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Curves in projective spaces: questions and remarks

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Abstract

We discuss several questions on the geometry of curves in projective spaces: existence or non-existence for prescribed degrees and genera, their Hilbert function and their gonality.

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We collect some open questions (often raised several times in the literature) concerning non-degenerate curves $X \subset \mathbb{P}^r$, $r \geq 3$. As a general suggestion when you get the existence of some curves with given degre and genera in a projective space, try to study the Hilbert scheme of all such curves, the cohomological properties of the ideal sheaves of such curves and the Brill–Noether theory of them. In each irreducible component Γ of such Hilbert scheme there is the general Hilbert function and general gonality (the ones of a general $[X] \in \Gamma$) and the ones achieved only by some $[X] \in \Gamma$. Thus the Hilbert functions (or the gonality or the higher gonalities) give an interesting stratification of Γ. Several questions collected here remain open for at least 40 years. We try to grade them according to their feasibility and their interest. From both points of view, the best question is 17. We discuss it in Remark 19. Some of these questions are related to the construction of "intrinsic" subvarieties of $\overline{\mathcal{M}}_g$, i.e. the compactification of the moduli space of stable genus g curves, in order to determine its Kodaira dimension [2, 24, 29, 37, 38]. For a century (at least) the stratification by gonality of $\mathcal{M}_g$ has been considered (see Def. 10). A key step for studying $\overline{\mathcal{M}}_g$ was to study the moduli space $\overline{\mathcal{M}}_{q,n}$ of stable n-pointed curves for all $q < g$; motivated G. Farkas and coworkers to study the Brill–Noether theory of pointed curves [25, 27, 30, 31]. For syzygies of curves with prescribed gonality, see [3, 26]. Our main topic is the geography of curves in projective spaces. On one side we can ask:

Question 1. For which triples (r, d, g) of positive integers, $r \geq 3$, there is a smooth, connected and non-degenerate curve $X \subset \mathbb{P}^r$ such that $\deg(X) = d$ and $p_a(X) = g$?

On the other side, we can ask:

Question 2. Given a triple (r, d, g), find an integral and non-degenerate curve $X \subset \mathbb{P}^r$ such that $\deg(X) = d$ and $p_a(X) = g$.

This is known only for $r = 3$ by Gruson and Peskine [35,39] and known, except for very large g, in $\mathbb{P}^4$ and $\mathbb{P}^5$ [64]. Refined versions, prescribing that the curve is linearly normal appear in [49, 61, 62]. We recall:

Theorem 3. *[8, Thm. 1.4] For every integer $r \geq 3$ there is a real number $c_r > 0$ such that for all integers $d \geq r$ and g satisfying the inequalities $d - r \leq g \leq c_r d^2$ there is a smooth and connected linearly normal curve $X \subset \mathbb{P}^r$ of degree d and genus g.*

The classification of curves with maximal genus [36, Ch. 3] shows that $c_r < \frac{1}{2r-2}$. For $r = 3$ the best constant is $c_3 = 1/6$ by [35]. One could try to improve [8], but we consider that a more interesting problem is the following.

Conjecture 4. *There exists a real number $\alpha > 0$ such that for all integers $d \geq r \geq 3$, $d - r \leq g \leq \frac{\alpha}{r}d^2$ there is a smooth, connected and linearly normal curve $X \subset \mathbb{P}^r$ of degree d and genus g.*

Probably the constants c_r and α may be improved if we prescribe the existence only for large d. Better constants c_r and α are obtained if instead of taking $g \leq c_r d^2$ or $g \leq \frac{\alpha}{r}d^2$ we take $g \leq c_r d^2 + \gamma(d, r)$ or $g \leq \frac{\alpha}{r}d^2 + \gamma(d, r)$ with $\lim_{d \to n+\infty} \gamma(d, r)/d^2 = 0$. We discuss these topics and related results in Section 1. At the end of Section 1 we consider the Brill–Noether theory of embedded curves. Here we fix the abstract curve X and look at its embeddings in projective spaces.

In Section 2 we take $r = 3$: space curves. Section 2 consider a refined version of the problem (due to Halphen) where we also fix, roughly speaking, the minimal degree s of a surface containing the space curve X. We just recall that Guido Castelnuovo's part started around 140 years ago, while Halphen's works predate Castelnuovo's ones by about 15 years. Brill–Noether paper was published in 1873. In Sections 1 and 2 we also discuss the Hilbert function of the embedded curves $X \subset \mathbb{P}^r$. We need the following notation and definition.

For any $k \in \mathbb{N}$ and any $X \subset \mathbb{P}^r$ let $\rho_{X,k} : H^0(\mathcal{O}_{\mathbb{P}^r}(k)) \to H^0(\mathcal{O}_X(k))$ denote the restriction map.

Definition 5. Fix $k \in \mathbb{N}$. The curve $X \subset \mathbb{P}^r$ is said to be *k-normal* if $\rho_{X,k}$ is surjective. X is said to be *strongly k-normal* if it is j-normal for all $0 \leq j \leq k$. Furthermore X is said to have *maximal rank* if for each $t \in \mathbb{N}$ the linear map $\rho_{X,t}$ is either injective or surjective, i.e. if either $h^0(\mathcal{I}_X(t)) = 0$ or $h^1(\mathcal{I}_X(t)) = 0$.

Let $\alpha(X)$ be the first integer t such that $h^0(\mathcal{I}_X(\alpha(X))) = 0$ (this restriction does no depend on t). We say that X has maximal rank if and only if $h^1(\mathcal{I}_X(t)) = 0$ for all $t \geq \alpha(X)$. For each integer $s \in \{2, \dots, r-1\}$ let $\alpha_s(X)$ be the first positive integer t such that $|\mathcal{I}_X(t)|$ has base locus of codimension $\geq s$. Obviously $\alpha_2(X) \geq \alpha(X)$. Since X is integral, $\alpha(X) = \alpha_2(X)$ if and only if $h^0(\mathcal{I}_X(\alpha(X)) \geq 2$.

Remark 6. Let $H(r,d,g)$ denote the open part of the Hilbert scheme of $\mathbb{P}^r$ parametrizing all smooth, connected and non-degenerate curves of degree d and genus g. The following argument explains the big difference between $r = 3$ and $r > 3$ for the smoothness of $H(r,d,g)$.

Fix a curve $X \in H(r,d,g)$ and let N_X denote the normal bundle of $X \in \mathbb{P}^r$. The vector space $H^0(N_X)$ is the tangent space of $H(r,d,g)$ at $[X]$ and $H^1(N_X)$ is the obstruction space for $H(r,d,g)$ at $[X]$.

Since $\deg(N_X) = (r+1)d + 2 - 2g$ and N_X is a rank $(r-1)$ vector bundle on X, $\chi(N_X) = (r+1)d + (3-r)(1-g)$. Thus, $\chi(N_X) = 4d$ for $r = 3$, i.e. $h^0(N_X)$ does not depend on g for $r = 3$, while $\chi(N_X) < 0$ (and hence $h^1(N_X) > 0$) if $r \geq 4$ and, say, $g > 1 + \frac{r+1}{r-3}d$.

We always have $h^1(N_X) = 0$ if $h^1(\mathcal{O}_X(1)) = 0$ by the Euler's sequence of $T\mathbb{P}^r$ and it may be proved that $h^1(N_X) = 0$ for a general embedding of fixed degree of a general curve of genus g. However, to prove maximal rank using the Horace Method one needs $h^1(N_X(-S)) = 0$ for a general $S \subset X$ such that $s := \#S$ is as large as possible [10, 11, 54–57].

Since N_X has rank $r-1$, $\chi(N_X) = (r+1)d + (r-3)(1-g) - (r-1)s$. It is important to consider the case $\#S = d$ with the stronger requirement $h^1(N_X(-1)) = 0$ which is equivalent to saying that for a fixed hyperplane H, there exists a general subset of H whose cardinality is the intersection with some $X \in H(r,d,g)$.

We have $\chi(N_X(-1)) = 2d + (r-3)(1-g)$ and hence for $r > 3$ we cannot have $h^1(N_X(-1)) = 0$ in the range $d \geq g + r$ associated to non-special embeddings. Thus, the range of (r,d,g), $r > 3$, for which $h^1(N_X(-1)) = 0$ is true does not cover the Brill–Noether range $(r+1)d \geq rg + r(r+1)$, i.e. the integers d such that a general smooth curve of genus g has a non-degenerate degree d embedding in $\mathbb{P}^r$ (to get an embedding, not just a g^r_d on X one may quote [20]). This is, up to now, the main technical problem to get maximal rank theorems using smoothings of reducible curves.

For any positive integer d let $\alpha(d)$ be the maximum genus of a smooth, connected non-degenerate degree d curve $X \subset \mathbb{P}^3$ such that $h^1(N_X) = 0$. Let $\beta(d)$ be the maximum integer such that for all $0 \leq g \leq \beta(d)$ there is a smooth, connected and non-degenerate curve of degree d and genus g $X \subset \mathbb{P}^3$ such that $h^1(N_X) = 0$. G. Ellingsrud and A. Hirschowitz proved that $\alpha(d)$ and $\beta(d)$ grow like $d^{3/2}$ [24], which roughly for the same (d,g) also proved $h^1(N_X(-2)) = h^0(N_X(-2)) = 0$. Note that in $\mathbb{P}^3$ the Brill–

Noether range grows like $\frac{4}{3}d$, while the maximum genus grows like $d^2/4$.

Remark 7. In the body of the paper we assume characteristic 0. In positive characteristic we point out [8], where the authors wrote if it is proved in arbitrary characteristic or not and in the latter case in the proof it is described the key missing ingredient for a characteristic free proof. For Castelnuovo's upper bound for the genera and for curves near this large genus in positive characteristic, see [9, 63].

Remark 8. A different picture arises when one looks at non-reduced projective curves without embedded points, i.e. locally Cohen–Macaulay schemes $X \subset \mathbb{P}^r$ with X of pure dimension 1. The numerical invariants are the degree $d := \deg(X)$ and the arithmetic genus $p_a(X) = 1 - \chi(\mathcal{O}_X)$. Even for $d = 2$ (double lines) the integer $p_a(X)$ may be an arbitrarily negative integer. For upper bounds on the integer $p_a(X)$ in terms of d, see [14, 15]. R. Hartshorne conjectured that the Hilbert scheme of locally Cohen–Macaulay curves in $\mathbb{P}^3$ of degree d and arithmetic genus g is connected [43, 58].

1 Castelnuovo's side

Let $A(r,d,g)$ (resp. $A'(r,d,g)$, resp. $A_1(r,d,g)$) be the set of all triples $(r,d,g) \in \mathbb{N}^3$ such that $r \geq 3$ and there is a smooth and connected (resp. integral, resp. smooth, connected and linearly normal) curve $X \subset \mathbb{P}^r$ such that $\deg(X) = d$ and $p_a(X) = g$. For any fixed integer $r \geq 3$, the classical theory of Castelnuovo gives a quadratic upper bound for the genus g in terms of d and a description of the curves achieving the upper bound or very near to it [36, Ch. 3]. Curves $X \subset \mathbb{P}^r$ with very high genus are contained in a low degree surface. Several results with g of quadratic order with respect to d appear in [36, Ch. 3]. D. Eisenbud and J. Harris made a conjecture [36, Main Conjecture at p. 132], but it seems that there are no tools to solve it (in the first 20 years after [36] a few papers solved the very first cases of it using very long calculations).

There is no complete full description of the sets $A(r,d,g)$, $A'(r,d,g)$ and $A_1(r,d,g)$ for $r \geq 6$. For $r = 3$, see [35,39,62]. For $r = 4,5$ see [61,64]. The query about $A_1(r,d,g)$ was raised by R. Hartshorne [41, Prob 4d.4]. It was recently studied again, because in some (but not all) cases the open subset of the Hilbert scheme parametrizing smooth and linearly normal curves with fixed degree and genus is irreducible [13, 22, 23, 46, 50–52].

Question 9. [8, Question 1.3] Fix a positive integer k. Is there a real number $c_{r,k}$ such that for all integers $d \geq r$ and $d - r \leq g \leq c_{r,k}d^2$ there is a smooth and connected k-normal (or strongly k-normal) curve $X \subset \mathbb{P}^r$ of degree d and genus g?

See [8, Thm. 1.4] for a positive answer in characteristic 0 and see [8, Thm. 1.5] for a similar characteristic free result. These quoted theorems do not only give an asymptotic result, but they give explicit quadruples (d, g, r, k) for which degree d and genus g strongly k-normal curves $X \subset \mathbb{P}^r$ exist. Moreover, they give other numerical and cohomological informations on the curves X lying in these surfaces. But for the same quadruple (d, g, r, k) there should exist very different solutions X (maybe even lying in different irreducible components of the Hilbert scheme of $\mathbb{P}^r$).

Definition 10. Let X be a smooth projective curve of genus $g \geq 3$. The *gonality* of X is the first integer k such that $h^0(L) = 2$ for some $L \in \operatorname{Pic}^k(X)$.

Fix $L \in \operatorname{Pic}^k(X)$ such that $h^0(L) = 2$. There are two different definitions of higher gonalities.

Definition 11. Fix any integer $r \geq 2$. The *r-gonality* $k(r)$ of X is the minimal degree of a line bundle L on X such that $h^0(L) \geq r + 1$.

Definition 12. The *birational r-gonality* $b(r)$ of X is the minimal degree of a non-degenerate integral curve $Y \subset \mathbb{P}^r$ birational to X.

Note that $k(r) = b(r) = r + g$ for all $r \geq g$. Often $k(r) = b(r)$. For instance, this is true for curves with general moduli. For $r \geq 3$ we suggest [20] and Severi theory of nodal plane curves for $r = 2$. However, when $g \gg k$ often $b(r) > k(r)$ for low r. For instance, if X is not a multiple covering of a curve of positive genus and $g > (k-1)(2k-1)$, then $k(2) = 2k$ and the only $R \in \operatorname{Pic}^{2k}(X)$ such that $h^0(R) = 3$ is $L^{\otimes 2}$ by the Castelnuovo–Severi inequality [1], [48, Cor. p. 26].

Remark 13. A conjecture on the ratios of the integers $b(r+1)/b(r)$ was disproved in [5], but the former conjecture is true in many cases and there is no guess for a universally true good bound for these ratios or the difference $b(r+1) - b(r)$.

The paper [5] is another use of singular curves contained in easy surfaces to get results for smooth curves. Often one can compute the gonality and the Brill–Noether theory of X if it is known a low degree plane model of X. For instance, M. Coppens described all degrees of the base point free line bundles of a smooth plane curve [17]. The gonality of complete intersection curves is known [45]. Moreover The gonality of X is associated to a morphism $X \to \mathbb{P}^1$ of minimal degree. If we drop the minimality condition and we vary X in $\mathcal{M}_g$ we get the Hurwitz space (see [16] and references therein).

2 Halphen's side

Fix integers $s \geq 2$ and $d \geq 3$. Let $G(d,s)$ be the maximal genus of a smooth and connected curve $X \subset \mathbb{P}^3$ such that $h^0(\mathcal{I}_X(s-1)) = 0$. A classical problem is the computation of the integer $G(d,s)$ [40, Prob. 3.1]. For this problem the set of all $(d,s) \in \mathbb{N}^2$ was divided in the following 4 regions (ranges $\emptyset$, **A**, **B** and **C**) [7,32,33,40]. If $d < \frac{s^2+4s+6}{6}$, then no such curve exists [40, Thm 3.3]. Hence this is called the Range $\emptyset$.

Range **A** corresponds to the case $\quad \frac{s^2+4s+6}{6} \leq d < \frac{s^2+4s+6}{3}$.

Range **B** is the range $\quad \frac{s^2+4s+6}{3} \leq d \leq s(s-1)$,

Range **C** is the range $\quad d > s(s-1)$.

Fix (d,s) in Range **C** and set $G_C(d,s) := 1 + [d(d+s^2-4s) - e(s-e)(s-1)]/2s$, where e is the only integer such that $d+e \equiv 0 \pmod{s}$ and $0 \leq e < s$. L. Gruson and C. Peskine proved a 100 years old guess (with an incomplete proof) of Halphen. They proved that $G(d,s) = G_C(d,s)$ in Range **C** and classified all curves with maximal genus (they are all arithmetically Cohen–Macaulay) [34]. There are integers $g < G_C(d,s)$ such that there is no curve $X \subset \mathbb{P}^3$ of degree d and genus g such that $h^0(\mathcal{I}_X(s-1)) = 0$ [18, 53]; these triples (d,s,g) are called *Halphen's gaps* or g is called a Halphen's gap for (d,s). It is conjectured:

Conjecture 14. *No $g \leq G_C(d,s+1)$ is a Halphen's gap for (d,s).*

In Range **C** all maximal rank curves are arithmetically Cohen–Macaulay

[19] and, in particular, they have maximal rank. See part (b) of Remark 19 for this problem in the ranges **A** and **B**.

Remark 15. We expect that for "many" $g \le G_C(d, s+1)$ there is a smooth degree d and genus g curve $X \subset \mathbb{P}^3$ such that $h^0(\mathcal{I}_X(s-1)) = 0$ and $h^1(\mathcal{I}_X(t)) = 0$ for all $t \ge \alpha_2(X)$.

We cannot be more precise in Guess 15 at the moment. Each attempt to cover a part of genera $g \le G_C(d, s+1)$ should give as a byproduct a lot of curves X with nice Hilbert function, with the only restriction that $h^0(\mathcal{I}_X(s-1)) = 0$. Many curves may be constructed using linkage. For instance, the maximal genus curves are arithmetically Cohen–Macaulay (aCM) because they are linked to a plane curve. Nearby genus are obtained taking the linkage of a small degree curve not contained in the plane. The linkage construction gives the Hilbert function of the curve X in the Range **C** in terms of the data of the small degree curve used for the linkage. For instance non aCM curves of submaximal genus and degree $sk-2$, $k \ge s$, are obtained linking two skew lines by a complete intersection of a surface of degree s and a surface of degree k.

Now we consider Range **B**. Note that if (d, s) belongs to Range **B**, then $s \ge 5$. For any $c \in \mathbb{Z}$ set $\delta(c) := 3$ if $c \in \{1, 3\}$, $\delta(c) := 2$ if $c \equiv 2 \pmod 3$ and $\delta(c) := 0$ otherwise. For all integers s and f such $s \ge 5$ and $s-1 \le f \le 2s-5$ set

$$A(s,f) := \frac{s^2 - sf + f^2 - 2s + 7f + 1 + \delta(2s-f-6)}{3},$$

$$B(s,f) := \frac{s^2 - sf + f^2 + 6f + 11 + 1 + \delta(2s-f-7)}{3}.$$

The function $A(s,f)$ is an increasing function of f in the interval $s-1 \le f \le 2s-5$ and it partitions the genera in Range **B** with $A(s, s-1) = (s^2+4s+6)/3$ and $A(s, 2s-5) = s(s-1)+1$. Moreover, $A(s,f) < B(s,f) \le A(s, f+1)$. Fix d such that (d, s) is in Range **B** and take f such that $A(s,f) \le d < A(s, f+1)$. Set $h(d) := 0$ if $A(s,f) \le d \le B(s,f)$ and $h(d) := (d - B(s,f))(d - B(s,f)+1)/2$ if $B(s,f) \le d < A(s, f+1)$. Set

$$G_{\mathrm{HH}}(d) := d(s-1) + 1 - \binom{s+2}{3} + \binom{f-s+4}{3} + h(d).$$

In Range **A** it is easy to see that $G(d,s) \le G_A(d,s) := 1 + d(s-1) - \binom{s+2}{3}$ [40, Thm 3.3] and it was conjectured that equality holds [42, p. 364]. This conjecture is known to be true for all d, s such that $\frac{s^2+4s+6}{4} \le d < \frac{s^2+4s+6}{3}$ [32], [7, Cor. 2.4]. See [12] for the case $s \gg 0$.

In Range **B**, R. Hartshorne and A. Hirschowitz proved that $G(d,s) \ge G_{\mathrm{HH}}(d,s)$ and conjectured that equality holds [42]. We also mention [33] and [42] which settle a few cases just on the right of the last inequality of the Range **A**.

The following questions were also raised (with less hints) in [6, p. 22].

Question 16. Take (d,s) in Range **A**. Fix an integer g with $0 \le g \le G(d,s)$. Prove the existence of a smooth connected curve $X \subset \mathbb{P}^3$ such that: $\deg(X) = d$, $p_a(X) = g$, $h^0(\mathcal{I}_X(s-1)) = 0$ and X has maximal rank.

Question 17. Take (d,s) in Range **B**. Fix any integer g such $0 \le g \le G_{\mathrm{HH}}(d,s)$. Prove the existence of a smooth and connected curve $X \subset \mathbb{P}^3$ such that: $\deg(X) = d$, $p_a(X) = g$ and $h^0(\mathcal{I}_X(s-1)) = 0$.

Question 18. Take (d,s) in Range **C**. Fix any integer g such that $0 \le g \le G_C(d,s+1)$. Prove the existence of a smooth and connected curve $X \subset \mathbb{P}^3$ such that:
$\deg(X) = d$, $p_a(X) = g$ and $h^0(\mathcal{I}_X(s-1)) = 0$.

Remark 19. The more promising part of these questions is the one in Range **B**, because we believe that there are tools to study it and these efforts should give other results.

(a) The legacy of [42] was overshadowed by the Hartshorne–Hirschowitz conjecture that $G(d,s) = G_{\mathrm{HH}}(d,s)$ in the Range **B**. However, [42] and the previous papers by Hartshorne on stable reflexive sheaves may be adapted to produce many lower genus curves. The key point is to generalize [44], which is technically very sophisticated and it was taylor-made to get the input case of [42] constructing certain sheaves with the best possible cohomology (semi-natural cohomology).

(b) Curves in the Range **A** with genus $G_A(d,s)$ have maximal rank. Curves in the Range **C** with maximal genus have maximal rank (they are even aCM). Any curve in the Range **B** with genus $G_{\mathrm{HH}}(d,s)$ has very nice cohomology and this is a reason to suspect that $G(d,s) = G_{\mathrm{HH}}(d,s)$. The construction in [42] proved the existence of curves with good cohomology. While it is almost obvious that any curve X in the Range **A** with genus $G_A(d,s)$ has maximal rank, the positivity of several integers $h^1(\mathcal{O}_X(t))$

does not allow us to say that every X in the Range **B** with genus $G_{\mathrm{HH}}(d,s)$ has maximal rank. In our opinion this is the main difficulty to prove that $G_{\mathrm{HH}}(d,s) = G(d,s)$ in the Range **B**. It is reasonable to hope that covering a large slice of genera below $G_{\mathrm{HH}}(d,s)$ with the approach of [42] will give many curves with very good Hilbert function. Thus the first step should be to generalize [44], which was a key input for [42].

Recently another tool was used to attach the Range **B**: Bridgeland stability. In [59] E. Macrì and B. Schmidt reproved Gruson–Peskine upper bound for the genus result in Range **C** (and as in [42] and for other authors got the very right part of Range **B**). But the most promising part coming from [59] is that they were able to translate the discrete invariants of space curves to other threefolds and get non-existence results. Thus, instead of $\mathbb{P}^3$, one could use a different threefold W, say a Fano threefold or a blowing-up of a Fano threefold. For the existence parts perhaps a few surfaces embedded in the threefold may cover large parts of the numerical invariants. Compare the very good paper [60] (spanned line bundles on an ample divisor, which of course helps for non-existence results for curves) with [59] to see the advances in the technology.

Remark 20. Many existence theorems (e.g. the ones on Castelnuovo's theory in [36, Ch. III], the case of the cubic surfaces in [35, 39], singular quartics in [35,39] and some of the proofs in [8]) have the following set-up. Let S be an integral surface defined over $\mathbb{Q}$ and L be a line bundle on X defined over $\mathbb{Q}$. Thus $|L|$ is a projective space defined over $\mathbb{Q}$. There is a non-empty open subset U of $|L|(\mathbb{C})$ such that each $X \in U$ is a smooth and connected projective curve of degree d and genus g and, in the set-up of [8], satisfying some cohomological conditions. Since $|L|$ is a projective space defined over $\mathbb{Q}$, $|L|(\mathbb{Q})$ is Zariski dense in $|L|(\mathbb{C})|$. Thus, there exists $X \in U$ defined over $\mathbb{Q}$.

Question 21. Find other cases where solutions are curves defined over $\mathbb{Q}$.

We recall that only a few years ago E. Arbarello, A. Bruno, G. Farkas and G. Saccà constructed Gieseker–Petri curves defined over $\mathbb{Q}$ [4].

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