



Some Open Questions in Quiver Gauge Theory

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Abstract

Quivers, gauge theories and singular geometries are of great interest in both mathematics and physics. In this note, we collect a few open questions which have arisen in various recent works at the intersection between gauge theories, representation theory, and algebraic geometry. The questions originate from the study of supersymmetric gauge theories in different dimensions with different supersymmetries. Although these constitute merely the tip of a vast iceberg, we hope this guide can give a hint of possible directions in future research.

Keywords: Quiver gauge theory, symplectic singularity, Gorenstein singularity, moduli space

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1 Motivation

Quivers, introduced to algebraic geometry in [1] and to theoretical physics in [2], have been a useful tool in the study of supersymmetric gauge theories. In particular, moduli spaces have been extensively studied in the past few decades. The explorations of different branches of the moduli spaces connect various areas in string theory, algebraic geometry, symplectic geometry, tropical geometry, cluster algebra and representation theory.

For instance, brane setups and transitions naturally give rise to symplectic singularities, especially through affine Grassmannians as shown recently in [3]. The notion of magnetic quivers [4] and manipulations of quivers have then been introduced, revealing connections among different theories and geometries. Of late, it has been discussed that superconformal field theories (SCFTs) in higher dimensions, in particular their Higgs branches (i.e., the moduli of matter sectors), can be analyzed via certain spaces of dressed monopole operators¹. We should also emphasize that the corresponding spaces of dressed monopole operators and magnetic quivers, as a feature of the theories, are studied for their own sake, which in this sense is broader than Coulomb branches (i.e., moduli of gauge field sectors) of some 3d dual theories.

To analyze the moduli spaces in more details, phase diagrams (aka Hasse diagrams) are introduced to determine the structures of the singularities. On the other hand, Hilbert series (HS) as well as highest weight generating functions (HWGs) have been standard and helpful concepts to enumerate gauge invariant operators (GIOs). Given a variety $X = \text{Spec}(I)$ in some ring I , the HS of the ring counts holomorphic functions (which are GIOs physically) at different degrees. The HWGs provide a more concise notion for HS using highest weight Dynkin labels of the symmetry groups. The HS can be refined or unrefined. When we have multivariables, this gives multi-gradings mathematically and they represent different fugacities for different symmetry groups in our supersymmetric theories. When we perform unrefinement, they would encode less information but have lighter expressions which might be easier to compute.

As a rational function, the HS can be written as a Taylor expansion in

¹For more details on (both bare and dressed) monopole operators, readers are referred to [5].

the form

$$\text{HS} = \sum_{k=0}^{\infty} H(k)t^k, \quad (1.1)$$

where $H(k)$ is known as the Hilbert function. It is also useful to take the Laurent expansion of the (unrefined) HS as

$$\text{HS} = \sum_{i=0}^{\infty} \frac{\gamma_i}{(1-t)^{d-i}}, \quad (1.2)$$

where d is the (Krull) dimension of the ring. When our variety X is Gorenstein, the leading coefficient γ_0 is the volume of the Sasaki-Einstein (SE) base and is physically related to the central charges of the supersymmetric gauge theory [6, 7]. In general, γ_0 is the degree of X . For some special cases, γ_0 and γ_1 can be related to the a -invariant (i.e., the degree of the numerator minus the degree of the denominator of HS) or even the order and pseudoreflections for a finite group [8, 9]. These two coefficients are also shown to be crucial in the study of K-stability [10, 11].

Example 1. Let us consider $\mathbb{C}^3/\mathbb{Z}_3$ (with coordinates $z_{1,2,3}$) as an example. Hence, a monomial $z_1^{n_1} z_2^{n_2} z_3^{n_3}$ is invariant under the $\mathbb{Z}_3$ action if and only if $n_1 + n_2 + n_3 \equiv 0 \pmod{3}$. As a result, the multi-variate HS is

$$\text{HS}(t_1, t_2, t_3) = \sum_{\substack{n_1, n_2, n_3=0 \\ n_1+n_2+n_3 \equiv 0}}^{\infty} t_1^{n_1} t_2^{n_2} t_3^{n_3} = 1 + t_1^3 + t_2^3 + t_3^3 + t_1^2 t_2 + \dots \quad (1.3)$$

If we take the fugacity map $t_1 = tx_1, t_2 = tx_2/x_1, t_3 = t/x_2$, we have

$$\text{HS}(x_1, x_2, t) = 1 + t^3 x_1^3 + t^3 x_2^3/x_1^3 + \dots = \sum_{k=0}^{\infty} \chi([3k, 0])t^{3k}, \quad (1.4)$$

where $\chi([m_1, m_2])$ is the character of irreducible representations of $\text{SU}(3)$ (in the highest weight notation) with variables $x_{1,2}$. By setting $x_{1,2} = 1$ and redefining $t^3 \rightarrow t$, we get

$$\text{HS}(t) = \frac{1 + 7t + t^2}{(1-t)^3} = \frac{9}{(1-t)^3} - \dots \quad (1.5)$$

However, more generally, it has been a long-standing problem to fully understand and determine all the Laurent coefficients of any HS.

Question 1. What are the mathematical and physical interpretations of all the Laurent coefficients of any HS?

2 Theories with 8 Supercharges

Let us begin with rigid theories with 8 supercharges. Such theories have vector multiplets and hypermultiplets. The vector multiplets transform under adjoint reps of the gauge groups while the hypers/matters can transform under any reps. In the context of quivers, we will restrict to (bi)fundamental reps for the hypers². In the language of quivers, we can express the information of the supersymmetric theories with 8 supercharges using

Round Node $\bigcirc$	Gauge group and vector multiplet under its adjoint rep
Square Node $\square$	Flavour group
Line ———	Hyper(s) transforming under the bifundamental rep of the groups the line connecting

It is also possible to other types of edges in the quivers such as ones with orientations in non-simply laced quivers [12–16] and squiggles denoting charge 2 hypers [17].

2.1 The Higgs and Coulomb Branches

In recent years, the 3d $\mathcal{N} = 4$ Coulomb branches, realized as spaces of dressed monopoles (to be precise), play a key role in the study of quivers and SCFTs in various dimensions. In particular, various tools have been developed including HS [5, 18] and HWGs [19], Kraft-Procesi (KP) transitions and transverse slices [20–23], quiver subtractions [24] and quiver additions [3], discrete gauging and quiver origami [25–28], and magnetic quivers and phase diagrams [4, 17, 29–34]. There are many interesting perspectives which can be found in these references. Here we will only pick several examples.

The HS is a generating function of a ring/variety/scheme. Physically, it enumerates the GIOs at each degree for the moduli space (which will mainly be the Higgs or Coulomb branch here in our discussion). Importantly, magnetic quivers (as fundamental properties of the higher dimensional theories) allow us to formally equate Coulomb branches for certain

²In 6d with $\mathcal{N} = (1, 0)$, there are also tensor multiplets. There are also interesting questions for 6d theories and tensor multiplets, but we will not discuss them here.

3d theories with Higgs branches for higher dimensional theories as moduli spaces. Therefore, we are able to analyze the moduli spaces in 5d or 6d using the knowledge of 3d theories³. Since they are *symplectic singularities*, they are composed of *symplectic leaves* and these leaves form a poset by inclusions of their closures. Every smaller leaf (in the sense of the partial order) has a *transverse slice* to it in a larger leaf. Every transverse slice can be viewed as a moduli space for some theory which can be obtained from (partial) Higgsing from the unbroken theory.

On the other hand, it is convenient to use phase diagrams to encode the poset structure of the symplectic singularities. In a phase diagram, the nodes are often labelled by the (quaternionic) dimensions of the leaves and the edges have numbers denoting the dimensions of the neighbouring transverse slices (aka *elementary slices*). Going from the bottom of the phase diagram to the top is the process of Higgsing the theory, where the top corresponds to the fully Higgsed theory at the origin of the full Higgs branch while each segment starting from the top to some node in the middle (or at the bottom) corresponds to certain effective theory where the gauge group is partially (or completely) broken.

With these two powerful tools, it would be natural to ask the following question.

Question 2. *Is there a way to compute the phase diagrams from the HS/HWGs?*

The phase diagrams for the full moduli spaces (including mixed branches) were studied in [32], and it was found that certain theories have Higgs and Coulomb branches which are related by *inversion* of their phase diagrams. On the other hand, a duality known as *symplectic duality* between Higgs and Coulomb branches of a 3d $\mathcal{N} = 4$ theory was conjectured in [35] and has been extensively studied in [36–43]. Therefore, for phase diagrams, another question could be raised.

Question 3. *Is there any relation between symplectic duality and inversions of phase diagrams?*

All the related questions either mentioned or not mentioned here would

³As aforementioned, we should really treat this as a property of the 5d or 6d theory itself.

help us get a better understanding of the moduli space. Besides, there is a fundamental question we should always mention:

Question 4. *What is the mathematical definition of Coulomb branches (for 3d $\mathcal{N} = 4$ theories)?*

Nakajima and collaborators have quite a few inspiring works on this including [36–38, 44]. This problem would be of great interest to both mathematicians and physicists.

Recently, Nakajima and Weekes have also generalized this to more general symmetrizable theories [45]. In particular, quiver folding allows one to get non-simply laced quivers from simply-laced ones. This relates Coulomb branches for unfolded and folded quivers via finite group actions. For these non-simply laced quivers, the HS and HWGs have been calculated in [28] for unitary quivers and in [16] for orthosymplectic quivers. However, it is still not clear on how to compute the Higgs branches.

Question 5. *What is the method to compute HS and HWGs for the Higgs branches of folded theories?*

2.2 Canonical Singularities and 5d SCFTs

Besides the techniques introduced above, a parallel study for 5d $\mathcal{N} = 1$ SCFTs appeared in [46]. In that work, Xie and Yau conjectured that every three dimensional canonical singularity would give rise to a 5d SCFT with $\mathcal{N} = 1$. In particular, the chamber structure and prepotential of the Coulomb branch are studied using the Nef cones from crepant resolutions.

Following the works [47–49], *dot diagrams* (aka *generalized toric polygons*) which were first introduced in [50] were used to study those 5d $\mathcal{N} = 1$ SCFTs. The aforementioned tools including magnetic quivers and phase diagrams can also be successfully applied under various manipulations of the dot diagrams. Therefore, one may raise the following question.

Question 6. *Does M-theory on any 3d canonical singularity always define a 5d $\mathcal{N} = 1$ SCFT (conjecture 1 in [46])? Can we use the tools including dot diagrams and magnetic quivers to verify this conjecture?*

Incidentally, the (*generalized*) *s-rule* can also be described in terms of dot diagrams following [48, 50]. It is straightforward to incorporate the

s-rule using tessellations of the polygon, but it is hard to write a more sufficient algorithm in the sense of coding (and hence there is a strong s-rule in [48]) so that none of the possible tessellations would be missed. On the other hand, the s-rule can be determined by the self-intersection number of the holomorphic curve that M2 wraps for irreducible brane junctions [51, 52].

Question 7. *What is the criterion of generalized s-rule in terms of self-intersection numbers for reducible junctions?*

Question 8. *Is there a faster way/(coding) algorithm to determine generalized s-rule using dot diagrams?*

2.3 Affine Grassmannians

In a recent paper [3], the *affine Grassmannian* (which is an ind-scheme) has been introduced to the study of brane setups in string theory, where Coulomb branches (for framed quivers) are slices in it [53]. The authors worked out the transverse slices and phase diagrams for different affine Grassmannians by studying the corresponding brane systems and quivers. Here, we will only give the definition of affine Grassmannians. See [3] for detailed calculations and examples.

Definition 2.1. *Let k be a separably closed field. The ring of formal power series and the formal Laurent series over it are denoted as $k[[t]]$ and $k((t))$ respectively. The affine Grassmannian Gr_G of a connected reductive group G is the coset space $G(k((t)))/G(k[[t]])$.*

In [3], the authors focused on $k = \mathbb{C}$ and any finite dimensional Lie group G . In this case, Gr_G is the quotient $G(\mathbb{C}((t)))/G(\mathbb{C}[[t]])$. Equivalently, it is defined to be the set of all *lattices* in $k((t))^n$, where a lattice in $k((t))^n$ is a free $k[[t]]$ -module of rank n .

In the language of category theory, the definition can be reformulated as follows [54]. Let **k -Algebra** and **Grp** be the category of commutative k -algebra and the category of (small) groups. The *loop group* is the group functor $L_t G : \mathbf{k}\text{-Algebra} \rightarrow \mathbf{Grp}$, $A \mapsto G(k(t))$. Likewise, the *positive loop group* is the group functor $L_t^+ G : \mathbf{k}\text{-Algebra} \rightarrow \mathbf{Grp}$, $A \mapsto G(k[[t]])$. Then the affine Grassmannian is the (fpqc-)quotient $\mathrm{Gr}_G = L_t G / L_t^+ G$.

Question 9. *Study the Coulomb branch for “ugly” and “bad” theories (in the sense of [55]) as (generalized) affine Grassmannian slices.*

As discussed above, besides the Coulomb branches of the quivers, the Higgs branches can be obtained from inversions of phase diagrams.

Question 10. *What is the appropriate notion of symplectic duals for the affine Grassmannians?*

The unitary quivers without loops for elementary slices including those in the study of minimal nilpotent orbits [56–58] and in the study of affine Grassmannian are listed in [3, Table 1]. Nevertheless, the list is still incomplete. Therefore, a classification of elementary slices (including those not from affine Grassmannians) is still not known.

Question 11. *Classify all the elementary slices.*

It is also worth noting that the notion of quiver addition was also introduced in [3]. Together with quiver subtractions, we may ask:

Question 12. *What is the rigorous mathematical description of quiver subtractions and quiver additions?*

3 Theories with 4 Supercharges

Next, we move on to theories with fewer supersymmetries, viz, with four supercharges which exist in dimension less than or equal to 4. In particular, we will mainly focus on 4d $\mathcal{N} = 1$ theories here whose quivers can be obtained from decomposing $\mathcal{N} = 2$ multiplets using the following rules:

4d $\mathcal{N} = 2$ (8 supercharges)	4d $\mathcal{N} = 1$ (4 supercharges)
Round Node 	Round Node with Directed Loop 
Vector multiplet	Vector multiplet & Adjoint Chiral Multiplet
Line 	Bidirectional Line 
Hypers	Chiral and Anti-Chiral Multiplets

For 3d $\mathcal{N} = 2$ theories, the quivers can be obtained from $\mathcal{N} = 4$ theories with the same rules. As we can see, the quivers for 4d and 3d theories with 4 supercharges share some common features. Both of them have R-symmetry $U(1)$, yet the underlying physics could still be quite different.

3.1 Minimized Volumes and Their Bounds

A very-well studied class of 4d theories is the worldvolume theories of D3-branes probing toric Gorenstein singularities [59–72]⁴. The forward and inverse algorithms allow us to go between toric diagrams and quivers as well as brane tilings easily. The holographic dual of such a theory would live in $AdS_5 \times Y_5$ where the Sasaki-Einstein manifold Y_5 (of real dimension 5) is exactly the base of the 3d toric Gorenstein singularity. Even more remarkably, this SE base has a close relation to the determination of R-symmetry of our 4d theory. Under RG trajectory, the R-symmetry at IR fixed point may become a linear combination of the original $U(1)_R$ and some abelian flavour symmetries preserved through the flow. To determine the R-charges of the operators in our theory, Intriligator and

⁴In literature, this is often referred to as Calabi-Yau (CY) n -folds. However, to be strict, we will save the name CY for compact (smooth) manifolds and call the unresolved singularities Gorenstein. For other types of branes probing Gorenstein singularities, see for example [73–78].

Wecht proposed a procedure known as the a -maximization in [79] where a is one of the central charges in 4d. The idea that a decreases along RG flow was first conjectured by Cardy in [80] and was proven over two decades later by Komargodski and Schwimmer in [81].

For the worldvolume theories of D3s discussed here, Gubser showed in [82] that $a = \frac{1}{4V_n}$ where V_n is the volume function of the SE base Y normalized by the volume of the 5-sphere⁵. Therefore, a -maximization has now been translated to the problem of *volume minimization* for Sasaki-Einstein manifolds. See also [83, 84].

From a lattice polygon, we not only have a non-compact 3d toric Gorenstein singularity, but can also construct a compact toric variety (which is not necessarily CY) from the inner normal fan of the polygon. In general, the discussions so far do not need to be restricted to 3d Gorenstein singularities but any dimension n . Every Y_{2n-1} still has a minimized volume and we can always construct some compact toric variety X of complex dimension $(n-1)$ from the corresponding $(n-1)$ -tope. Physically, they may also have some string theory picture in terms of different Dp -branes in certain dimensions.

In [85, 86], systematic computations of minimized volumes were performed. In particular, one can relate the minimized volumes to the topological invariants of $\tilde{X}$ where the tilde denotes the *complete* (crepant) resolution of X , as X may not be smooth⁶. In particular, we shall always take *fine* triangulations, that is, triangulations involving all the lattice points. The authors conjectured that the minimized volumes have certain bounds in terms of the Chern numbers of X . This then naturally relates geometric and topological quantities for different objects. The most updated conjecture can now be written as follows.

Conjecture 3.1. *For an n -dimensional toric Gorenstein singularity associated with a polytope Δ_{n-1} (either reflexive or non-reflexive), the minimized volume $V_{n,\min}$ of the SE base manifold Y is bounded by*

$$\frac{1}{\chi} \leq V_{n,\min} < m_n \int c_1^{n-1}, \quad (3.1)$$

⁵In general for n -dimensional Gorenstein, V_n is the normalized volume for Y_{2n-1} , viz, $\text{vol}(Y_{2n-1})/\text{vol}(S^{2n-1})$.

⁶Following [87, 88], X whose dimension is no less than 4 may not be completely desingularized. In such cases, we will only focus on toric varieties that admit complete resolutions.

where χ and c_1 are the Euler number and first Chern class of the complete resolution $\tilde{X}$. Moreover, the left bound is saturated when the Gorenstein singularity is an Abelian orbifold of $\mathbb{C}^n$.

In particular for reflexive cases, the coefficients m_n are conjectured to be positive and satisfy $m_3 \sim 3^{-3}$, $m_4 \sim 4^{-4}$ and $m_n > m_{n+1}$.

It is worth noting that such bounds for 2d lattice polygons in terms of their areas are obtained in [89]:

$$\frac{1}{A} \leq V_{3,\min} < \frac{4\pi^2}{27A}, \quad (3.2)$$

where A is the normalized area of the polygon. Moreover, the lower bound is saturated for triangles while the upper bound is the case for ellipses (as limit shapes of polygons) and hence can never be saturated. If we compare (3.2) with (3.1) with $n = 3$, we find that the lower bounds agree: χ is the Euler number for a complete resolution which corresponds to a fine triangulation, and hence $\chi = A$. Furthermore, they take equalities under the same condition. For the upper bound, (3.1) becomes $m_3 C_1 = m_3(12 - \chi)$, which is more subtle to understand its connection to (3.2). This leads to the question below.

Question 13. *Prove or disprove Conjecture 3.1. If the conjecture is true, are these the best bounds? If the conjecture is not true, find the correct bounds. Also, can we find the connections between (3.1) and (3.2) (in particular for their upper bounds)?*

Furthermore, conjecture 5.4 in [85] gives rise to another question for minimized volumes and Euler numbers. In particular, the authors observed that maximum value of $V_{n,\min}\chi$ is reached at dP_3 for $n = 3$ and some fibrations of dP_3 for $n = 4$ [85, Figure 15]. One may then ask:

Question 14. *Is the maximum value for $V_{n,\min}\chi$ for reflexive polytopes attained by various (not necessarily uniquely) dP_3 fibrations?*

3.2 K-Stability and Chiral Rings

As discussed above, a -maximization is closely related to volume minimization. Following [10, 11], K -stability naturally appears in the study of a -maximization. More specifically, the authors showed that K-(semi)stability for product test configurations is equivalent to volume minimization. Therefore, in [90], K-stability for general test configurations was regarded as some “generalized a -maximization”.

Given a chiral ring I which is a ring composed of chiral operators under operator product expansions, we can compute the HS of the associated variety $X = \text{Spec}(I)$. Given the (unrefined) HS of some X graded by the one-dimensional symmetry⁷ ζ , the strategy is to “perturb” it with some test symmetry η such that the new HS is graded by $\zeta + \epsilon\eta$ for sufficiently small $\epsilon > 0$. Then one can compute the (*Donaldson-*)*Futaki invariant* for this test symmetry using the (leading and subleading) coefficients in the Laurent expansions of HS. Now X is said to be K-semistable if $F \geq 0$ for any test symmetry η . For it to be K-stable, F can be zero only when the norm vanishes. For the expression of Futaki invariants and the definition of norm, one is referred to [91, 92]. More details of defining K-stability along with its calculations can also be found in [92].

In [90], it was then conjectured that a ring is the chiral ring for an SCFT iff X is K-stable. Indeed, for some theories (such as the worldvolume theory of D3s probing Gorenstein singularities), one can show that K-stability recovers the unitarity bounds or irrelevance of superpotential terms [90]. However, a counterexample was found in [92]. Therefore, it is natural to ask:

Question 15. *What is the precise description on how K-stability is related to chiral rings and SCFTs?*

Computationally, it is still not known how to fully determine K-stability for a general variety as one needs to check infinitely many test symmetries in principle. For hypersurface singularities, especially for those having $U(1)^{n-1}$ isometry, not only the number but also exactly which test symmetries can be determined to compute Futaki invariants. See [93, 94] for more details. Nevertheless, the problem for a general variety is still unsolved.

⁷Physically, ζ is treated as our $U(1)$ R-symmetry.

Question 16. *How to determine the number of test symmetries one needs to check to determine K -stability for a general variety? Can we further determine exactly which test symmetries should be checked?*

Of course, it is reasonable to wonder whether there is any analogue in other dimensions, such as F -theorem in 3d or c -theorem in 2d etc.

Question 17. *Is it possible to apply K -stability to chiral rings in other dimensions? Can K -stability cope with accidental symmetries that might appear under RG flow?*

Moreover, Benvenuti and Giacomelli introduced another chiral ring stability in [95] in terms of dropping certain terms in the superpotential.

Question 18. *Is there any relation between the two stabilities for chiral rings?*

3.3 Graded Quivers: from $m = 1$ to general m

Now let us relax the restriction of fixed number of supercharges and take a quick look at *graded quivers*, aimed at providing a unified mathematical framework for gauge theories in even dimensions. Graded quivers have been extensively studied in [96–102]. As the name suggests, the arrows/fields in an m -graded quiver are associated with a grading by some quiver degree $c \in \{0, 1, \dots, m\}$. These different types of fields are denoted by $\Phi_{ij}^{(c)}$ for an arrow pointing from node i to node j with c ranging from 0 to $\lfloor m/2 \rfloor$. When a field has degree $m/2$ (for even m), it is unoriented. For those of degree $(m - c)$, we can use the notion of conjugate arrow with an opposite direction: $\bar{\Phi}_{ji}^{(m-c)} \equiv \overline{\Phi_{ij}^{(c)}}$. Physically, such conjugation refers to CPT conjugate fields. Superpotentials, W , are always crucial for a graded quiver. They are linear combinations of gauge invariant terms of degree $(m - 1)$ satisfying $\{W, W\} = 0$, where $\{-, -\}$ is the *Kontsevich bracket* which is a generalization of the Poisson bracket [96].

In the string theory picture, graded quivers generalize the D3s probing 3d Gorenstein singularities to $D(5 - 2m)$ -branes transverse to $(m + 2)$ -dimensional Gorenstein singularities in Type IIB string theory. Usually, we consider $m = 0, 1, 2, 3$ whose theory has 2^{3-m} supercharges in dimension $(6 - 2m)$ (see for example the table in (1.2) in [97]).

Analogous to Seiberg duality in 4d, graded quivers also enjoy certain dualities in other dimensions such as triality in 2d and quadrality in 0d [99]. Moreover, the generalized brane tilings on $\mathbb{T}^{m+1}$ for m -graded quivers have also been developed in [98]. More recently, a notion of product of toric diagrams was introduced in [100] which allows one to produce the quiver theory for an $(m+n+3)$ -dimensional Gorenstein singularity from a pair of theories for $(m+2)$ - and $(n+2)$ -dimensional Gorenstein singularities. Here we list some of the questions in the study of graded quivers.

Question 19. *Study the dualities for graded quivers involving adjoint fields.*

Question 20. *For brane tilings on $\mathbb{T}^2$, zig-zag paths and perfect matchings are important concepts. Reveal their further mathematical and physical implications for general m .*

Question 21. *Graded quivers can be obtained from the topological B -model. Is there any correspondence between the product of toric diagrams and the B -model approaches?*

3.4 Dessins d'Enfants

Dessins d'enfants are bipartite graphs, whose primary role is in the study of Riemann surfaces X . Specifically they represent the degeneracies and ramifications of so-called Belyĭ maps which take X to the Riemann sphere $\mathbb{P}^1$ [103]. They received particular popularity due to the faithful action on them by the absolute Galois group over the rational numbers: $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, whereby making them a point d'appui in geometry and number theory.

Let us begin with the classic theorem of Belyĭ [104].

Theorem 3.2 (Belyĭ). *A Riemann surface X has an algebraic model over $\overline{\mathbb{Q}}$ if and only if there exists a (surjective) map $\beta : X \rightarrow \mathbb{P}^1$ which is ramified at exactly 3 points.*

In the theorem, β is the Belyĭ map, and after Möbius transformation on the $\mathbb{P}^1$, the three ramification points can be mapped to: $\{0, 1, \infty\}$. The surface together with the map, (X, β) , is called the Belyĭ pair.

The components of the dessin d'enfant are then identified with these points via:

$$\beta^{-1}(0) \mapsto \circ, \quad \beta^{-1}(1) \mapsto \bullet, \quad \beta^{-1}((0, 1)) \mapsto - \quad (3.3)$$

such that the preimages of the $[0, 1]$ interval on the Riemann surface, X , form the dessin d'enfant, and in addition the preimages of ∞ are associated to the dessin d'enfant's faces.

3.4.1 Brane Tilings and Modular Parameters

The bipartite nature of dessin d'enfants allows for a natural association to brane tilings (*a.k.a.* dimer models). Brane tilings encode quiver gauge theories with toric moduli spaces as bipartite graphs drawn on the torus, $\mathbb{T}^2$ [63, 64, 72, 105]; the bipartite nature of these theories are considered in [106–109] and the relation to amoebae and tropical geometry, where the dual of the toric diagrams are so-called brane webs [110] are considered in [105].

Brane tilings are an alternative graphical method of representing the information of a quiver gauge theory with superpotential. They are related to the quiver dual graph, whereby tiling faces associate to $U(N)$ gauge groups, tiling edges to chiral multiplets, and tiling vertices to superpotential terms. The bipartite nature of the tiling identifies the orientation about which to write the vertex's incident edges for the vertex's monomial term in the superpotential. Interpreting genus 1 dessins d'enfants as brane tiling bipartite graphs, and vice versa, lead to a series of intriguing questions about the connections between them.

Quite interestingly a specific limit of the brane tiling's field theory can be graphically represented in a specific drawing of the brane tiling [111]. In the SCFT, performing a -maximisation for the central charge a as a function of the fields' R-charges, fixes the latter. These R-charge values are encoded in the tiling through *isoradial embedding*; which draws all nodes as lying on a unit circle centred on the face they border, whilst at the centre the angle subtended by a bordering edge corresponding to R-charge R_i , is $\pi(1 - R_i)$. The field theory maximisation is performed under the following conditions: (i) all superpotential terms maintain an R-charge of 2 (s.t. Lagrangian in superspace is well-defined) corresponding geometrically to the face centre's total angle equaling 2π ; and (ii) β -functions vanishing (to ensure conformality) corresponding to the faces forming a rhombus tiling [112]. There is a notion of “consistency” of whether these conditions suffice physically and mathematically [68, 78, 113], and a question emerges as to

Question 22. *Given an inconsistent tiling, what is the infra-red physics, and the geometry of the moduli space? On the other hand, for consistent theories, it would be important to check that the renormalization group flow in the field theory takes one to the correct geometry.*

An isoradial brane tiling dictates a specific form of the underlying torus. Its modular parameter, and hence, complex structure, can be extracted. We denote it as τ_R (to emphasize the R-charge origin). On the other hand, the Belyi pair fixes the complex structure of X , which we denoted as τ_B (to emphasize the Belyi origin).

The equality of these two complex structures was first conjectured in [106], and further work has found an array of examples where this equality holds ($\mathbb{C}^3$, conifolds, and their orbifolds). However, a counter-example in [111], viz., $L^{2,2,2}$, showed that these two complex structures need not be equal in general.

Now, Seiberg duality preserves τ_R from a physical point of view, as shown in [114]. However, its action on τ_B , and indeed on the dessin, is not clear. These discussions naturally lead to questions about the scope of the equality and the parameters' respective interpretations.

Question 23. *In what scenarios does $\tau_B = \tau_R$ hold?*

Question 24. *What is the physical interpretation of τ_B in the brane tiling's SCFT?*

Question 25. *Does τ_R have use in the theory of dessins d'enfants, or further in Galois theory?*

Another modular parameter also appears in this context, this time associated to a torus arising directly from the geometry. In the cases where the quiver gauge theory's toric vacuum moduli space is Gorenstein in nature, this Gorenstein singularity can be reformulated as a torus fibration. From there, the action of the 3d mirror symmetry acts alike a T-duality with $U(1)^3$ symmetry [105, 115], however only a $U(1)^2$ subgroup preserves the Kähler form and holomorphic 3-form. This $U(1)^2$ may be used to define an invariant part of the Gorenstein's SLag (special Lagrangian), such that this part will be a torus. Finally, pulling-back the metric to this torus where the brane tiling exists provides a metric. This torus'

metric's complex structure is then another modular parameter, denoted τ_G to emphasise the geometric motivation [116].

It is then instinctive to ask where this τ_G fits in with the other modular parameters. Computations in [116] showed that $\tau_G = \tau_R$ in some simple cases ($\mathbb{C}^3$, conifold); whilst also that this equality holds only approximately in more complicated scenarios ($L^{1,2,1}$).

Question 26. *What is the precise relation between τ_G , τ_R and τ_B ?*

3.4.2 Galois Orbits and Seiberg Duality

The dessin d'enfant Belyi maps, through Belyi's theorem, may be defined using algebraic numbers exclusively. However which specific field extension of the rationals is sufficient to describe them depends on the Belyi pair in question. The roots used in the Galois extension can be considered as roots of some minimal separable polynomial over the rationals. Changing which of these roots is used in the field extension for the Belyi map takes the Belyi map, and respective dessin, through its Galois orbit. Through Galois orbits dessins manifestly group themselves, and how these orbits find use physically is an area of particular interest.

Contrastingly, on the physical side quiver gauge theories naturally sort themselves into duality classes, based on the action of Seiberg duality [117–120]. Seiberg duality is an IR equivalence between 4d $\mathcal{N} = 1$ theories, in our case this importantly also applies to toric quiver gauge theories. The brane tilings for these theories thus form groupings as bipartite graphs within the same duality trees, alike the groupings of dessins d'enfants under Galois conjugation. How these two groupings arise in each others fields is suggested to have some interesting significance.

Question 27. *Is there physical significance to the relation between brane tiling theories whose bipartite graphs as dessins d'enfants are in the same Galois orbit?*

Question 28. *How does Seiberg duality between brane tilings relate dessins d'enfants, and what Galois invariants (if any) can be inspired by these duality trees?*

Since a primary goal within Galois theory is identifying Galois invariants that can be used as tools in calculation, perhaps Seiberg duality trees

may reveal some physically-inspired invariants with use on the mathematical side.

3.4.3 Seiberg-Witten Curves and Modular Surfaces

Seiberg-Witten curves are exact descriptions of the Coulomb branch for the IR limit of $\mathcal{N} = 2$ gauge theories. Deforming these theories with tree-level superpotentials causes symmetry breaking to $\mathcal{N} = 1$ theories, where vacua produced are connected in phases based on the deforming superpotential's parameters [121].

Through tuning the deforming superpotential's parameters, the roots of the components of the $\mathcal{N} = 2$ hyperelliptic curve can be shown to coalesce, such that a drawing of the roots connected by the relevant branch cuts forms a bipartite graph. These bipartite graphs occur in the moduli space exclusively at the points the Seiberg-Witten curve develops isolated singularities, considered special points of the $\mathcal{N} = 1$ phases.

These bipartite graphs are *dessins d'enfants*, and it was shown in [122] that order parameters separating different branches of the $\mathcal{N} = 1$ vacua may be considered as Galois invariants. This surprisingly link between the vacua phases of this symmetry breaking and *dessins d'enfants* inspires curiosity into the deeper connection between these objects.

Question 29. *How are $\mathcal{N} = 1$ vacua from Seiberg-Witten curves and *dessins d'enfants* explicitly linked?*

Their association would lead to powerful insight on both sides, identifying further Galois invariants, as well as deeper physical meaning. In addition to the relation to Seiberg-Witten hyperelliptic curves, *dessins d'enfants* also arise in relation to elliptic curves in another way.

Modular curves are formed through quotient action on the upper half plane, $\mathcal{H}$, by subgroups of the modular group, $\mathrm{PSL}(2, \mathbb{Z})$. After extension of this modular action to $\mathcal{H} \times \mathbb{C}$, the ‘modular surfaces’ formed for the index 24 subgroups are surprisingly K3 surfaces, and those for index 36 subgroups are Calabi-Yau 3-folds [123].

Examining the elliptic j -invariants for these surfaces in elliptic Weierstraß form, the fibration over $\mathbb{C}$ from the modular action extension turns these invariants into maps. These maps astoundingly turn out to be Belyĭ in nature, and thus these elliptic surfaces are related to *dessins d'enfants*

quite naturally. Moreover, the same dessins can be seen to arise from these subgroups more directly through their Schreier coset graphs [124].

This unexpected connection between modular surfaces and the dessins d'enfants through the modular group stimulates further questions about its scope for other modular subgroups.

Question 30. *To what extent do modular subgroups connect modular surfaces to dessins d'enfants?*

4 A Digression: Machine Learning

In recent years string and gauge theories have capitalised on a modern computational tool: machine-learning (ML). Its use initiated in this field with the examination of string landscapes [125–128], and has since quickly developed these techniques to a wide range of subfields related to gauge theories. In particular, ML has seen great success in topics discussed in this paper, examining: plethystics [129, 130], amoebae [131], Seiberg duality [132], and dessins d’enfants [133].

ML finds its use within this field in two primary scenarios. The first is for speed in computation since many problems have beyond polynomial time complexity to solve. Where these problems require minimal computation to check a solution, ML tools can quickly, and computationally cheaply, provide an array of predicted solutions which can easily be confirmed if valid or not. The second scenario is in conjecture formulation, the complexity of many ML tools makes them excellent at higher-dimensional pattern recognition, particularly useful when examining large datasets to spot relations and aid in forming conjectures.

The field of study of ML separates itself into 3 styles of problem: supervised, unsupervised, and reinforcement. Supervised learning uses tools such as neural networks, support vector machines, random forests to perform non-linear function fitting between known inputs and outputs. These tools are used with the aim that the function will have predictive power for both interpolation and extrapolation beyond the training dataset. Unsupervised learning uses tools such as clustering and autoencoders to identify patterns in data, and isolate degrees of freedom. Finally reinforcement learning trains an agent to efficiently search for optimum/desired solutions within a known state space, in a Markov Decision Process style.

Neural networks often use the ReLU function as the non-linear component, s.t. $\text{ReLU}(x) := \max(x, 0)$. The linear combination of these functions between neural network neurons makes the full function used to approximate the supervised data piecewise linear in nature. Interestingly, and surprisingly developed independently, the combination of linear action and maximisation finds itself at the heart of another field also, tropical geometry. Only recently has work been initiated to examine the interrelation between these areas [134], and calls for further exploration.

Question 31. *What is the complete correspondence between tropical geometry and neural networks?*

Since there is a natural connection between tropical geometries through amoebae and quiver gauge theories, the previous question would link into our primary topic:

Question 32. *Can the tropical functions describing ReLU NNs be re-described as quiver gauge theories?*

The action of neural network training has recently drawn inspiration from traditional quantum field theory techniques. The non-gaussianity of the training process has been related to renormalisation group flow, with different trained networks acting alike fixed points in the flow [135, 136] (q.v. [137]). Cementing these ideas is an interesting topic for further work.

Question 33. *Can the success of machine learning methods be explained through quantum field theory techniques?*

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