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On generating functions of modified Jacobi polynomials

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Abstract

In this paper, the present author has made an attempt to present a generalization of the result on bilateral generating functions involving modified Jacobi polynomials, found derived in [1, 2], by means of the theory of one parameter group of continuous transformations and using the notion of partial quasi bilateral generating function [3] involving some special functions.

Keywords: *Partial quasi-bilateral (or partial quasi-bilinear) generating relation, Partial quasi-bilateral (or partial quasi-bilinear) generating function, modified Jacobi polynomial.*

MSC Classification : *33C45.*

1. Preliminary concepts and introduction

Special functions are the solutions of a wide class of mathematically and physically relevant functional equations. Generating functions play a large role in the study of special functions. The generating functions which are available in the literature are generally unilateral or bilateral in nature. There is dearth of trilateral generating functions. But apart from these types, some other terms viz. quasi bilateral (or quasi bilinear) generating function, partial quasi bilateral (or partial quasi bilinear) generating function etc. are found in the literature. In this note we discuss a problem connected with partial quasi bilateral (or partial quasi bilinear) generating function [3] which is defined by the following generating relation:

$$G(x, z, w) = \sum_{n=0}^{\infty} a_n w^n P_{m+n}^{(\alpha)}(x) Q_r^{(m+n)}(z), \tag{1.1}$$

where the coefficients, a_n are arbitrary as well as independent of x and z and $P_{m+n}^{(\alpha)}(x), Q_r^{(m+n)}(z)$ are two special functions of orders $m+n, r$ and of parameters $\alpha, m+n$ respectively. If, in particular, the above two special functions are same i.e., $Q_r^{(m+n)}(z) \equiv P_r^{(m+n)}(z)$, we call the generating function as partial quasi-bilinear generating function. For example, if we replace $P_{m+n}^{(\alpha)}(x), Q_r^{(m+n)}(z)$ by $L_{m+n}^{(\alpha)}(x), C_r^{(m+n)}(z)$ respectively, then (1) is called partial quasi bilateral generating relation involving Laguerre and Gegenbauer polynomials [4] and if we replace $P_{m+n}^{(\alpha)}(x), Q_r^{(m+n)}(z)$ by the same type of polynomials $L_{m+n}^{(\alpha)}(x), L_r^{(m+n)}(z)$ (say), we call the relation (1) as partial quasi bilinear generating relation involving Laguerre polynomials [5] etc.

In the investigation of generating functions of various special functions, group-theoretic method seems to be a potent one in comparison with analytical method and has been receiving much attention in recent years. The idea of group theoretic method in the study of generating functions for various special functions was introduced by L. Weisner [6, 7, 8] while investigating Hypergeometric, Hermite and Bessel functions.

In [9], Das obtained the following result on bilateral generating function involving modified Jacobi polynomials.

Theorem 1. *If there exists a generating relation of the form:*

$$G(x,w)=\sum_{n=0}^{\infty} a_n w^n P_n^{(\alpha-n,\beta)}(x) \tag{1.2}$$

then

$$\begin{aligned}(1+w)\quad & \alpha\{1+\tfrac{w}{2}(1-x)\}^{-1-\alpha-\beta} G\left(\tfrac{x-\frac{w}{2}(1-x)}{1+\frac{w}{2}(1-x)}, \tfrac{vw}{(1+w)}\right)\\ & =\textstyle\sum_{n=0}^{\infty} w^n g_n(v) P_n^{(\alpha-n,\beta)}(x)\end{aligned}$$

where

$$g_n(v)=\sum_{p=0}^{n} a_p \binom{n}{p} v^p.$$

Subsequently, Chongdar [1], Chongdar and Mukherjee [2] obtained the following extension of the above theorem by analytical method.

Theorem 2. *If there exists a generating relation of the form:*

$$G(x,w)=\sum_{n=0}^{\infty} a_n w^n P_{n+r}^{(\alpha-n,\beta)}(x) \tag{1.3}$$

then

$$\begin{aligned}(1+w)\quad & \alpha\{1+\tfrac{w}{2}(1-x)\}^{-1-\alpha-\beta-r} G\left(\tfrac{x-\frac{w}{2}(1-x)}{1+\frac{w}{2}(1-x)}, \tfrac{vw}{(1+w)}\right)\\ & =\textstyle\sum_{n=0}^{\infty} w^n g_n(v) P_{n+r}^{(\alpha-n,\beta)}(x)\end{aligned}$$

where

$$g_n(v)=\sum_{p=0}^{n} a_p \binom{n+r}{p+r} v^p.$$

The above theorems are not only important but also very much useful for generalizing the known results.

The object of the present paper is to further generalize the theorem 2 by means of group-theoretic method based on the theory of one parameter group of continuous transformations and by using the notion of partial quasi bilateral generating function as defined in [3]. The main result obtained in this paper is given in the form of the following theorem.

Theorem 3. *If there exists a generating relation of the form:*

$$G(x,u,w)=\sum_{n=0}^{\infty}a_n P_{n+r}^{(\alpha-n,\beta)}(x)P_m^{(n+r,\beta)}(u)w^n \tag{1.4}$$

then

$$\begin{aligned}(1-w)\quad &-1-\beta-r-m(1-2w)^{\alpha}\{1-w(1-x)\}^{-1-\alpha-\beta-r}\\ &\times G\left(\tfrac{x+w(1-x)}{1-w(1-x)},\tfrac{u+w}{1-w}\tfrac{vw}{(1-2w)(1-w)}\right)\\ &=\textstyle\sum_{n,p,q=0}^{\infty}a_n\frac{w^{n+p+q}}{p!q!}v^n(-2)^p(n+r+1)_p(1+n+r+m+\beta)_q\\ &\times P_{n+r+p}^{(\alpha-n-p,\beta)}(x)P_m^{(n+r+q,\beta)}(u)\end{aligned}$$

which did not seem to appear in the earlier works.

2. Proof of the Theorem 3

To prove the theorem 3, we first consider the following generating relation:

$$G(x,u,w)=\sum_{n=0}^{\infty}a_n P_{n+r}^{(\alpha-n,\beta)}(x)P_m^{(n+r,\beta)}(u)w^n.$$

Replacing w by $wytv$ on both sides of the above relation, we get

$$G(x,u,wytv)=\sum_{n=0}^{\infty}a_n\left(P_{n+r}^{(\alpha-n,\beta)}(x)y^n\right)\left(P_m^{(n+r,\beta)}(u)t^n\right)(vw)^n. \tag{2.1}$$

Now for the above special functions, we consider the following two linear partial differential operators each of which generates one parameter continuous transformations group:

$$R_1=(1-x^2)y\tfrac{\partial}{\partial x}+2y^2\tfrac{\partial}{\partial y}-[(1+\alpha+\beta+r)(x-1)+2\alpha]y$$
$$R_2=(1+u)t\tfrac{\partial}{\partial u}+t^2\tfrac{\partial}{\partial t}+(1+r+\beta+m)t$$

such that

$$R_1\left(P_{n+r}^{(\alpha-n,\beta)}(x)y^n\right)=-2(n+r+1)P_{n+r+1}^{(\alpha-n-1,\beta)}(x)y^{n+1}$$
$$R_2\left(P_m^{(n+r,\beta)}(u)t^n\right)=(1+n+r+\beta+m)P_m^{(n+r+1,\beta)}(u)t^{n+1}$$

and

$$e^{wR_1}f(x,y)=(1-2wy)^{\alpha}\{1-wy(1-x)\}^{-1-\alpha-\beta-r}f\left(\tfrac{x+w(1-x)y}{1-w(1-x)y},\tfrac{y}{1-2wy}\right)$$
$$e^{wR_2}f(u,t)=(1-wt)^{-1-\beta-r-m}f\left(\tfrac{u+wt}{1-wt},\tfrac{t}{1-wt}\right).$$

Now operating $e^{wR_1}e^{wR_2}$ on both sides of (8), we get

$$e^{wR_1}e^{wR_2}G(x,u,wytv) = e^{wR_1}e^{wR_2}\sum_{n=0}^{\infty} a_n\left(P_{n+r}^{(\alpha-n,\beta)}(x)y^n\right)$$
$$\times\left(P_m^{(n+r,\beta)}(u)t^n\right)(vw)^n.$$

Left hand side of (13), with the help of (11) and (12), becomes

$$e^{wR_1}e^{wR_2}G(x,u,wytv) = (1-wt)^{-1-\beta-r-m}(1-2wy)^{\alpha}$$
$$\times\{1-wy(1-x)\}^{-1-\alpha-\beta-r}G\left(\frac{x+w(1-x)y}{1-w(1-x)y}, \frac{u+wt}{1-wt}, \frac{wytv}{(1-2wy)(1-wt)}\right).$$

Right hand side of (13), with the help of (9) and (10), becomes

$$e^{wR_1}\quad e^{wR_2}\sum_{n=0}^{\infty} a_n\left(P_{n+r}^{(\alpha-n,\beta)}(x)y^n\right)\left(P_m^{(n+r,\beta)}(u)t^n\right)(vw)^n$$
$$= \sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} a_n\frac{w^{n+p+q}}{p!q!}v^n(-2)^p(n+r+1)_pP_{n+r+p}^{(\alpha-n-p,\beta)}(x)y^{n+p}$$
$$\times(1+n+r+m+\beta)_qP_m^{(n+r+q,\beta)}(u)t^{n+q}$$

Equating (14) and (15) and then putting $y=t=1$, we get

$$(1-w)^{-1-\beta-r-m}(1-2w)^{\alpha}\{1-w(1-x)\}^{-1-\alpha-\beta-r}$$
$$\times G\left(\frac{x+w(1-x)}{1-w(1-x)}, \frac{u+w}{1-w}, \frac{vw}{(1-2w)(1-w)}\right)$$
$$= \sum_{n,p,q=0}^{\infty} a_n\frac{w^{n+p+q}}{p!q!}v^n(-2)^p(n+r+1)_p(1+n+r+m+\beta)_q$$
$$\times P_{n+r+p}^{(\alpha-n-p,\beta)}(x)P_m^{(n+r+q,\beta)}(u),$$

which is relation (7). This completes the proof of theorem 3.

Corollary 1: Putting $r=0$ in theorem 3, we get the following result:

Result 1: If there exists a quasi-bilinear generating relation [10]:

$$G(x,u,w) = \sum_{n=0}^{\infty} a_nP_n^{(\alpha-n,\beta)}(x)P_m^{(n,\beta)}(u)w^n \tag{2.2}$$

then

$$(1-w)^{-1-\beta-m}(1-2w)^{\alpha}\{1-w(1-x)\}^{-1-\alpha-\beta}$$
$$\times G\left(\frac{x+w(1-x)}{1-w(1-x)}, \frac{u+w}{1-w}, \frac{vw}{(1-2w)(1-w)}\right)$$
$$= \sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} a_n\frac{w^{n+p+q}}{p!q!}v^n(-2)^p(n+1)_p(1+n+m+\beta)_q$$
$$\times P_{n+p}^{(\alpha-n-p,\beta)}(x)P_m^{(n+q,\beta)}(u)$$

which shows that the existence of a quasi-bilinear generating relation involving modified Jacobi polynomial implies the existence of a more general generating relation.

We now proceed to show that theorem 3 is a generalization of theorem 2 by discussing the particular case of theorem 3 when $m = 0$.

Particular Case: Putting $m = 0$ in (7), we get

$$(1-w)^{-1-\beta-r}(1-2w)^{\alpha}\{1-w(1-x)\}^{-1-\alpha-\beta-r} \times G\left(\frac{x+w(1-x)}{1-w(1-x)}, \frac{vw}{(1-2w)(1-w)}\right)$$
$$= \sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} a_n \frac{w^{n+p+q}}{p!q!} v^n (-2)^p (n+r+1)_p (1+n+r+\beta)_q P_{n+r+p}^{(\alpha-n-p,\beta)}(x)$$
$$= \sum_{n=0}^{\infty}\sum_{p=0}^{\infty} a_n \frac{(-2)^p (n+r+1)_p}{p!} v^n w^{n+p} P_{n+r+p}^{(\alpha-n-p,\beta)}(x) \sum_{q=0}^{\infty} \frac{(1+n+r+\beta)_q}{q!} w^q$$
$$= \sum_{n=0}^{\infty}\sum_{p=0}^{\infty} a_n \frac{(-2)^p (n+r+1)_p}{p!} v^n w^{n+p} P_{n+r+p}^{(\alpha-n-p,\beta)}(x)(1-w)^{-1-n-r-\beta}$$

or,

$$(1-2w)^{\alpha}\{1-w(1-x)\}^{-1-\alpha-\beta-r} G\left(\frac{x+w(1-x)}{1-w(1-x)}, \frac{vw}{(1-2w)(1-w)}\right)$$
$$= \sum_{n=0}^{\infty}\sum_{p=0}^{\infty} a_n \frac{(-2)^p (n+r+1)_p}{p!} v^n w^{n+p} (1-w)^{-n} P_{n+r+p}^{(\alpha-n-p,\beta)}(x)$$
$$= \sum_{n=0}^{\infty}\sum_{p=0}^{\infty} a_n \frac{(-2)^p (n+r+1)_p}{p!} \left(\frac{v}{1-w}\right)^n w^{n+p} P_{n+r+p}^{(\alpha-n-p,\beta)}(x)$$
$$= \sum_{n=0}^{\infty}\sum_{p=0}^{\infty} a_n \frac{(n+r+1)_p}{p!} \left(\frac{v}{-2(1-w)}\right)^n (-2w)^{n+p} P_{n+r+p}^{(\alpha-n-p,\beta)}(x).$$

Now replacing $(-2w)$ by w and $\frac{v}{-2(1-w)}$ by v, we get

$$(1+w)^{\alpha}\{1+\tfrac{w}{2}(1-x)\}^{-1-\alpha-\beta-r} G\left(\frac{x-\frac{w}{2}(1-x)}{1+\frac{w}{2}(1-x)}, \frac{vw}{(1+w)}\right)$$
$$= \sum_{n=0}^{\infty}\sum_{p=0}^{\infty} a_n \frac{(n+r+1)_p}{p!} w^{n+p} v^n P_{n+r+p}^{(\alpha-n-p,\beta)}(x)$$
$$= \sum_{n=0}^{\infty}\sum_{p=0}^{n} a_{n-p} \frac{(n-p+r+1)_p}{p!} w^n v^{n-p} P_{n+r}^{(\alpha-n,\beta)}(x)$$
$$= \sum_{n=0}^{\infty} w^n \left(\sum_{p=0}^{n} a_{n-p} \frac{(n-p+r+1)_p}{p!} v^{n-p}\right) P_{n+r}^{(\alpha-n,\beta)}(x).$$

Thus we see that theorem 3 at $m = 0$ yields the following result:

If there exists a generating relation of the form:

$$G(x,w)=\sum_{n=0}^{\infty} a_n w^n P_{n+r}^{(\alpha-n,\beta)}(x)$$

then

$$\begin{aligned}(1+w) \quad & \alpha\{1+\tfrac{w}{2}(1-x)\}^{-1-\alpha-\beta-r} G\left(\frac{x-\frac{w}{2}(1-x)}{1+\frac{w}{2}(1-x)}, \frac{vw}{(1+w)}\right) \\ & = \textstyle\sum_{n=0}^{\infty} w^n g_n(v) P_{n+r}^{(\alpha-n,\beta)}(x)\end{aligned}$$

where

$$g_n(v)=\sum_{p=0}^{n} a_p \left(\begin{array}{c} n+r \\ p+r \end{array}\right) v^p,$$

which is theorem 2.

Corollary 2: If we put $r=0$ in theorem 2 we get theorem 1.

Now we would like to mention that by using the symmetry relation [11]:

$$P_n^{(\alpha,\beta)}(-x)=(-1)^n P_n^{(\beta,\alpha)}(x), \tag{2.3}$$

one can easily obtain the following analogous results from the above theorems (1-3) and result 1.

Theorem 4. If there exists a partial quasi-bilinear generating relation of the form:

$$G(x,u,w)=\sum_{n=0}^{\infty} a_n P_{n+r}^{(\alpha,\beta-n)}(x) P_m^{(\alpha,n+r)}(u) w^n \tag{2.4}$$

then

$$\begin{aligned}&(1+w)^{-1-\alpha-r-m}(1+2w)^{\beta}\{1+w(1+x)\}^{-1-\alpha-\beta-r} \\ &\times G\left(\frac{x+w(1+x)}{1+w(1+x)}, \frac{u+w}{1+w}, \frac{vw}{(1+2w)(1+w)}\right) \\ &= \textstyle\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} a_n \frac{w^{n+p+q}}{p!q!} v^n (-2)^p (n+r+1)_p (-1)^q (1+n+r+m+\alpha)_q \\ &\times P_{n+r+p}^{(\alpha,\beta-n-p)}(x) P_m^{(\alpha,n+r+q)}(u)\end{aligned}$$

which is analogous to theorem 3.

Result 2: If there exists a quasi-bilinear generating relation of the form:

$$G(x,u,w)=\sum_{n=0}^{\infty}a_nP_n^{(\alpha,\beta-n)}(x)P_m^{(\alpha,n)}(u)w^n \tag{2.5}$$

then

$$\begin{aligned}&(1+w)^{-1-\alpha-m}(1+2w)^{\beta}\{1+w(1+x)\}^{-1-\alpha-\beta}\\ &\times G\left(\tfrac{x+w(1+x)}{1+w(1+x)},\tfrac{u+w}{1+w},\tfrac{vw}{(1+2w)(1+w)}\right)\\ &=\textstyle\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}a_n\frac{w^{n+p+q}}{p!q!}v^n(-2)^p(n+1)_p(-1)^q(1+n+m+\alpha)_q\\ &\times P_{n+p}^{(\alpha,\beta-n-p)}(x)P_m^{(\alpha,n+q)}(u)\end{aligned}$$

which is analogous to result 1.

Theorem 5. If there exists a generating relation of the form:

$$G(x,w)=\sum_{n=0}^{\infty}a_nw^nP_{n+r}^{(\alpha,\beta-n)}(x) \tag{2.6}$$

then

$$\begin{aligned}(1-w)\quad &^{\beta}\{1-\tfrac{w}{2}(1+x)\}^{-1-\alpha-\beta-r}G\left(\tfrac{x-\frac{w}{2}(1+x)}{1-\frac{w}{2}(1+x)},\tfrac{vw}{(1-w)}\right)\\ &=\textstyle\sum_{n=0}^{\infty}w^ng_n(v)P_{n+r}^{(\alpha,\beta-n)}(x)\end{aligned}$$

where

$$g_n(v)=\sum_{p=0}^{n}a_p\begin{pmatrix}n+r\\p+r\end{pmatrix}v^p,$$

which is analogous to theorem 2.

Theorem 6. If there exists a generating relation of the form:

$$G(x,w)=\sum_{n=0}^{\infty}a_nw^nP_n^{(\alpha,\beta-n)}(x) \tag{2.7}$$

then

$$\begin{aligned}(1-w)^{\beta}\quad &\{1-\tfrac{w}{2}(1+x)\}^{-1-\alpha-\beta}G\left(\tfrac{x-\frac{w}{2}(1+x)}{1-\frac{w}{2}(1+x)},\tfrac{vw}{(1-w)}\right)\\ &=\textstyle\sum_{n=0}^{\infty}w^ng_n(v)P_n^{(\alpha,\beta-n)}(x)\end{aligned}$$

where

$$g_n(v) = \sum_{p=0}^{n} a_p \begin{pmatrix} n \\ p \end{pmatrix} v^p,$$

which is analogous to theorem 1 and is found derived in [12].

3. Observation

It is observed that though the theorem 3 has been proved by group-theoretic method, still the result stated in theorem 3 owes its existence to the following generating functions:

$$\begin{aligned}(1+t)^{\alpha-n} \quad & \{1+\tfrac{t}{2}(1-x)\}^{-1-\alpha-\beta-r} P_{n+r}^{(\alpha-n,\beta)}\left(\frac{x-\frac{t}{2}(1-x)}{1+\frac{t}{2}(1-x)}\right) \\ & = \textstyle\sum_{p=0}^{\infty} \frac{(n+r+1)_p}{p!} P_{n+r+p}^{(\alpha-n-p,\beta)}(x) t^p, \\ (1-t) \quad & -1-n-\beta-m-r P_m^{(n+r,\beta)}\left(\frac{u+t}{1-t}\right) \\ & = \textstyle\sum_{q=0}^{\infty} \frac{(1+n+r+\beta+m)_q}{q!} P_m^{(n+r+q,\beta)}(u) t^q,\end{aligned}$$

as well as to the partial quasi bilinear generating function assumed in theorem 3.

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