σ. Let $\Omega_X^1(\log D)$ be the rank two vector bundle on X of *logarithmic differentials* with poles in D. Let

$$c_i = c_i(\Omega_X^1(\log D)^*)$$

be the Chern classes of the dual of $\Omega^1_X(\log D)$ (see e.g. [54, §1.4 and §3.2]). The logarithmic Chern numbers of the pair (X, D) are $\bar{c}_1^2 = c_1 \cdot c_1$ and $\bar{c}_2 = c_2$. In [57, §2 and §4] it is explained this process in more generality for arrangements of curves in algebraic surfaces.

An analog of the well-studied geography of surfaces of general type (cf. [41], [25]) would be to find restrictions on logarithmic Chern numbers and to construct line arrangements for any given admissible pair. This has not been solved for surfaces of general type as far as the author knows! We will now summarize what is known about constraints on $(\bar{c}_1^2, \bar{c}_2)$, which depend on the field k, and results on the distribution of the *Chern slope* $\bar{c}_1^2/\bar{c}_2$. For any details on what follows we refer to [15].

If $\mathcal{A}$ has $t_d = t_{d-1} = 0$, then

$$\bar{c}_1^2 > 0 \quad \text{and} \quad \bar{c}_2 > 0,$$

and so $\bar{c}_1^2/\bar{c}_2$ is a well-defined positive rational number. Moreover

$$\frac{2d-6}{d-2} \leq \frac{\bar{c}_1^2}{\bar{c}_2} \leq 3, \tag{4}$$

and equality holds on the left if and only if the arrangement is in general position; equality holds on the right if and only if $\mathcal{A}$ is a finite projective plane arrangement. The inequality on the left is combinatorial, and the proof can be found in [50, Theorem (5.1)]. The inequality on the right is exactly the inequality (3). Therefore, we have that independently of the field k

$$\bar{c}_1^2/\bar{c}_2 \in [1, 3].$$

Arrangements with low slopes seem to be easier to construct. They include all arrangements in general position for $\bar{c}_1^2/\bar{c}_2 < 2$, and for that range we cannot have accumulation points since we are forced to have $d \to \infty$.

Question 6. Is it possible to classify arrangements with $\bar{c}_1^2/\bar{c}_2 < 2$? Do we have restrictions on the field k for the realization of line arrangements with $\bar{c}_1^2/\bar{c}_2 < 2$?

In [50] Sommese characterizes arrangements for some few low slopes.

Arrangements with $\bar{c}_1^2/\bar{c}_2 \geq 2$ have more complexity. For example we have two zones in characteristic zero: any arrangement d lines $\mathcal{A}$ with $t_d = t_{d-1} = 0$ satisfies

(ℝ) If k is in $\mathbb{R}$, then $\bar{c}_1^2 \leq \frac{5}{2}\bar{c}_2$. Equality holds if and only if $\mathcal{A}$ is simplicial.

(ℂ) If k is in $\mathbb{C}$, then $\bar{c}_1^2 \leq \frac{8}{3}\bar{c}_2$. Equality holds if and only if $\mathcal{A}$ is the dual Hesse arrangement.

The inequality in (ℝ) is precisely the inequality (1), and in (ℂ) is the inequality (2). This is a beautiful and surprising connection! We recall that, in general, we do have logarithmic BMY inequalities in the complex case (see [33]), bypassing Hirzebruch's trick with coverings, and even we have a characterization in the case of equality (see [53, Theorem 3.1]).

In this way, the interval $]8/3, 3]$ contains only Chern slopes for arrangements defined in positive characteristics. We know that accumulation points of Chern slopes are only possible in $[2, 3]$. There is a strategy in [15] to fabricate accumulation points, which can be used to produce the following distribution results.

- For any given p, the set of accumulation points of Chern slopes of arrangements over $\overline{\mathbb{F}}_p$ is the interval $[2, 3]$.
- The set of accumulation points of Chern slopes of arrangements over $\mathbb{R}$ is the interval $[2, \frac{5}{2}]$.

Hence we have complete answers in those cases. Over $\mathbb{Q}$ we know that the Chern slope is dense in $[2, 2.375]$, because we have an infinite family of arrangements over $\mathbb{Q}$ whose Chern slopes tend to 2.375 (see [15]).

Question 7. Is the set of accumulation points of Chern slopes of arrangements over $\mathbb{Q}$ equal to the interval $[2, \frac{5}{2}]$?

This could in principle be solved if someone answers positively the question on the existence of infinitely many non-quasi trivial simplicial arrangements over $\mathbb{Q}$ (see the previous section).

Over $\mathbb{C}$ there is more mystery. There are no known accumulation points in the purely complex zone $]5/2, 8/3]$. The inequality in (ℂ) does not consider explicitly any properties of the field k, and, by the Hilbert Nullstellensatz theorem, the existence of an arrangement over $k \subset \mathbb{C}$ would imply the existence of an arrangement with the same point-line incidences over some number field (see at the end of [15]).

Question 8. Let k be a complex number field. Is there an inequality as in (2) which includes some properties of k?

Question 9. Are Chern slopes of complex line arrangements discrete in $]5/2, 8/3]$? i.e., is there any accumulation point in that interval?

We do not even know if there is some $\epsilon > 0$ such that the only arrangement in $[8/3 - \epsilon, 8/3]$ is the dual Hesse arrangement!

As explained in [15], there is a direct connection with Harbourne constants. They were introduced in [4] to study the bounded negativity conjecture (BNC) for blow-ups of the complex projective plane. In general, the BNC states:

"Let X be a smooth complex projective surface. Then there is an integer $b(X)$ such that for every curve (reduced, irreducible) $C \subset X$ we have $C^2 \geq b(X)$".

It is known that it fails in positive characteristic via a suitable use of the Frobenious morphism. One way to approach to the BNC problem is to explore an asymptotic point of view via H-contants and h-indices. The H-constant $H(X)$ measures how negative the self-intersection of reduced curves over s could be in a blow-up of X at s distinct points (if it is not over s, then it would be trivially $-\infty$). The h-index $h(X)$ (Harbourne index) measures the same but over the singular set of the reduced curve. We point out that BNC for reduced irreducible curves is equivalent to the analogue BNC for reduced curves (see [20]), and so if $H(X)$ is bounded then BNC holds for blow-ups of X at distinct points. This is mainly to treat blow-ups of surfaces X were the BNC is trivially true, e.g. $X = \mathbb{P}^2_{\mathbb{C}}$, but where the question on blow-ups is unknown (and important). The h-index is slightly weaker than the H-constant, since

$$-2 \geq h(X) \geq H(X),$$

but for complex surfaces there is no example with $h(X) \to -\infty$. (For rational surfaces in char $p > 0$, there are examples coming from the finite projective plane line arrangements.) The BNC for complex surfaces is related to logarithmic BMY type of inequalities. For example, using the work of Miyaoka and the Zariski decomposition of pseudo-effective divisors, one can show a bound for C^2 which depends on the genus of the components [19], i.e. BNC is true for reduced curves with components of bounded genus.

The *linear H-constant* for a line arrangement $\mathcal{A}$ is defined as

$$H_L(\mathcal{A}) := \frac{d^2 - \sum_{m\geq 2} m^2 t_m}{\sum_{m\geq 2} t_m},$$

or equivalently

$$H_L(\mathcal{A}) = \frac{3 - (\bar{c}_1^2 - 2\bar{c}_2)}{d - (\bar{c}_1^2 - 3\bar{c}_2)} - 2.$$

We then have a bijection between the above open questions on density of Chern slopes and the analogue for linear H-constants, which can be found in [15, Section 4]. See [9] for more on this topic.

3 Arrangements of rational plane curves

We close this note with some thoughts in relation to the study of arrangements of rational plane curves. As it was already mentioned in the introduction, these arrangements have been used to answer difficult questions on the existence of special 4-manifolds (algebraic surfaces or not), and to test the bounded negativity conjecture. The author believes that we know very little about them, even the case of configurations of conics has not been explored much. How about nodal cubic curves or a mix of cubics, conics, and lines? (For example, they naturally appear from singular fibers of rational elliptic fibrations. There is no systematic study of them.)

A key point in that study would be to find restrictions for the existence of special configurations, and a starting point could be to understand optimally constraints coming from their log Chern invariants. The goal would be to prove theorems as we described in the previous section for line arrangements. We recall that we can assign these Chern invariants to any arrangement of curves in any nonsingular projective surface over any field k, by considering the logarithmic differentials in a log resolution of the arrangement. (See e.g. [55, 57] for some relevant results which use arrangements that are pairwise transversal, producing a bridge between geography and log geography of surfaces.) Let us point out that these more general arrangements introduce all sorts of plane curve singularities, which makes the problem more difficult to handle.

Here is one explicit motivation. In the theory of algebraic surfaces, we still do not know how to classify simply-connected Godeaux surfaces. The work of Lee-Park [34] gave us a way to explore instead its Kollár–Shephered-Barron–Alexeev compactification via singular surfaces with only T-singularities. (See many examples in [51].) It turns out that most of these singular surfaces are rational, and the images of the exceptional divisors are arrangements of rational plane curves. It is possible to classify these arrangements? There is no description yet of the KSBA boundary of rational surfaces for the moduli space of simply-connected Godeaux surfaces. In addition, it is an open related question: What is the list of T-singularities that show up in this KSBA boundary? Configurations of rational curves in K3 surfaces are relevant to construct surfaces of general type with geometric genus 1. A very recent work on that connection is [44], which is opening a complete new world to be explored!

An algebraic treatment of arrangements of rational plane curves can be found here [5]. For general facts on arrangements of rational smooth plane curves, i.e. lines and conics, one can check [23, Section 9], [36], [48] to mention some. Recently there has been an interest on special arrangements of conics, as one can check here [43], [29], [45], [13], [10], [11].

Instead of developing further any details on Chern invariants and applications of these more general arrangements, we finish with the following questions.

Question 10. What is the optimal interval for Chern slopes of conic-line arrangements?

Question 11. Which are the arrangements of complex rational plane curves with the highest Chern slopes?

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