Proyecciones Journal of Mathematics
Vol. 42, N^o 3, pp. 663-680, June 2023.
Universidad Católica del Norte
Antofagasta - Chile

10.22199/issn.0717-6279-5261

Periodic orbits of linear flows on connected Lie groups

S. N. Stelmastchuk
Universidade Federal do Paraná, Brazil
Received : December 2021. Accepted : October 2022

Abstract

Our main goal is to study the periodic orbits of linear flows on a real, connected Lie group. Since each linear flow φ_t has a derivation associated $\mathcal{D}$, we show that the existence of periodic orbits of φ_t is based on the eigenvalues of the derivation $\mathcal{D}$. From this, we study periodic orbits of a linear flow on noncompact, semisimple Lie groups, and we work with periodic orbits of a linear flow on a connected, simply connected, solvable Lie groups of dimension 2 or 3.

AMS 2020 subject classification: *22E46, 34A05, 34D20, 37B99, 37C27, 37C10, 22E99.*

Keywords: *periodic orbits, linear flows, connected Lie groups.*

1. Introduction

The study of periodic orbits is a recurrent issue. In sprayed literature, many theories and techniques are presented to find and classify periodic orbits. For a fuller treatment the reader we refer the reader to [4], [5], and [8] among others. In [11], the author study the periodic orbits of a linear flow on a real, connected, compact, semisimple Lie group. Here, our wish is to study periodic orbits of a linear flow in a real, connected Lie group.

Let G be a real, connected Lie group and $\mathbf{g}$ its Lie algebra. We recall that a vector field $\mathcal{X}$ on G is called linear if its flow $(\varphi_t)_{t\in\mathbf{R}}$ is a family of automorphism of the Lie group G. Namely, the linear flow φ_t is the solution of dynamical system

$$\dot{g} = \mathcal{X}(g),\ g \in G. \tag{1.1}$$

Furthermore, the linear vector field $\mathcal{X}$ yields the derivation $\mathcal{D} = -ad(\mathcal{X})$ of the Lie algebra $\mathbf{g}$.

In our work, we begin by seeing that the periodicity of the flow $e^{t\mathcal{D}}$ is equivalent to a condition over eigenvalues of derivation $\mathcal{D}$. After, we show that periodicity of the flow $e^{t\mathcal{D}}$ implies one of the linear flow φ_t. A little more, assuming that G is an exponential Lie group we show that the latter is an equivalence. Also, if we adopt an inner derivation on a simple connected, triangular Lie group, we show that there are not periodic orbits of the linear flow φ_t.

When G is noncompact, semisimple Lie group, we have an Iwasawa's decomposition $\mathbf{g} = k \oplus a \oplus n$. Thus, given an inner derivation $\mathcal{D} = -ad(H + X)$ with $H \in a$ and $X \in n$, we show that there are not periodic orbits $\varphi_t(x)$ if $x \in AN$. Furthermore, we show a necessary condition to exist periodic orbits in $Sl(2, \mathbf{R})$.

Finally, we study periodic orbits of a linear flow on a real, simply connected, connected, solvable Lie groups of dimension 2 or 3. In fact, we present the derivations of these Lie algebra and apply our results to classify the periodic orbits of the linear flow in these Lie groups.

This paper is organized as follows. Section 2 briefly reviews the notions of linear vector fields. Section 3 works with periodic orbits of the linear flows on a real, connected Lie group. Section 4 studies periodic orbits on noncompact, semisimple Lie groups. Section 5 works with periodic orbits of a linear flow φ_t on a connected, simply connected, solvable Lie group of dimension 2 or 3.

2. Linear vector fields

In this section we recall some basic facts about linear vector fields. For a fuller treatment we refer to [1], [3], and [6]. Let G be a connected Lie group, and let us denote by $\mathbf{g}$ its Lie algebra. A vector field $\mathcal{X}$ on G is called linear if its flow $(\varphi_t)_{t\in\mathbf{R}}$ is a family of automorphisms of the Lie group G. For a linear vector field $\mathcal{X}$ it is possible to associated a derivation of Lie algebra $\mathbf{g}$:

$$\mathcal{D}(Y) = -[\mathcal{X}, Y],\ Y \in \mathbf{g}.$$

For the convenience of the reader we resume some facts about a linear vector field $\mathcal{X}$ and its flow φ_t. The proof of these facts can be found in [6].

Proposition 2.1. *Let $\mathcal{X}$ be a linear vector field and φ_t its flow. The following assertions are equivalent:*

(i) the linear flow φ_t is an automorphism of Lie groups for each t;

(ii) for $g, h \in G$ it follows that $\mathcal{X}$ is linear iff $\mathcal{X}(gh) = R_{h*}\mathcal{X}(g) + L_{g*}\mathcal{X}(h)$;

(iii) at identity e, we have $(d\varphi_t)_e = e^{t\mathcal{D}}$ for all $t \in \mathbf{R}$.

It is known that $G = \mathbf{R}^n$ is a Euclidean Lie Group. For any $n \times n$ matrix A it is true that $\mathcal{X} = A$ is a linear vector field. Furthermore, $\mathcal{D}_x(b) = -[Ax, b] = Ab$. In this sense, we can view the dynamical system

$$\dot{g} = \mathcal{X}(g), \quad g \in G,$$

as a generalization of dynamical system on $\mathbf{R}^n$ given by

$$\dot{x} = Ax, \quad x \in \mathbf{R}^n.$$

Let $\mathcal{X}$ be a linear vector field on G. In a natural way, we define a derivation $\mathcal{D} = -ad(\mathcal{X})$ on the Lie algebra $\mathbf{g}$ associated to $\mathcal{X}$. On contrary, it is not true that a derivation yields a linear vector field if G is only connected. However, if G is connected and simply connected, then there is a one-to-one relation between derivations and linear vector fields (see [10] for more details).

3. Periodic orbits

In this section we assume that G is a connected Lie group. Let $\mathcal{X}$ be a vector linear vector field on G. Let us denote by $\mathcal{D}$ the associated derivation of $\mathcal{X}$ and by φ_t the linear flow associated of $\mathcal{X}$. We going to show that if the flow $e^{t\mathcal{D}}$ is periodic then the linear flow φ_t is too. Furthermore, we going to present a condition for the flow $e^{t\mathcal{D}}$ to be periodic.

Theorem 3.1. *Let G be a connected Lie group. Assume that $\mathcal{X}$ is a linear vector field on G, and denote by $\mathcal{D}$ and φ_t their derivation and flow, respectively. If the flow $e^{t\mathcal{D}}$ is periodic with period $T > 0$, then every orbit $\varphi_t(g)$ is too if $g \in G$ is not fixed point.*

Proof. Suppose that the flow $e^{t\mathcal{D}}$ is periodic with period $T > 0$. It means that $e^{(t+T)\mathcal{D}} = e^{t\mathcal{D}}$ for any $t \in \mathbf{R}$. Since G is connected, for $g \in G$ we have $g = \exp(Y_1)\exp(Y_2)\ldots\exp(Y_k)$ for $Y_1, Y_2, \ldots, Y_k \in \mathbf{g}$. Thus, using the fact that φ_t is automorphism we have

$$\begin{aligned}\varphi_{t+T}(g) &= \varphi_{t+T}(\exp(Y_1)\exp(Y_2)\ldots\exp(Y_k)) \\ &= \varphi_{t+T}(\exp(Y_1))\varphi_{t+T}(\exp(Y_2))\ldots\varphi_{t+T}(\exp(Y_k)) \\ &= \exp(e^{(t+T)\mathcal{D}}(Y_1))\exp(e^{(t+T)\mathcal{D}}(Y_2))\ldots\exp(e^{(t+T)\mathcal{D}}(Y_k)) \\ &= \exp(e^{t\mathcal{D}}(Y_1))\exp(e^{t\mathcal{D}}(Y_2))\ldots\exp(e^{t\mathcal{D}}(Y_k)) = \varphi_t(g),\end{aligned}$$

which proves the theorem. □

We can improve the result above if we ask that G be an exponential Lie group. We remember that a Lie group G is said to be exponential if the exponential mapping $\exp : \mathbf{g} \Rightarrow G$ is a diffeomorphism. As example of exponential Lie groups we have any simply-connected nilpotent Lie group.

Theorem 3.2. *Let G be an exponential Lie group. Let $\mathcal{X}$ be a linear vector field on G and denote by $\mathcal{D}$ and φ_t their derivation and flow, respectively. The flow $e^{t\mathcal{D}}$ is periodic with period $T > 0$ if and only if every orbit $\varphi_t(g)$ is too if $g \in G$ is not fixed point.*

Proof. We begin by observing that every $g \in G$ can be written as $g = \exp(X)$ for some $X \in \mathbf{g}$ because G is exponential Lie group. Then

$$\varphi_t(g) = \varphi_t(\exp(X)) = \exp(e^{t\mathcal{D}}X).$$

Suppose that the flow $e^{t\mathcal{D}}$ is periodic with period $T > 0$, that is, $e^{(t+T)\mathcal{D}} = e^{t\mathcal{D}}$ for all $t \in \mathbf{R}$. Then

$$\varphi_{t+T}(g) = \exp(e^{(t+T)\mathcal{D}}X) = \exp(e^{t\mathcal{D}}X) = \varphi_t(g),$$

which means that $\varphi_t(g)$ is periodic orbit with period $T > 0$ since $g \in G$ is not fixed point.

On contrary, suppose that $\varphi_t(g)$ is a periodic with period $T > 0$, that is, $\varphi_{(t+T)}(g) = \varphi_t(g)$ for any $t \in \mathbf{R}$. Thus

$$\exp(e^{(t+T)\mathcal{D}}X) = \exp(e^{t\mathcal{D}}X) \Rightarrow e^{(t+T)\mathcal{D}}X = e^{t\mathcal{D}}X,\ \forall X \in \mathbf{g}.$$

It implies that $e^{(t+T)\mathcal{D}} = e^{t\mathcal{D}}$, which means that $e^{t\mathcal{D}}$ is periodic orbit with period $T > 0$. □

Our next step is to find some conditions to the flow $e^{t\mathcal{D}}$ be periodic. Before we need to introduce some concepts. Following [5], if for an eigenvalue μ all complex Jordan blocks are one-dimensional, i.e., a complete set of eigenvectors exists, it is called semisimple. Equivalently, the corresponding real Jordan blocks are one-dimensional if μ is real, and two-dimensional if μ and $\bar{\mu} \in \mathbf{C} \setminus \mathbf{R}$.

Proposition 3.3. *Let* $\mathbf{g}$ *be a Lie algebra. If* $\mathcal{D}$ *is a derivation of* $\mathbf{g}$*, then the following conditions are equivalent:*

i) *the flow* $e^{t\mathcal{D}}$ *is periodic;*

ii) *the eigenvalues of the derivation* $\mathcal{D}$ *are semisimple and they are null or* $\pm\alpha_1 i, \dots, \pm\alpha_r i$ *with rational quotient* α_i/α_j *for* $i, j = 1, \dots r$.

Proof. We first suppose that the flow $e^{t\mathcal{D}}$ is periodic. It means that there exists $T > 0$ such that

$$e^{(t+T)\mathcal{D}} = e^{t\mathcal{D}}.$$

A simple calculus shows that $e^{T\mathcal{D}} = Id$. If J is the Jordan form of $\mathcal{D}$, then $e^{TJ} = Id$.

We break the proof in two steps:

i) real eigenvalues: Let μ be a real eigenvalue of derivation $\mathcal{D}$. Let us denote by J_μ the m-dimensional Jordan block of μ. We thus get $e^{TJ_\mu} = I_m$. Consider $m > 1$. From Jordan block J_μ we see that $e^{T\mu}(T) = 0$. It implies that $T = 0$, a contradiction. Therefore $m = 1$. It means that μ is a semisimple eigenvalue. Consequently, $e^{TJ_\mu} = I_1$, which gives $e^{T\mu} = 1$. We

thus get $T\mu = 0$. Since $T > 0$, it follows that $\mu = 0$. It shows that unique real eigenvalue of derivation $\mathcal{D}$ is 0.
ii) complex eigenvalues: suppose that $\mu = \alpha \pm i\beta$ are conjugate, complex eigenvalues of derivation $\mathcal{D}$. Write

$$R = R(t) = \begin{pmatrix} \cos(t\beta) & -\sin(t\beta) \\ \sin(t\beta) & \cos(t\beta) \end{pmatrix}.$$

Let J_μ denote the 2m-dimensional Jordan block of μ. Suppose that $m > 1$. From $e^{TJ_\mu} = I_{2m}$ we see that

$$e^{\alpha T} \cdot T \cdot R = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

A simple account shows that $T = 0$, a contradiction. Therefore $m = 1$. It means that μ is semisimple. Hence the Jordan block J_μ is two dimensional. It gives $e^{TJ_\mu} = I_2$. From this equality we conclude that

$$e^{\alpha T} \cos(\beta T) = 1 \quad \text{and} \quad e^{\alpha T} \sin(\beta T) = 0.$$

The second equality implies that $\sin(\beta T) = 0$. It follows that $\beta T = n\pi$ for any $n \in Z$. Substituting this in the first equation gives

$$e^{\alpha T} \cos(n\pi) = 1.$$

Because $e^{\alpha T} > 0$, it follows that $n = 2m$ for some $m \in Z$. Hence $e^{\alpha T} = 1$. So $\alpha T = 0$. Since $T > 0$, we have $\alpha = 0$. It means that complex eigenvalues of derivation $\mathcal{D}$ are of the form $\mu = \pm\beta i$.

Now we proved that nonnull complex eigenvalues $\pm\alpha i$ of $\mathcal{D}$ yield rational quotient.

Suppose that there are non null complex eigenvalues $\pm\alpha_i i$, $i = 1, \ldots, r$. By proved above, its real Jordan blocks are

$$\begin{pmatrix} \cos(t\alpha_i) & -\sin(t\alpha_i) \\ \sin(t\alpha_i) & \cos(t\alpha_i) \end{pmatrix}, \ i = 1, \ldots r.$$

As $e^{TJ} = Id$ we have $\alpha_i \cdot T = p_i \cdot 2\pi$ for some $p_i \in Z$, $i = 1, \ldots, r$. It means for any $i, j = 1, \ldots, r$ that $\alpha_i/\alpha_j = p_i/p_j$ is a rational number.

Reciprocally, assume that the eigenvalues of $\mathcal{D}$ are semsimple and they are 0 or $\pm\alpha_1 i, \ldots, \pm\alpha_r i$ with $\alpha_i \neq 0$, $i = 1, \ldots, r$, and α_i/α_j is rational for $i, j = 1, \ldots n$. Trivially the solution is constant for the eigenvalue 0. We thus work with the eigenvalues $\pm\alpha_i i$ with $\alpha_i \neq 0$. By assumption, $\pm\alpha_i i$ is semisimple eigenvalue for $i = 1, \ldots, r$. It implies that every real Jordan block has dimension two and the solution applied at this block gives the following matrix

$$\begin{pmatrix} \cos(t\alpha_i) & -\sin(t\alpha_i) \\ \sin(t\alpha_i) & \cos(t\alpha_i) \end{pmatrix}.$$

On the other hand, we know that there exists $p_{ij}, q_{ij} \in Z$ with $q_{ij} > 0$ such that $\alpha_i/\alpha_j = p_{ij}/q_{ij}$ for $i, j = 1, \ldots r$. In particular, we can written $\alpha_i = (p_{i1}/q_{i1})\alpha_1$ for $i = 2, \ldots, r$. Supposing that $\alpha_1 > 0$ it is sufficient to take $T = q_{21}q_{31} \ldots q_{r1}(2\pi/\alpha_1)$ to get satisfies $e^{TJ} = Id$, where J is the Jordan form. This gives $e^{T\mathcal{D}} = Id$. We thus conclude that $e^{t\mathcal{D}}$ is periodic with period $T > 0$. □

We now presents a necessary condition to a linear flow to have periodic orbits.

Theorem 3.4. *Let G be a connected Lie group. Suppose that $\mathcal{X}$ is a linear vector field on G, and denote by $\mathcal{D}$ and φ_t their derivation and flow, respectively. If the eigenvalues of the derivation $\mathcal{D}$ are semisimple and they are null or $\pm\alpha_1 i, \ldots, \pm\alpha_r i$ with rational quotient α_i/α_j for $i, j = 1, \ldots r$, then there exist periodic orbits for the linear flow φ_t.*

Proof. It follows directly from Proposition 3.3 and Theorem 3.1. □

In the sequel, we use the results above to study periodic orbits on invariant flow. Let X be a right invariant vector field on G. Define a vector field by $\mathcal{X} = X + I_*X$, where I_*X is the left invariant vector field associated to X. Here I_* is the differential of inverse map $i(g) = g^{-1}$ (more details is founded in [10]). It is possible to show that $\mathcal{X}$ is linear and its associated derivation is given by $\mathcal{D} = -ad(\mathcal{X}) = -ad(X)$. Furthermore, the differential equation (1.1) is written as

$$\dot{g} = X(g) + (I_*X)(g), \ g \in G.$$

It is possible to show that linear flow φ_t is solution of (1.1) if and only if $\varphi_t(g) \cdot \exp(tX)$ is solution of $\dot{g} = X(g)$.

Proposition 3.5. *Let G be a connected Lie group and X be a right invariant vector field on G. If there exists a periodic orbit for the right invariant flow $\exp(tX)$, then the eigenvalues of the derivation $\mathcal{D} = -ad(X)$ are semisimple and they are null or $\pm\alpha_1 i, \ldots, \pm\alpha_r i$ with rational quotient α_i/α_j for $i, j = 1, \ldots r$.*

Proof. Assume that there exists a $g \in G$ such that $\exp(tX)g$ is periodic with period $T > 0$. It is equivalent to $\exp(tX)$ be periodic with period $T > 0$. From this we deduce that

$$\begin{aligned} \exp((t+T)X) = \exp(tX) \ &\Rightarrow \ Ad(\exp(-(t+T)X)) = Ad(\exp(-tX)) \\ &\Rightarrow \ e^{(t+T)\mathcal{D}} = e^{t\mathcal{D}}, \end{aligned}$$

which means that the flow $e^{t\mathcal{D}}$ is periodic. By Proposition 3.3, the eigenvalues of the derivation $\mathcal{D}$ are semisimple and they are 0 or $\pm\alpha_1 i, \ldots, \pm\alpha_r i$ where $\alpha_i \neq 0$, $i = 1, \ldots, r$, with α_i/α_j a rational number for $i, j = 1, \ldots r$. □

Next, we show a class of linear and invariant flow that does not have periodic obits.

Theorem 3.6. *Let G be a simply connected, triangular Lie group. Let X be an invariant vector field and $\mathcal{D} = -ad(X)$ its inner derivation. Then:*

1. *there is not periodic orbits for the linear flow φ_t associated to derivation $\mathcal{D}$.*
2. *there is not periodic orbits for the invariant flow $\exp(tX)$.*

Proof. We begin by remember that G is an exponential Lie group. Furthermore, by definition of triangular Lie group, for any $X \in \mathbf{g}$ we have that eigenvalues of $ad(X)$ are real. It implies that $e^{t\mathcal{D}}$ is not periodic from Proposition 3.3. Respectively, Theorem 3.2 and Theorem 3.5 assures that the linear flow φ_t and the invariant flow $\exp(tX)$ do not have periodic orbits. □

The next example show that a case that there exists periodic orbit to linear flow and the condition ii) of Proposition 3.3 fails.

Example 3.1. *Let $SO(3)$ be the orthogonal Lie group of dimension 3. The canonical basis of Lie algebra $\mathbf{so}_3$ is given by*

$$e_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It follows directly that $[e_1, e_2] = e_3$, $[e_3, e_1] = e_2$, and $[e_2, e_3] = e_1$.

Let α an irrational real number. Consider in the Lie group $G = SO(3) \times SO(3)$ the right-invariant vector field $X = (e_1, \alpha e_1)$. The derivation associated to X, $\mathcal{D} = -ad((e_1, \alpha e_1))$, has as eigenvalue 0, $\pm i$, and $\pm \alpha i$. periodic by Proposition 3.3.

Now the linear flow φ_t is given by

$$\varphi_t(g_1, g_2) = (\exp(te_1)g_1 \exp(-te_1), \exp(t\alpha e_1)g_1 \exp(-t\alpha e_1)).$$

It is easily to see that $\varphi_t(g_1, e)$ and $\varphi_t(e, g_2)$ are periodic orbits for any (g_1, e) and (e, g_2) in G. it is not sufficient to assures that e^{tD} is periodic to conclude that φ_t has periodic orbit.

4. Semisimple Lie group

In [11], the author study periodic orbits on compact, semisimple Lie groups. We now presents some results about the noncompact case. Let G be a noncompact, semisimple Lie group. From Iwasawa's decomposition there exists three Lie subalgebra k, a, and n such that $\mathbf{g} = k \oplus a \oplus n$. Let us denote by $G = K \cdot (AN)$ the global decomposition of G following the Iwasawa's decomposition. Our first purpose is to study periodic orbits of the linear flow φ_t acting only in AN.

Proposition 4.1. *Under assumptions above, if $H \in a, X \in n$ and if $\mathcal{D} = -ad(H + X)$ is a inner derivation, then for any $x \in AN$ that is not fixed point of φ_t we have that the orbit $\varphi_t(x)$ is not periodic.*

Proof. We first write for any $x \in AN$ the linear flow as

$$\varphi_t(x) = \exp(t(H + X)) \cdot x \cdot \exp(-t(H + X)).$$

Since $H + X \in a \oplus n$, it follows that $\exp(t(H + X)) \in A \cdot N$. Therefore $\varphi_t(x) \in AN$. It means that AN is φ_t-invariant. It is well known that AN is simply connected and that the Lie subalgebra $a \oplus n$ is a triangular Lie algebra. Now from Proposition 3.5 we see that $\varphi_t(x)$ is not periodic. □

Under assumptions of Proposition above, it is direct that the invariant flow restrict to AN does not have periodic orbits from Theorem 3.6. This invariant case was first proved by Kawan, Rocio and Santana in [7].

In the general case, that is, when $g \in G = K \cdot (AN)$ and the derivation is arbitrary, some answer about periodic orbit is not known. To contribute in this question we present a result about periodic orbits on special linear group $Sl(2,\mathbf{R})$. Let us denote by $sl(2,\mathbf{R})$ the Lie algebra of $Sl(2,\mathbf{R})$. Since $Sl(2,\mathbf{R})$ is a semisimple non-compact Lie group, its Lie algebra has the following Iwasawa's decomposition:

$$sl(2,\mathbf{R}) = \left\{ \begin{pmatrix} 0 & -\alpha \\ \alpha & 0 \end{pmatrix}, \alpha \in \mathbf{R} \right\} \oplus \left\{ \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}, a \in \mathbf{R} \right\}$$

$$\oplus \left\{ \begin{pmatrix} 0 & \nu \\ 0 & 0 \end{pmatrix}, \nu \in \mathbf{R} \right\}.$$

It is clear that

$$\beta = \left\{ Y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, Z = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}$$

form a basis of $sl(2,\mathbf{R})$. Furthermore, they brackets are given by

$$[Y,H] = 2YX + 4Z,\ [Y,Z] = -H,\ [H,Z] = 2Z.$$

Let $\mathcal{X}$ be a linear vector field on $Sl(2,\mathbf{R})$ and $\mathcal{D}$ its derivation. Then there exist a right invariant vector field $X \in sl(2,\mathbf{R})$ such that $\mathcal{D} = -ad(X)$. Write X as

$$X = aY + bH + cZ,\ a,b,c \in \mathbf{R}.$$

In the basis β, the matrix of derivation is written as

$$\mathcal{D} = -ad(X) = \begin{pmatrix} 2b & -2a & 0 \\ -c & 0 & a \\ 4b & -4a+2c & -2b \end{pmatrix}.$$

From this we see that the eigenvalues of $\mathcal{D}$ are

$$\left\{0,\ -2\sqrt{-a^2+ac+b^2},\ 2\sqrt{-a^2+ac+b^2}\right\}. \tag{4.1}$$

We are in a position to show a condition to an orbit of linear flow be periodic.

Proposition 4.2. *Let $\mathcal{X}$ be a linear vector field on $Sl(2,\mathbf{R})$ and X its associated right invariant vector field. Writing $X = aY + bH + cZ$ in the basis β of $sl(2,\mathbf{R})$, we have that:*

1. *orbits of the linear flow φ_t of $\mathcal{X}$ has periodic if $a^2 > ac + b^2$.*
2. *there are not periodic orbits of the invariant flow $\exp(tX)$ if $a^2 \leq ac + b^2$.*

Proof. It is a direct application of Theorem 3.4 and Proposition 3.5 with eigenvalues (4.1). □

The meaning of the Proposition above is that to study periodic orbits of linear or invariant flows on $Sl(2,\mathbf{R})$ is necessary to consider the compact, abelian, and nilpotent parts of Iwasawa's decomposition. In other words, we need to consider derivations of the $\mathcal{D} = -ad(Y+H+X)$ with $Y \in k, H \in a$ and $X \in n$.

5. Solvable Lie groups of dimension 3

In this section we study periodic orbits of linear flow on connected, simply connected, and solvable Lie groups of 3. In the semisimple case, $SL(2,\mathbf{R})$ was studied in the section above, and the unitary group $SU(2)$ and the orthogonal group $SO(3)$ were studied in [11].

5.1. Dimension 2

Let $\mathbf{g}$ be a Lie algebra of dimension 2. It is well know that there exists two possibilities of Lie algebras: abelian and solvable.

5.1.1. Abelian case

The simply-connected, abelian Lie group of dimension 2 is $\mathbf{R}^2$, which is exponential Lie group. In this case any derivation is given by

$$\mathcal{D} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

A simple account shows that the eigenvalues are

$$\left\{ \frac{1}{2}\left(a+d-\sqrt{(a-d)^2+4bc}\right), \frac{1}{2}\left(a+d+\sqrt{(a-d)^2+4bc}\right) \right\}.$$

By Proposition 3.3 and Theorem 3.2, the linear flow associated for the derivation $\mathcal{D}$ has periodic orbits if and only if $a+d=0$ and $(a-d)^2+4bc<0$.

5.1.2. Solvable case

The solvable, connected, simply connected Lie group of dimension 2 is the affine group $Aff_0(2)$. A first fact about $Aff_0(2)$ is that it is exponential Lie group. In [6], it is showed that derivations are inner and they are given by

$$\mathcal{D} = \begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix}.$$

It is easy to see that eigenvalues are $\{0, d\}$. By Proposition 3.3 and Theorem 3.2, any linear flow in $Aff_0(2)$ do not have periodic orbit.

5.2. Dimension 3

Let G be a connected Lie group of dimension 3. In [9] it is classified all connected Lie groups of dimension 3, and in [2] we find a clear presentation of this classification. In [2], we find that for an (appropriate) ordered basis $(E1, E2, E3)$ of $\mathbf{g}$ the Lie bracket of a connected Lie group of dimension 3 are given by

$$\begin{array}{rcl} [E_1, E_2] & = & n_3 E_3 \\ [E_3, E_1] & = & aE_1 + n_2 E_2 \\ [E_2, E_3] & = & n_1 E_1 - aE_2, \end{array} \tag{5.1}$$

where a, n_1, n_2 e n_3 are given by table 1 in [2]. In each case bellow, we give values of a, n_1, n_2 and n_3.

Let $\mathcal{D}$ be a derivation on the Lie algebra $\mathbf{g}$. In the basis (E_1, E_2, E_3), we can write the derivation $\mathcal{D}$ as

$$\begin{array}{rcl} \mathcal{D}(E_1) & = & x_1 E_1 + y_1 E_2 + z_1 E_3 \\ \mathcal{D}(E_2) & = & x_2 E_1 + y_2 E_2 + z_2 E_3 \\ \mathcal{D}(E_3) & = & x_3 E_1 + y_3 E_2 + z_3 E_3 \end{array}$$

with $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in \mathbf{R}$. Applying Lie Brackets (5.1) at equalities

$$\mathcal{D}[E_i, E_j] = [\mathcal{D}(E_i), E_j] + [E_i, \mathcal{D}(E_j)], \quad \text{for } i, j = 1, 2, 3 \text{ and } i \neq j,$$

we obtain the following equation system

$$
(5.2) \qquad \begin{cases}
n_3x_3 + n_1z_1 + az_2 & = & 0 \\
n_3y_3 + n_2z_3 - az_1 & = & 0 \\
n_3z_3 - n_3x_1 - n_3y_2 & = & 0 \\
n_2x_2 + n_1y_1 - az_3 & = & 0 \\
n_2y_2 - n_2x_1 - n_2z_3 & = & 0 \\
n_2z_2 + n_3y_3 + az_1 & = & 0 \\
n_1x_1 - n_1y_2 - n_1z_3 & = & 0 \\
n_1y_1 + n_2x_2 + az_3 & = & 0 \\
-n_2x_2 + n_3x_3 - ay_2 & = & 0.
\end{cases}
$$

Thus, taking values to a, n_1, n_2 and n_3 in Table 1 in [2] we can find derivations in each class of connected, simply connected, solvable Lie groups of dimension 3.

5.2.1. Type $3\mathbf{g}_1$ (Abelian Groups)

In this class the simply connected Lie group G with Lie algebra $\mathbf{g} \cong 3\mathbf{g}_1$ is isomorphic to $\mathbf{R}^3$. It is clear that G is an exponential Lie group. Here, Lie brackets (5.1) assume values $a = n_1 = n_2 = n_3 = 0$. Trivially, by linear system (4), the matrix of any derivation is written as

$$
\mathcal{D} = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_3 & z_3 \end{pmatrix}.
$$

The characteristic polynomial of $\mathcal{D}$ is $p(\lambda) = -\lambda^3 + tr(\mathcal{D})\lambda^2 + A\lambda + \det(\mathcal{D})$ where $A = x_2y_1 - x_1y_2 + x_3z_1 - x_1z_3 + y_3z_2 - y_2z_3$. By Theorem 3.2, we want to work with derivation with one null eigenvalue and two conjugate complex eigenvalues. It is direct that $\det(\mathcal{D}) = 0$. Then $p(\lambda) = \lambda(-\lambda^2 + tr(\mathcal{D})\lambda + A)$ where $A = x_2y_1 - x_1y_2 + x_3z_1 - x_1z_3 + y_3z_2 - y_2z_3$. Now consider the polynomial $q(\lambda) = -\lambda^2 + tr(\mathcal{D})\lambda + A$. A simple account shows that the roots are $\lambda = (tr(\mathcal{D}) \pm \sqrt{tr(\mathcal{D})^2 + 4A})/2$. By Theorem 3.2 again, we want that $tr(\mathcal{D}) = 0$, whit implies that $\lambda_1 = \sqrt{A}$ and $\lambda_2 = -\sqrt{A}$. Summarizing, we have the conditions to the linear flow has periodic orbits.

Proposition 5.1. *Let φ_t be a linear flow on the simply connected with Lie algebra of type $3\mathbf{g}_1$. Then φ_t has periodic orbit if and only if $tr(\mathcal{D}) = \det(\mathcal{D}) = 0$ and $x_2y_1 - y_1x_2 + x_3z_1 - x_1z_3 + y_3z_2 - y_2z_3 < 0$.*

Now let X be a invariant vector field. Since $3\mathbf{g}_1$ is abelian, it follows that $ad(X) = 0$. Consequently, $\mathcal{D} = -ad(X) = 0$. From Proposition 3.5 we conclude that the invariant flow $\exp(tX)$ is not periodic.

5.3. Type $\mathbf{g}_{2,1} \oplus \mathbf{g}_1$

If a Lie algebra $\mathbf{g} \cong \mathbf{g}_{2,1} \oplus \mathbf{g}_1$, then the semisimple Lie group G is isomorphic to $Aff(\mathbf{R})_0 \times \mathbf{R}$. It is directed that this group is exponential Lie group. In this case, the Lie bracket (5.1) is characterized by $a = 1$, $n_1 = 1$, $n_2 = -1$, and $n_3 = 0$. From linear system (4) it follows that the matrix of any derivation is

$$\mathcal{D} = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_2 & x_1 & y_3 \\ 0 & 0 & 0 \end{pmatrix},$$

and its eigenvalues are $\{0, x_1 - x_2, x_1 + x_2\}$.

Proposition 5.2. *Any linear flow on a Lie group with Lie algebra of type $\mathbf{g}_{2,1} \oplus \mathbf{g}_1$ do not have periodic orbits.*

5.3.1. Type $\mathbf{g}_{3,1}$

The simply connected, matrix group with Lie algebra of type $\mathbf{g}_{3,1}$ is isomorphic to Heisenberg group H_3, which is trivially an exponential Lie group. In this case, we adopt $a = 0$, $n_1 = 1$, $n_2 = 0$, and $n_3 = 0$. From linear system (4) we see that the matrix of any derivation is written as

$$\mathcal{D} = \begin{pmatrix} y_2 + z_3 & x_2 & x_3 \\ 0 & y_2 & y_3 \\ 0 & z_2 & z_3 \end{pmatrix}.$$

The eigenvalues of derivation $\mathcal{D}$ are

$$\begin{aligned} \{y_2 + z_3, \quad & \tfrac{1}{2}\left(y_2 + z_3 - \sqrt{(y_2 - z_3)^2 + 4x_3 z_2}\right), \\ & \tfrac{1}{2}\left(y_2 + z_3 + \sqrt{(y_2 - z_3)^2 + 4x_3 z_2}\right)\right\}. \end{aligned}$$

Proposition 5.3. *Let φ_t be a linear flow on a simply connected Lie group with Lie algebra of type $\mathbf{g}_{3,1}$. Then φ_t has periodic orbit if and only if $y_2 + z_3 = 0$ and and $(y_2 - z_3)^2 + 4x_3 z_2 < 0$.*

5.4. Type $\mathbf{g}_{3,2}$

The simply connected Lie group G with Lie algebra $\mathbf{g} \cong \mathbf{g}_{3,2}$ is isomorphic to $G_{3,2}$, which is an exponential Lie group (see for instance [9]). The Lie bracket is given by $a = 1$, $n_1 = 1$, $n_2 = 0$, and $n_3 = 0$. It follows, by linear system (4), that the matrix of any derivation is written as

$$\mathcal{D} = \begin{pmatrix} 0 & x_2 & x_3 \\ 0 & 0 & y_3 \\ 0 & 0 & 0 \end{pmatrix},$$

and its eigenvalues are $\{0, 0, 0\}$.

Proposition 5.4. *Let φ_t be a linear flow on a simply connected Lie group with Lie algebra of type $\mathbf{g}_{3,2}$. Then φ_t does not have periodic orbits.*

5.5. Type $\mathbf{g}_{3,3}$

The simply connected Lie group G with Lie algebra of $\mathbf{g} \cong \mathbf{g}_{3,3}$ is isomorphic to $G_{3,3}$ and its Lie bracket is given by $a = 1$, $n_1 = 0$, $n_2 = 0$, and $n_3 = 0$. Furthermore, $G_{3,3}$ is an exponential Lie Group (see for instance [9]). From linear system (4) we deduce that the matrix of any derivation is given by

$$\mathcal{D} = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 0 & 0 & 0 \end{pmatrix},$$

and its eigenvalues are

$$\left\{0, \frac{1}{2}\left(x_1 + y_2 - \sqrt{(x_2 - y_2)^2 + 4x_2 y_1}\right), \frac{1}{2}\left(x_1 + y_2 + \sqrt{(x_2 - y_2)^2 + 4x_2 y_1}\right)\right\}.$$

Proposition 5.5. *Let φ_t be a linear flow on a simply connected Lie group with Lie algebra of type $\mathbf{g}_{3,3}$. Then φ_t has periodic orbit if and only if $x_1 + y_2 = 0$ and $(x_2 - y_2)^2 + 4x_2 y_1 < 0$.*

5.6. Type $\mathbf{g}_{3,4}^0$

In this class, any simply connected Lie group G with Lie algebra $\mathbf{g} \cong \mathbf{g}_{3,4}^0$ is isomorphic to $SE(1,1)$. Furthermore, Lie bracket is obtained by $a = 0$, $n_1 = 1$, $n_2 = -1$, and $n_3 = 0$, and it is an exponential Lie group (see for instance [9]). From linear system (4) we see that the matrix of any derivation is

$$\mathcal{D} = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_2 & x_1 & y_3 \\ 0 & 0 & 0 \end{pmatrix},$$

and its eigenvalues are $\{0, x_1 - x_2, x_1 + x_2\}$.

Proposition 5.6. *Any linear flow on a Lie group with Lie algebra of type* $\mathbf{g}^0_{3,4}$ *do not have periodic orbits.*

5.7. Type $\mathbf{g}^a_{3,4}$

Here, we have a family of Lie algebra of type $\mathbf{g}^a_{3,4}$ given by conditions $a > 0$ and $a \neq 1$, $n_1 = 1$, $n_2 = -1$, and $n_3 = 0$. Furthermore, a simply connected Lie group G with Lie algebra $\mathbf{g}^a_{3,4}$ is isomorphic to $G^a_{3,4}$, which is an exponential Lie group (see for instance [9]). By linear system (4), the matrix of any derivation is written as

$$\mathcal{D} = \begin{pmatrix} -y_2 & ay_2 & x_3 \\ ay_2 & y_2 & y_3 \\ 0 & 0 & 0 \end{pmatrix},$$

which yields as eigenvalues $\left\{0, -\sqrt{(1+a)y_2^2}, \sqrt{(1+a)y_2^2}\right\}$.

Proposition 5.7. *For $a > 0$ and $a \neq 1$, any linear flow on a simply connected Lie group with Lie algebra of type* $\mathbf{g}^a_{3,4}$ *do not have periodic orbits.*

5.8. Type $\mathbf{g}^a_{3,5}$

A family of Lie algebra $\mathbf{g}^a_{3,5}$ is characterize by $a > 0$, $n_1 = 1$, $n_2 = 1$, and $n_3 = 0$. In this case, simply connected Lie groups with Lie algebra are isomorphic to $G^a_{3,5}$, which is an exponential Lie group (see for instance [9]). Solving linear system (4) we can write the matrix of any derivation as

$$\mathcal{D} = \begin{pmatrix} y_2 & -ay_2 & x_3 \\ ay_2 & -ay_2 & y_3 \\ 0 & 0 & 0 \end{pmatrix},$$

which yields as eigenvalues

$$\begin{aligned} \{0, \quad & \tfrac{1}{2}\left((-a-1)y_2 - \sqrt{(-5a+1)y_2^2}\right), \\ & \tfrac{1}{2}\left((-a-1)y_2 + \sqrt{(-5a+1)y_2^2}\right)\}. \end{aligned}$$

By Proposition 3.3, a first condition to for the flow $e^{t\mathcal{D}}$ to be periodic is that $(-a-1)y_2 = 0$. Since $a > 0$, it implies that $-a-1 \neq 0$. We must have $y_2 = 0$, which implies that eigenvalues are null.

Proposition 5.8. *i) A linear flow on a simply connected Lie group with Lie algebra of type $\mathbf{g}_{3,5}^{a}$ for some $a > 0$ does not have periodic orbits.*
ii) Any invariant flow on a simply connected Lie group with Lie algebra of type $\mathbf{g}_{3,5}^{a}$ for some $a > 0$ does not have periodic orbits

References

[1] V. Ayala and J. Tirao, "Linear control systems on Lie groups and controllability", *Proceedings of Symposia in Pure Mathematics,* vol. 64, pp. 47-64, 1999.

[2] R. Biggs and C. Remsing, "Control systems on three-dimensional Lie groups: equivalence and controllability", *Journal of Dynamical and Control Systems,* vol. 20, 2014. doi: 10.1007/s10883-014-9212-0

[3] F. Cardetti and D. Mittenhuber, "Local controllability for linear control systems on Lie groups", *Journal of Dynamical and Control Systems*, vol. 11, no. 3, 353-373, 2005. doi: 10.1007/s10883-005-6584-1

[4] C. Chicone, *Ordinary differential equations with applications*. New York: Springer, 2006.

[5] F. Colonius and W. Kliemman, *Dynamical System and Linear Algebra*. Providence: AMS, 2014.

[6] Ph. Jouan, "Equivalence of Control Systems with Linear Systems on Lie Groups and Homogeneous Spaces", *Journal of Dynamics and Control Systems*, vol. 17, pp. 591-616, 2011.

[7] C. Kawan, O. G. Rocio and A.J. Santana, "On topological conjugacy of left invariant flows on semisimple and affine Lie groups", *Proyecciones (Antofagasta),* vol. 30, no. 2, pp. 175-188, 2011. doi: 10.4067/s0716-09172011000200004

[8] C. Robinson, *Dynamical Systems. Stability, Symbolic Dynamics, and Chaos.* London: CRC Press, 1999.

[9] A.L Onishchik and E.B. Vinberg, *Lie Groups and Lie Algebras III- Structure of Lie Groups and Lie Algebras.* Berlin: Springer, 1990.

[10] L.B. San Martin, *Grupos de Lie*. Editora da Unicamp, Campinas, 2017.

[11] S. N. Stelmastchuk, “Linear flows on compact, semisimple Lie groups: stability and periodic orbits”, *Electronic Journal of Qualitative Theory of Differential Equations,* no. 84, pp. 1-12, 2021. doi: 10.14232/ejqtde.2021.1.84

S. N. Stelmastchuk
Universidade Federal do Paraná
Jandaia do Sul,
Brazil
e-mail: simnaos@gmail.com