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Ulam type stability of second-order linear differential equations with constant coefficients having damping term by using the Aboodh transform

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Abstract

The main aim of this paper is to investigate various types of Ulam stability and Mittag-Leffler stability of linear differential equations of second order with constant coefficients having damping term using the Aboodh transform method. We also obtain the Hyers-Ulam stability constants of these differential equations using the Aboodh transform and some examples to illustrate our main results are given.

Subjclass [2010]: *34K20, 26D10, 44A10, 39B82, 34A40, 39A30.*

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1. Introduction

The concept of Ulam type stability began with a question "Give conditions in order for a linear mapping near an approximately linear mapping to exist." in Ulam's speech [1] at the University of Wisconsin in 1940. In the following year, Hyers [2] gave the first positive answer to Ulam's question in Banach spaces by proving the stability of the additive functional equation. Since then, a number of variations and generalizations of this result have been used for many types of differential equations, (see [3, 4, 5, 6, 7] and references therein).

It seems that Obłoza is the first author who has investigated the Hyers–Ulam stability of first order linear differential equations, (see [40, 41] and references therein). Thereafter, Alsina and Ger [31] published their paper, which handles the Hyers–Ulam stability of the linear differential equation $y'(t) = y(t)$.

Those previous results were extended to the Hyers–Ulam stability of linear differential equations of first order and higher order with constant coefficients in [23, 38, 46, 47] and in [8, 39], respectively. Furthermore, Jung has also proved the Hyers–Ulam stability of linear differential equations (see [34, 36]). Rus investigated the Hyers–Ulam stability of differential and integral equations using the Gronwall lemma and the technique of weakly Picard operators (see [44, 45]). Recently, the Hyers–Ulam stability problems of linear differential equations of first order and second order with constant coefficients were studied by using the method of integral factors (see [37, 48]). The results given in [35, 37, 38] have been generalized by Cimpean and Popa [32] and by Popa and Raşa [42, 43] for the linear differential equations of nth order with constant coefficients.

The result of Hyers-Ulam stability for second-order linear differential equations has been generalized by many researchers. For example, Jung [35] investigated the Hyers-Ulam stability of the differential equation of the form

$$t^2y''(t) + \alpha ty'(t) + \beta y(t) = 0.$$

Many researchers considered the Hyers-Ulam stability of linear differential equations of second order of the form

$$x''(t) + \alpha x'(t) + \beta x(t) = 0 \tag{1.1}$$

and

$$x''(t) + \alpha x'(t) + \beta x(t) = f(t) \tag{1.2}$$

where $\alpha, \beta \in \mathbf{R}$. For example, Li and Shen [37] considered the two roots of the characteristic equation $r^2+\alpha r+\beta = 0$ for the Hyers-Ulam stability (1.1) and (1.2). Also Xue [50] studied the Ulam stability of the same equation and its non-homogeneous form regardless of the characteristic equation of the same equations. In 2017, Nejati et al. [51] improved a result of Li and Shen.

Recently, Razaei et al. [49] and Jung [10] studied the Hyers-Ulam stability problems by using the Laplace transform method of linear differential equations with constant coefficients of the form

$$x^{(n)}(t) + \sum_{k=0}^{n-1} \alpha_k x^{(k)}(t) = f(t), \tag{1.3}$$

where α_k are scalars and $x(t)$ is an n times continuously differentiable function and of exponential order (see also [27]). Murali et al. [24] investigate the Hyers–Ulam stability of the linear differential equations by using the Fourier transform method (see also [24, 25, 33]). Mohanapriya et al. [54] studied the Mittag-Leffler-Ulam stability for second order differential equations with constant coefficient by using the Fourier transform method. In 2014, Alqifiary and Jung [10] proved the Hyers-Ulam stability of linear differential equation (1.3).

In recent years, many authors are studying the Hyers-Ulam stability of differential equations, and a number of mathematicians are paying attention to new results of the Hyers-Ulam stability of differential equations (see [9, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23] and references therein). Recently, Murali et al. [24] and Aruldass et al. [52] have investigated the Hyers-Ulam stability of the linear differential equation using Fourier transform and Mahgoub transform methods respectively (see also [25, 26, 29, 33] and references therein).

Kalvandi et al. [53] discussed Mittag-Leffler–Hyers–Ulam stability for equations (1.1). Mohanapria at al. [54] discussed Mittag-Leffler–Hyers–Ulam stability for equations (1.3) under the condition $x(t) \to 0$ for $|t| \to \infty$ by using the Fourier transform.

Very recently, Murali et al. [55] have established the Hyers–Ulam stability and the Mittag-Leffler–Hyers–Ulam stability of the following second order linear differential equations:

$$z''(t) + \gamma^2 z(t) = 0$$

and

$$z''(t) + \gamma^2 z(t) = q(t)$$

with the help of the Aboodh transform under the condition $u^2 + \gamma^2 = (u-l)(u-m)$ with $l+m=0$ and $lm=\gamma^2$.

However, the Hyers-Ulam stability of second order differential equation having damping term has not been fully reported with the help of Aboodh transform method.

Motivated by above results, the aim of this paper is to more efficiently prove the Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam stability of the second-order linear differential equations of the forms (1.1) and (1.2) by the help of the Aboodh transform method, where α and β are given real constants and $x(t) \in C^2(I)$ is a continuously twice differentiable function of exponential order. Where $I = [a,b]$, $-\infty < a < b < \infty$.

2. Preliminaries and basic notations

In this section, we introduce some standard notations and definitions which will be useful to prove our main results. Throughout this paper, $\mathcal{F}$ denotes the real field $\mathbf{R}$ or the complex field $\mathbf{C}$.

A function $f : [0,\infty) \to \mathcal{F}$ is of exponential order if there exist constants $M, \alpha \in \mathbf{R}$ such that $|f(t)| \le Me^{\alpha t}$ for all $t \ge 0$.

Let us define

$$B = \left\{ f(t) : \exists M, \quad k_1, k_2 > 0, \quad |f(t)| < M\ e^{-ut},\ k_1 < t < k_2 \right\}.$$

For a given function in the set B, the constant M must be finite number; k_1, k_2 may be finite or infinite, see [27].

Integral transforms are the most useful techniques of the mathematics which are used to find the solutions of differential equations, partial differential equations, integro-differential equations, partial integro-differential equations, delay differential equations, fractional differential equations and population growth.

The Aboodh transform [27, 28] is a new integral transform denoted by the operator $\mathcal{A}(.)$ and defined for a function $f \in B$ of exponential order as follows

$$\mathcal{A}\{f(t)\} = \frac{1}{u}\int_0^\infty e^{-ut} f(t)\ dt = F(u), \qquad t \ge 0,$$

provided that the integral exists for some u, where $u \in (k_1, k_2)$. Here $\mathcal{A}$ is called the Aboodh (integral) transform operator and if $\mathcal{A}\{f(t)\} = F(u)$, then $f(t)$ is called the inverse Aboodh (integral) transform of $F(u)$ and is denoted as $f(t) = \mathcal{A}^{-1}\{F(u)\}$, where $\mathcal{A}^{-1}$ is the inverse Aboodh transform operator [30].

For a given function $f(t)$ in the set B, M must be a finite number, k_1 and k_2 may be finite or infinite [27]. The sufficient conditions for the existence of Aboodh transform are that $f(t)$ for $t \geq 0$ be piecewise continuous and of exponential order, otherwise Aboodh transform may or may not exist.

The Aboodh transformation has linearity properties: If c_1 and c_2 are any constants and $f, g \in B$, then

$$\mathcal{A}\{c_1 f(t) + c_2 g(t)\} = c_1 \mathcal{A}\{cf(t)\} + c_2 \mathcal{A}\{g(t\}).$$

For given functions $f(t)$, $g(t)$ the convolution of these two functions denoted by $f(t) * g(t)$ and is defined by

$$f(t) * g(t) = (f * g)(t) = \int_0^t f(s)g(t-s)ds.$$

It is easy to show that convolution have commutativity property

$$f(t) * g(t) = g(t) * f(t).$$

In 2018, Osu and Sampson proved the convolution theorem for Aboodh transform [30] such as: Let $f(t)$ and $g(t)$ are given functions defined for $t \geq 0$ and $\mathcal{A}\{f(t)\} = F(u)$ and $\mathcal{A}\{g(t)\} = G(u)$ are exist, then

$$\mathcal{A}\{f(t) * g(t)\} = u\mathcal{A}\{f(t)\}\mathcal{A}\{g(t)\} = uF(u)G(u).$$

The Mittag-Leffler function [53] of one parameter is denoted by $E_\xi(t)$ and it is defined as

$$E_\xi(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\xi k + 1)},$$

where $t, \xi \in \mathbf{C}$ and $\mathrm{Re}\,\xi > 0$. If we put $\xi = 1$, then the above equation becomes

$$E_1(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{t^k}{k!} = e^t.$$

The generalization of $E_\xi(t)$ is defined as a function

$$E_{\xi,\eta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\xi k + \eta)},$$

where $t, \xi, \eta \in \mathbf{C}$, $\mathrm{Re}\,\xi > 0$ and $\mathrm{Re}\,\eta > 0$ in [53].

Now we give the definitions of the Hyers-Ulam stability and the Hyers-Ulam-Rassias stability of the differential equations (1.1) and (1.2).

Let $\varepsilon > 0$, $\phi \in C([0,\infty),(0,\infty))$ be an increasing function and $E_\xi(t)$ is Mittag-Leffler function. Assume that for a continuously twice differentiable function of exponential order $x(t) \in C^2(I)$ satisfies one of the following inequations

$$|x''(t) + \alpha x'(t) + \beta x(t)| \leq \varepsilon \tag{2.1}$$

$$|x''(t) + \alpha x'(t) + \beta x(t)| \leq \varepsilon \phi\,(t) \tag{2.2}$$

$$|x''(t) + \alpha x'(t) + \beta x(t)| \leq \varepsilon E_\xi(t) \tag{2.3}$$

$$|x''(t) + \alpha x'(t) + \beta x(t)| \leq \varepsilon \phi\,(t)\, E_\xi(t) \tag{2.4}$$

and

$$|x''(t) + \alpha x'(t) + \beta x(t) - f(t)| \leq \varepsilon \tag{2.5}$$

$$|x''(t) + \alpha x'(t) + \beta x(t) - f(t)| \leq \varepsilon \phi\,(t) \tag{2.6}$$

$$|x''(t) + \alpha x'(t) + \beta x(t) - f(t)| \leq \varepsilon E_\xi(t) \tag{2.7}$$

$$|x''(t) + \alpha x'(t) + \beta x(t)| \leq \varepsilon \phi\,(t)\, E_\xi(t). \tag{2.8}$$

Definition 2.1. (*i*) *Equation (1.1) is Hyers-Ulam stable, if there exists a constant $K > 0$ such that for every $\varepsilon > 0$ and for each solution $x : [0,\infty) \to \mathcal{F}$ satisfies the inequality (2.1) for all $t \geq 0$, then there exists a solution the $y : [0,\infty) \to \mathcal{F}$ of the differential equation (1.1) such that*

$$|x(t) - y(t)| \leq K\varepsilon$$

for all $t \geq 0$. Then the constant K is called a Hyers-Ulam constant. (*ii*) *Equation (1.1) is Hyers-Ulam-Rassias stable, if there exists a constant $K > 0$ such that for every $\varepsilon > 0$ and for each solution $x : [0,\infty) \to \mathcal{F}$ satisfies the inequality (2.2) for all $t \geq 0$, then there exists a solution $y : [0,\infty) \to \mathcal{F}$ of the differential equation (1.1) such that*

$$|x(t) - y(t)| \leq K\phi(t)\varepsilon$$

for all $t \geq 0$. Then the constant K is called a Hyers-Ulam-Rassias constant. (iii) *Equation (1.1) is Mittag-Leffler-Hyers-Ulam stable, if there exists a constant $K > 0$ such that for every $\varepsilon > 0$ and for each solution $x : [0, \infty) \to \mathcal{F}$ satisfies the inequality (2.3) for all $t \geq 0$, then there exists a solution $y : [0, \infty) \to \mathcal{F}$ of the differential equation (1.1) such that*

$$|x(t) - y(t)| \leq K\varepsilon E_{\xi}(t)$$

for all $t \geq 0$. Then the constant K is called a Mittag-Leffler-Hyers-Ulam constant. (iv) *Equation (1.1) is Mittag-Leffler-Hyers-Ulam-Rassias stable if there exists a constant $K > 0$ such that for every $\varepsilon > 0$ and for each solution $x : [0, \infty) \to \mathcal{F}$ satisfies the inequality (2.4) for all $t \geq 0$, then there exists a solution $y : [0, \infty) \to \mathcal{F}$ of the differential equation (1.1) such that*

$$|x(t) - y(t)| \leq K\phi(t)\varepsilon E_{\nu}(t)$$

for all $t \geq 0$. For this case, we call K a Mittag-Leffler-Hyers-Ulam-Rassias constant.

Definition 2.2. (i) *Equation (1.2) is Hyers-Ulam stable if there exists a constant $K > 0$ such that for every $\varepsilon > 0$ and for each solution $x : [0, \infty) \to \mathcal{F}$ satisfies the inequality (2.5) for all $t \geq 0$, then there exists a solution $y : [0, \infty) \to \mathcal{F}$ of the differential equation (1.2) such that*

$$|x(t) - y(t)| \leq K\varepsilon$$

for all $t \geq 0$. Then the constant K is called a Hyers-Ulam constant. (ii) *Equation (1.2) is Hyers-Ulam-Rassias stable, if there exists a constant $K > 0$ such that for every $\varepsilon > 0$ and for each solution $x : [0, \infty) \to \mathcal{F}$ satisfies the inequality (2.6) for all $t \geq 0$, then there exists a solution $y : [0, \infty) \to \mathcal{F}$ of the differential equation (1.2) such that*

$$|x(t) - y(t)| \leq K\phi(t)\varepsilon$$

for all $t \geq 0$. For this case, we call the constant K a Hyers-Ulam-Rassias constant.

(iii) *Equation (1.2) is Mittag-Leffler-Hyers-Ulam stable, if there exists a constant $K > 0$ such that for every $\varepsilon > 0$ and for each solution $x : [0, \infty) \to \mathcal{F}$ satisfies the inequality (2.7) for all $t \geq 0$, then there exists a solution $y : [0, \infty) \to \mathcal{F}$ of the differential equation (1.2) such that*

$$|x(t) - y(t)| \leq K\varepsilon E_{\xi}(t)$$

for all $t \geq 0$. We call the constant K a Mittag-Leffler-Hyers-Ulam constant.

(iv) *Equation (1.2) is Mittag-Leffler-Hyers-Ulam-Rassias stable if there exists a constant $K > 0$ such that for every $\varepsilon > 0$ and for each solution $x : [0, \infty) \to \mathcal{F}$ satisfies the inequality (2.8) for all $t \geq 0$, then there exists a solution $y : [0, \infty) \to \mathcal{F}$ of the differential equation (1.2) such that*

$$|x(t) - y(t)| \leq K\phi(t)\varepsilon E_\nu(t)$$

for all $t \geq 0$. For this case, we call K a Mittag-Leffler-Hyers-Ulam-Rassias constant.

3. Ulam type stability of (1.1)

In this section, we prove several types of Hyers-Ulam stability of the homogeneous second-order linear differential equation (1.1) using the Aboodh transform.

It should be noted that in this and the next sections we investigate various types of Hyers-Ulam stability in the class of continuously differentiable functions of exponential order.

Assume that $x : [0, \infty) \to \mathcal{F}$ is a continuously differentiable function of exponential order for all $t \geq 0$. Let us define a function $z : [0, \infty) \to \mathcal{F}$ by $z(t) := x''(t) + \alpha x'(t) + \beta x(t)$ for all $t \geq 0$. Assume that $\mathcal{A}\{z(t)\} = Z(u)$. The Aboodh transform of $z(t)$ gives the following result:

$$\begin{aligned} Z(u) &= \mathcal{A}\{z(t)\} = \mathcal{A}\{x''(t) + \alpha x'(t) + \beta x(t)\} \\ &= \mathcal{A}\{x''(t)\} + \alpha\mathcal{A}\{x'(t)\} + \beta\mathcal{A}\{x(t)\} \\ &= u^2X(u) - \frac{x'(0)}{u} - x(0) + \alpha\left(uX(u) - \frac{x(0)}{u}\right) + \beta X(u), \end{aligned}$$

since $\mathcal{A}\{x''(t)\} = u^2X(u) - \dfrac{x'(0)}{u} - x(0)$ and $\mathcal{A}\{x'(t)\} = uX(u) - \dfrac{x(0)}{u}$ see [28]. Hence we get

$$\begin{aligned} \mathcal{A}\{x(t)\} &= X(u) = \frac{\dfrac{x'(0)}{u} + \dfrac{\alpha x(0)}{u} + x(0) + Z(u)}{u^2 + \alpha u + \beta} \\ &= \frac{x'(0) + \alpha x(0)}{u\left[\left(u + \dfrac{\alpha}{2}\right)^2 + \beta - \dfrac{\alpha^2}{4}\right]} + \frac{x(0) + Z(u)}{\left(u + \dfrac{\alpha}{2}\right)^2 + \beta - \dfrac{\alpha^2}{4}}. \end{aligned}$$

Thus we have

$$X(u)=\frac{x'(0)+\alpha x(0)}{u\left[\left(u+\frac{\alpha}{2}\right)^2+\beta-\frac{\alpha^2}{4}\right]}+\frac{x(0)}{\left(u+\frac{\alpha}{2}\right)^2+\beta-\frac{\alpha^2}{4}}+\frac{Z(u)}{\left(u+\frac{\alpha}{2}\right)^2+\beta-\frac{\alpha^2}{4}}. \tag{3.1}$$

Let $\lambda=\beta-\frac{\alpha^2}{4}$. Then if $4\beta>\alpha^2$ we can find a positive constant μ such that $\lambda=\mu^2$, but if $4\beta<\alpha^2$ then $\lambda=-\mu^2$ and if $4\beta=\alpha^2$ then $\lambda=0$. Let's examine these cases one by one.

Case I: Let $4\beta>\alpha^2$ then there exist a positive constant μ such that $\lambda=\mu^2$. In this case (3.1) becomes

$$X(u)=\frac{x'(0)+\alpha x(0)}{u\left[\left(u+\frac{\alpha}{2}\right)^2+\mu^2\right]}+\frac{x(0)}{\left(u+\frac{\alpha}{2}\right)^2+\mu^2}+\frac{Z(u)}{\left(u+\frac{\alpha}{2}\right)^2+\mu^2}.$$

If we put

$$y(t)=\left(\frac{2x'(0)+\alpha x(0)}{2\mu}\right)e^{-\frac{\alpha}{2}t}\sin\mu t+x(0)e^{-\frac{\alpha}{2}t}\cos\mu t,$$

then $y(0)=x(0)$, $y'(0)=x'(0)$ and $y(t)$ is a function of exponential order. The Aboodh transform of $y(t)$ gives the following result:

$$\mathcal{A}\{y(t)\}=Y(u)=\frac{x'(0)+\alpha x(0)}{u\left[\left(u+\frac{\alpha}{2}\right)^2+\mu^2\right]}+\frac{x(0)}{\left(u+\frac{\alpha}{2}\right)^2+\mu^2}. \tag{3.2}$$

Thus

$$\mathcal{A}\{y''(t)+\alpha y'(t)+\beta y(t)\}=u^2Y(u)-\frac{y'(0)}{u}-y(0)+\alpha\left(uY(u)-\frac{y(0)}{u}\right)+\beta Y(u).$$

Using (3.2), we have

$$\frac{x'(0)+\alpha x(0)}{u\left[\left(u+\frac{\alpha}{2}\right)^2+\mu^2\right]}\left(u^2+\alpha u+\beta\right)+\frac{x(0)}{\left(u+\frac{\alpha}{2}\right)^2+\mu^2}\left(u^2+\alpha u+\beta\right)-\frac{y'(0)}{u}-y(0)-\frac{\alpha y(0)}{u}=0.$$

This means that

$$\mathcal{A}\{y''(t)+\alpha y'(t)+\beta y(t)\}=0$$

Since $\mathcal{A}$ is a one-to-one operator, $y''(t)+\alpha y'(t)+\beta y(t)=0$. Hence $y(t)$ is a solution of the differential equation (1.1). By (3.1) and (3.2), we obtain

$$X(u)-Y(u)=\frac{Z(u)}{\left(u+\frac{\alpha}{2}\right)^2+\mu^2}=uZ(u)H(u)=\mathcal{A}\{z(t)^*h(t)\},$$

where

$$H(u)=\frac{1}{u\left[\left(u+\frac{\alpha}{2}\right)^2+\mu^2\right]},$$

which gives

$$h(t)=\mathcal{A}^{-1}\left\{\frac{1}{u\left[\left(u+\frac{\alpha}{2}\right)^2+\mu^2\right]}\right\}=\frac{1}{\mu}e^{-\frac{\alpha}{2}t}\sin\mu t.$$

Consequently, $\mathcal{A}\{x(t)-y(t)\}=\mathcal{A}\{z(t)*h(t)\}$ and thus $x(t)-y(t)=z(t)*h(t)$. Taking modulus on both sides, we have

$$(3.3)\quad |x(t)-y(t)|=|z(t)*h(t)|=\left|\int_0^t z(\zeta)h(t-\zeta)d\zeta\right|\leq\int_0^t|z(\zeta)|\,|h(t-\zeta)|\,d\zeta.$$

Assume that $x(t)$ satisfies inequality (2.1), then inequality (3.3) becomes

$$\begin{aligned}|x(t)-y(t)| &\leq \varepsilon\int_0^t|h(t-\zeta)|\,d\zeta=\varepsilon\int_0^t\left|\frac{1}{\mu}e^{-\frac{\alpha}{2}(t-\zeta)}\sin\mu\,(t-\zeta))\right|d\zeta\\ &\leq \frac{\varepsilon}{\mu}\int_0^t\left|e^{-\frac{\alpha}{2}(t-\zeta)}\right|d\zeta\leq\frac{\varepsilon}{\mu}e^{-\frac{\alpha}{2}t}\int_0^t e^{\frac{\alpha}{2}\zeta}d\zeta\\ &= \frac{2\varepsilon}{\alpha\mu}\left(1-e^{-\frac{\alpha}{2}t}\right)\leq K_1\varepsilon\end{aligned}$$

for all $t \geq 0$, where we set

$$K_1 = \frac{2}{\alpha\mu}. \tag{3.4}$$

If $x(t)$ satisfies inequality (2.2), then inequality (3.3) becomes

$$|x(t) - y(t)| \leq \int_0^t |z(\zeta)| \, |h(t-\zeta)| \, d\zeta \leq \varepsilon\phi(t) \int_0^t \left| e^{-\frac{\alpha}{2}(t-\zeta)} \right| d\zeta \leq \varepsilon K_1 \phi(t)$$

for all $t \geq 0$.

If $x(t)$ satisfies inequality (2.3), then inequality (3.3) becomes

$$|x(t) - y(t)| \leq \int_0^t |z(\zeta)| \, |h(t-\zeta)| \, d\zeta \leq \varepsilon E_\xi(t) \int_0^t \left| e^{-\frac{\alpha}{2}(t-\zeta)} \right| d\zeta \leq \varepsilon K_1 E_\xi(t)$$

for all $t \geq 0$.

Finally if $x(t)$ satisfies inequality (2.4), then inequality (3.3) becomes

$$|x(t) - y(t)| \leq \int_0^t |z(\zeta)| \, |h(t-\zeta)| \, d\zeta$$

$$\leq \varepsilon\phi(t) E_\xi(t) \int_0^t \left| e^{-\frac{\alpha}{2}(t-\zeta)} \right| d\zeta \leq \varepsilon K_1 \phi(t) E_\xi(t)$$

Using these results for $4\beta > \alpha^2$, we have the following theorem.

Theorem 3.1. *Let $4\beta > \alpha^2$ in the equation (1.1), then*
(i)Assume that $\alpha > 0$ is a constant. Then the differential equation (1.1) is stable in the sense of Hyers-Ulam with Hyers-Ulam constant K_1, which is given in (3.4) in the class of continuously differentiable functions of exponential order.
(ii) Assume that $\alpha > 0$ is a constant and $\phi(t)$ is a positive definitely increasing function. Then the differential equation (1.1) is stable in the sense of Hyers-Ulam-Rassias with Hyers-Ulam-Rassias constant K_1, which is given in (3.4) in the class of continuously differentiable functions of exponential order.
(iii)Assume that $\alpha > 0$ is a constant and $E_\xi(t)$, $\xi > 0$ is Mittag-Leffler function. The differential equation (1.1) is stable in the sense of Mittag-Leffler-Hyers-Ulam with Mittag-Leffler-Hyers-Ulam constant K_1, which is

given in (3.4) in the class of continuously differentiable functions of exponential order.
(iv)Assume that $\alpha > 0$ is a constant, $\phi(t)$ is a positive definitely increasing function and $E_\xi(t)$, $\xi > 0$ is Mittag-Leffler function. Then the differential equation (1.1) is stable in the sense of Mittag-Leffler-Hyers-Ulam-Rassias with Mittag-Leffler-Hyers-Ulam-Rassias constant K_1, which is given in (3.4) in the class of continuously differentiable functions of exponential order.

Case II: Let $4\beta < \alpha^2$ then there exist a positive constant μ such that $\lambda = -\mu^2$. In this case (3.1) becomes

$$X(u) = \frac{x'(0) + \alpha x(0)}{u\left[\left(u + \frac{\alpha}{2}\right)^2 - \mu^2\right]} + \frac{x(0)}{\left(u + \frac{\alpha}{2}\right)^2 - \mu^2} + \frac{Z(u)}{\left(u + \frac{\alpha}{2}\right)^2 - \mu^2}.$$

If we put

$$y(t) = \left(\frac{2x'(0) + \alpha x(0)}{2\mu}\right) e^{-\frac{\alpha}{2}t} \sinh \mu t + x(0) e^{-\frac{\alpha}{2}t} \cosh \mu t,$$

then $y(0) = x(0)$, $y'(0) = x'(0)$ and $y(t)$ is a function of exponential order. The Aboodh transform of $y(t)$ gives the following result:

$$\mathcal{A}\{y(t)\} = Y(u) = \frac{x'(0) + \alpha x(0)}{u\left[\left(u + \frac{\alpha}{2}\right)^2 - \mu^2\right]} + \frac{x(0)}{\left(u + \frac{\alpha}{2}\right)^2 - \mu^2}.$$

If we use the arguments of the Case I, we get

$$X(u) - Y(u) = \frac{Z(u)}{\left(u + \frac{\alpha}{2}\right)^2 - \mu^2} = uZ(u)H(u) = \mathcal{A}\{z(t) * h(t)\}$$

where

$$H(u) = \frac{1}{u\left[\left(u + \frac{\alpha}{2}\right)^2 - \mu^2\right]},$$

which gives

$$h(t) = \mathcal{A}^{-1}\left\{\frac{1}{u\left[\left(u + \frac{\alpha}{2}\right)^2 - \mu^2\right]}\right\} = \frac{1}{\mu} e^{-\frac{\alpha}{2}t} \sinh \mu t.$$

Consequently, $\mathcal{A}\{x(t)-y(t)\} = \mathcal{A}\{z(t)*h(t)\}$and thus $x(t)-y(t) = z(t)*h(t)$. Taking modulus on both sides, we have

$$|x(t)-y(t)| = |z(t)*h(t)| = \left|\int_0^t z(\zeta)h(t-\zeta)d\zeta\right| \leq \int_0^t |z(\zeta)|\,|h(t-\zeta)|\,d\zeta. \tag{3.5}$$

Similar to the Case I, if $x(t)$ satisfies inequality (2.1), then inequality (3.5) becomes

$$\begin{aligned}
|x(t)-y(t)| &\leq \varepsilon \int_0^t \left|\frac{1}{\mu}e^{-\frac{\alpha}{2}(t-\zeta)}\sinh\mu\,(t-\zeta)\right| d\zeta \\
&= \frac{\varepsilon}{\mu}\int_0^t \left|e^{-\frac{\alpha}{2}\theta}\sinh\mu\theta\right| d\theta \\
&= \frac{\varepsilon}{\mu}\int_0^t e^{-\frac{\alpha}{2}\theta}\sinh\mu\theta d\theta \\
&= \frac{\varepsilon}{4\mu^3-\alpha^2\mu}\left\{e^{-\frac{\alpha}{2}t}\left(\mu\cosh\mu t+\frac{\alpha}{2}\sinh\mu t\right)-\mu\right\} \\
&= \frac{\varepsilon}{\alpha^2-4\mu^2}+\varepsilon\frac{e^{-\frac{\alpha}{2}t}\left(\mu\cosh\mu t+\frac{\alpha}{2}\sinh\mu t\right)}{4\mu^3-\alpha^2\mu} \\
&= \frac{\varepsilon}{4\beta}+\varepsilon\frac{e^{-\frac{\alpha}{2}t}\left(\mu\cosh\mu t+\frac{\alpha}{2}\sinh\mu t\right)}{4\mu^3-\alpha^2\mu} \\
&= \frac{\varepsilon}{4\beta}+\varepsilon\frac{\left(\mu+\frac{\alpha}{2}\right)e^{-\frac{\alpha}{2}t}\sinh\mu t}{4\mu^3-\alpha^2\mu} \\
&= \frac{\varepsilon}{4\beta}+\varepsilon\frac{e^{-\frac{\alpha}{2}t}\sinh\mu t}{4\mu^2-\alpha^2}+\left(\frac{\alpha\varepsilon}{2\mu}\right)\frac{e^{-\frac{\alpha}{2}t}\sinh\mu t}{4\mu^2-\alpha^2} \\
&= \frac{\varepsilon}{4\beta}-\frac{e^{-\frac{\alpha}{2}t}\sinh\mu t}{4\beta}-\left(\frac{\alpha\varepsilon}{2\mu}\right)\frac{e^{-\frac{\alpha}{2}t}\sinh\mu t}{4\beta} \\
&= \frac{\varepsilon}{4\beta}\left(1-\left(1+\frac{\alpha}{2\mu}\right)e^{-\frac{\alpha}{2}t}\sinh\mu t\right) \qquad (3.6)
\end{aligned}$$

for all $t \geq 0$. Where we use the fact $e^{-\frac{\alpha}{2}\theta}\sinh\mu\theta > 0$ for all $t \geq 0$ and for all $\alpha \in \mathbf{R}$.

If $\beta > 0$ and $2\mu+\alpha > 0$ then (3.6) becomes

$$|x(t)-y(t)| \leq K_2\varepsilon$$

where

$$K_2 = \frac{1}{4\beta} \tag{3.7}$$

If $x(t)$ satisfies inequality (2.2), then inequality (3.4) becomes

$$\begin{aligned} |x(t) - y(t)| &\leq \varepsilon\phi(t) \int_0^t \left| \frac{1}{\mu} e^{-\frac{\alpha}{2}(t-\zeta)} \sinh\mu(t-\zeta) \right| d\zeta \\ &\leq \frac{\varepsilon}{\mu}\phi(t) \int_0^t e^{-\frac{\alpha}{2}\theta} \sinh\mu\theta d\theta \leq \varepsilon K_2 \phi(t) \end{aligned}$$

under the conditions $\beta > 0$ and $2\mu + \alpha > 0$.

If $x(t)$ satisfies inequality (2.3), then inequality (3.4) becomes

$$\begin{aligned} |x(t) - y(t)| &\leq \varepsilon E_\xi(t) \int_0^t \left| \frac{1}{\mu} e^{-\frac{\alpha}{2}(t-\zeta)} \sinh\mu(t-\zeta) \right| d\zeta \\ &\leq \frac{\varepsilon}{\mu} E_\xi(t) \int_0^t e^{-\frac{\alpha}{2}\theta} \sinh\mu\theta d\theta \leq \varepsilon K_2 E_\xi(t). \end{aligned}$$

under the conditions $\beta > 0$ and $2\mu + \alpha > 0$.

Finally if $x(t)$ satisfies inequality (2.3), then inequality (3.4) becomes

$$\begin{aligned} |x(t) - y(t)| &\leq \varepsilon\phi(t) E_\xi(t) \int_0^t \left| \frac{1}{\mu} e^{-\frac{\alpha}{2}(t-\zeta)} \sinh\mu(t-\zeta) \right| d\zeta \\ &\leq \frac{\varepsilon}{\mu}\phi(t) E_\xi(t) \int_0^t e^{-\frac{\alpha}{2}\theta} \sinh\mu\theta d\theta \leq \varepsilon K_2 \phi(t) E_\xi(t). \end{aligned}$$

under the conditions $\beta > 0$ and $2\mu + \alpha > 0$.

Using this results for $4\beta < \alpha^2$, we have the following theorem.

Theorem 3.2. *Let $4\beta < \alpha^2$ in the equation (1.1), then*
(i)Assume that $\beta > 0$ and $2\mu + \alpha > 0$. Then the differential equation (1.1) is stable in the sense of Hyers-Ulam with Hyers-Ulam constant K_2, which

is given in (3.7) in the class of continuously differentiable functions of exponential order.
(ii) Assume that $\beta > 0$ and $2\mu + \alpha > 0$ and $\phi(t)$ is a positive definitely increasing function. Then the differential equation (1.1) is stable in the sense of Hyers-Ulam-Rassias with Hyers-Ulam-Rassias constant K_2, which is given in (3.7) in the class of continuously differentiable functions of exponential order.
(iii)Assume that $\beta > 0$ and $2\mu + \alpha > 0$ and $E_\xi(t)$, $\xi > 0$ is Mittag-Leffler function. The differential equation (1.1) is stable in the sense of Mittag-Leffler-Hyers-Ulam with Mittag-Leffler-Hyers-Ulam constant K_2, which is given in (3.7) in the class of continuously differentiable functions of exponential order.
(iv)Assume that $\beta > 0$ and $2\mu + \alpha > 0$, $\phi(t)$ is a positive definitely increasing function and $E_\xi(t)$, $\xi > 0$ is Mittag-Leffler function. Then the differential equation (1.1) is stable in the sense of Mittag-Leffler-Hyers-Ulam-Rassias with Mittag-Leffler-Hyers-Ulam-Rassias constant K_2, which is given in (3.7) in the class of continuously differentiable functions of exponential order.

Case III: Let $4\beta = \alpha^2$ then (3.1) becomes

$$X(u) = \frac{x'(0) + \alpha x(0)}{u\left(u + \frac{\alpha}{2}\right)^2} + \frac{x(0)}{\left(u + \frac{\alpha}{2}\right)^2} + \frac{Z(u)}{\left(u + \frac{\alpha}{2}\right)^2}.$$

If we put

$$y(t) = \left(x'(0) + \frac{\alpha}{2}x(0)\right)e^{-\frac{\alpha}{2}t}t + x(0)e^{-\frac{\alpha}{2}t},$$

then $y(0) = x(0)$, $y'(0) = x'(0)$ and $y(t)$ is a function of exponential order. The Aboodh transform of $y(t)$ gives the following result:

$$\mathcal{A}\{y(t)\} = Y(u) = \frac{x'(0) + \alpha x(0)}{u\left(u + \frac{\alpha}{2}\right)^2} + \frac{x(0)}{\left(u + \frac{\alpha}{2}\right)^2}.$$

If we use the arguments of the Case I, we get

$$X(u) - Y(u) = \frac{Z(u)}{\left(u + \frac{\alpha}{2}\right)^2} = uZ(u)H(u) = \mathcal{A}\{z(t) * h(t)\}$$

where

$$H(u) = \frac{1}{u\left(u + \frac{\alpha}{2}\right)^2},$$

which gives

$$h(t) = \mathcal{A}^{-1}\left\{\frac{1}{u\left(u + \frac{\alpha}{2}\right)^2}\right\} = e^{-\frac{\alpha}{2}t}t.$$

Consequently, $\mathcal{A}\{x(t) - y(t)\} = \mathcal{A}\{z(t) * h(t)\}$ and thus $x(t) - y(t) = z(t) * h(t)$. Taking modulus on both sides, we have

$$|x(t) - y(t)| = |z(t) * h(t)| = \left|\int_0^t z(\zeta)h(t-\zeta)d\zeta\right| \leq \int_0^t |z(\zeta)|\,|h(t-\zeta)|\,d\zeta, \tag{3.8}$$

for all $t \geq 0$.

Similar to the Case I, if $x(t)$ satisfies inequality (2.1), then inequality (3.8) becomes

$$\begin{aligned} |x(t) - y(t)| &\leq \varepsilon \int_0^t |h(t-\zeta)|\,d\zeta = \varepsilon \int_0^t \left|e^{-\frac{\alpha}{2}(t-\zeta)}(t-\zeta)\right| d\zeta \\ &\leq \varepsilon \int_0^t e^{-\frac{\alpha}{2}\theta}\theta d\theta \leq \varepsilon\left(\frac{4}{\alpha^2} - e^{-\frac{\alpha}{2}t}\left(\frac{2\alpha t + 4}{\alpha^2}\right)\right) \leq \varepsilon K_3 \end{aligned}$$

for all $t \geq 0$, where we use $t - \zeta = \theta$ and $t \to \infty\lim\left(\frac{4}{\alpha^2} - e^{-\frac{\alpha}{2}t}\left(\frac{2\alpha t+4}{\alpha^2}\right)\right) = \frac{4}{\alpha^2}$ for $\alpha > 0$ and we set

$$K_3 = \frac{4}{\alpha^2}. \tag{3.9}$$

If $x(t)$ satisfies inequality (2.2), then inequality (3.8) becames

$$\begin{aligned} |x(t) - y(t)| &\leq \varepsilon\phi(t) \int_0^t |h(t-\zeta)|\,d\zeta = \varepsilon\phi(t) \int_0^t \left|e^{-\frac{\alpha}{2}(t-\zeta)}(t-\zeta)\right| d\zeta \\ &\leq \varepsilon\phi(t) \int_0^t e^{\frac{\alpha}{2}\theta}\theta d\theta \leq \varepsilon K_3 \phi(t), \end{aligned}$$

if $x(t)$ satisfies inequality (2.3), then inequality (3.8) becomes

$$\begin{aligned}|x(t)-y(t)| &\leq \varepsilon E_{\xi}(t)\int_{0}^{t}|h(t-\zeta)|\,d\zeta=\varepsilon E_{\xi}(t)\int_{0}^{t}\left|e^{-\frac{\alpha}{2}(t-\zeta)}(t-\zeta)\right|d\zeta\\ &\leq \varepsilon E_{\xi}(t)\int_{0}^{t}e^{\frac{\alpha}{2}\theta}\theta d\theta\leq\varepsilon K_{3}E_{\xi}(t),\end{aligned}$$

and finally if $x(t)$ satisfies inequality (2.4), then inequality (3.8) becomes

$$\begin{aligned}|x(t)-y(t)| &\leq \varepsilon\phi(t)\,E_{\xi}(t)\int_{0}^{t}|h(t-\zeta)|\,d\zeta\\ &= \varepsilon\phi(t)\,E_{\xi}(t)\int_{0}^{t}\left|e^{-\frac{\alpha}{2}(t-\zeta)}(t-\zeta)\right|d\zeta\\ &\leq \varepsilon\phi(t)\,E_{\xi}(t)\int_{0}^{t}e^{\frac{\alpha}{2}\theta}\theta d\theta\leq\varepsilon K_{3}\phi(t)\,E_{\xi}(t).\end{aligned}$$

Using this result for $4\beta=\alpha^2$, we have the following theorem.

Theorem 3.3. *Let $4\beta=\alpha^2$ in the equation (1.1), then*
(i)Assume that $\alpha>0$ is a constant. Then the differential equation (1.1) is stable in the sense of Hyers-Ulam with Hyers-Ulam constant K_3, which is given in (3.9) in the class of continuously differentiable functions of exponential order.
(ii) Assume that $\alpha>0$ is a constant and $\phi(t)$ is a positive definitely increasing function. Then the differential equation (1.1) is stable in the sense of Hyers-Ulam-Rassias with Hyers-Ulam-Rassias constant K_3, which is given in (3.9) in the class of continuously differentiable functions of exponential order.
(iii)Assume that $\alpha>0$ is a constant and $E_{\xi}(t)$, $\xi>0$ is Mittag-Leffler function. The differential equation (1.1) is stable in the sense of Mittag-Leffler-Hyers-Ulam with Mittag-Leffler-Hyers-Ulam constant K_3, which is given in (3.9) in the class of continuously differentiable functions of exponential order.
(iv)Assume that $\alpha>0$ is a constant, $\phi(t)$ is a positive definitely increasing

function and $E_{\xi}(t)$, $\xi > 0$ is Mittag-Leffler function. Then the differential equation (1.1) is stable in the sense of Mittag-Leffler-Hyers-Ulam-Rassias with Mittag-Leffler-Hyers-Ulam-Rassias constant K_3, which is given in (3.9) in the class of continuously differentiable functions of exponential order.

4. Ulam type stability of (1.2)

In this section, we prove several types of Hyers-Ulam stability of the non-homogeneous second-order linear differential equation (1.2) using the Aboodh transform.

Let $x, f : [0, \infty) \to \mathcal{F}$ be continuously differentiable functions of exponential order for all $t \geq 0$. Define the function $z : [0, \infty) \to \mathcal{F}$ by

$$z(t) := x''(t) + \alpha x'(t) + \beta x(t) - f(t)$$

for all $t \geq 0$. The Aboodh transform of $z(t)$ gives the following result: $\mathcal{A}\{z(t)\} = \mathcal{A}\{x''(t) + \alpha x'(t) + \beta x(t) - f(t)\}$. That is,

$$\begin{aligned} Z(u) &= \mathcal{A}\{x''(t) + \alpha x'(t) + \beta x(t)\} - \mathcal{A}\{f(t)\} \\ &= u^2 X(u) - \frac{x'(0)}{u} - x(0) + \alpha \left(X(u) - \frac{x(0)}{u} \right) + \beta X(u) - F(u) \\ &= X(u) \left(u^2 + \alpha u + \beta \right) - \frac{1}{u} \left(x'(0) + \alpha x(0) \right) - x(0) - F(u), \end{aligned}$$

which implies that

$$X(u) = \frac{Z(u)}{u^2 + \alpha u + \beta} + \frac{x'(0) + \alpha x(0)}{u \left(u^2 + \alpha u + \beta \right)} + \frac{x(0) + F(u)}{u^2 + \alpha u + \beta}. \tag{4.1}$$

If we use the same arguments of the Section 3 we can write the equation (4.1) as

$$X(u) = \frac{Z(u)}{\left(u + \frac{\alpha}{2}\right)^2 + \beta - \frac{\alpha^2}{4}} + \frac{x'(0) + \alpha x(0)}{u \left(\left(u + \frac{\alpha}{2}\right)^2 + \beta - \frac{\alpha^2}{4} \right)} + \frac{x(0) + F(u)}{\left(u + \frac{\alpha}{2}\right)^2 + \beta - \frac{\alpha^2}{4}} \tag{4.2}$$

and we must examine three cases, as we did in Section 3. Therefore, in this section we will obtain one of these states and give only the results for the others.

Let $4\beta > \alpha^2$, then equation (4.2) becomes

$$X(u)=\frac{Z(u)}{\left(u+\frac{\alpha}{2}\right)^2+\mu^2}+\frac{x'(0)+\alpha x(0)}{u\left(\left(u+\frac{\alpha}{2}\right)^2+\mu^2\right)}+\frac{x(0)+F(u)}{\left(u+\frac{\alpha}{2}\right)^2+\mu^2}, \tag{4.3}$$

where μ is given as Section 3.

If we set

$$\begin{aligned} y(t) &= \left(\frac{2x'(0)+\alpha x(0)}{2\mu}\right)e^{-\frac{\alpha}{2}t}\sin\mu t + x(0)e^{-\frac{\alpha}{2}t}\cos\mu t \\ &\quad + f(t) * \left(e^{-\frac{\alpha}{2}t}\cos\mu t - \frac{\alpha}{2\mu}e^{-\frac{\alpha}{2}t}\sin\mu t\right) \end{aligned}$$

then $y(0)=x(0)$, $y'(0)=x(0)$ and the Aboodh transform of $y(t)$ yields the following result:

$$\mathcal{A}\{y(t)\}=Y(u)=\frac{x'(0)+\alpha x(0)}{u\left(\left(u+\frac{\alpha}{2}\right)^2+\mu^2\right)}+\frac{x(0)}{\left(u+\frac{\alpha}{2}\right)^2+\mu^2}+\frac{F(u)}{\left(u+\frac{\alpha}{2}\right)^2+\mu^2}. \tag{4.4}$$

On the other hand,

$$\mathcal{A}\{y''(t)+\alpha y'(t)+\beta y(t)\}=Y(u)\left(u^2+\alpha u+\beta\right)-\frac{1}{u}\left(y'(0)+\alpha y(0)\right)-x(0),$$

since $x(0)=y(0)$ and $x'(0)=y'(0)$. Then, by (4.3),
we have

$$\mathcal{A}\{y''(t)+\alpha y'(t)+\beta y(t)\}=F(u)=\mathcal{A}\{f(t)\}$$

and thus $y''(t)+\alpha y'(t)+\beta y(t)=f(t)$. Hence $y(t)$ is a solution of the differential equation (1.2).

In addition, by applying (4.3) and (4.4), we can obtain

$$X(u)-Y(u)=\frac{Z(u)}{\left(u+\frac{\alpha}{2}\right)^2+\mu^2}=uZ(u)H(u)=u\mathcal{A}\{z(t)\}\mathcal{A}\{h(t)\},$$

where we set

$$H(u)=\frac{1}{u\left[\left(u+\frac{\alpha}{2}\right)^2+\mu^2\right]}$$

which gives

$$h(t)=\mathcal{A}^{-1}\left\{\frac{1}{u\left[\left(u+\frac{\alpha}{2}\right)^2+\mu^2\right]}\right\}=\frac{1}{\mu}e^{-\frac{\alpha}{2}t}\sin\mu t.$$

Therefore, we have

$$\mathcal{A}\{x(t)-y(t)\}=\mathcal{A}\left\{z(t)*\left(\frac{1}{\mu}e^{-\frac{\alpha}{2}t}\sin\mu t\right)\right\},$$

which yields $x(t)-y(t)=z(t)*\left(\frac{1}{\mu}e^{-\frac{\alpha}{2}t}\sin\mu t\right)$. Furthermore,

$$|x(t)-y(t)|=|z(t)^{*}h(t)|=\left|\int_0^t z(\zeta)h(t-\zeta)d\zeta\right|\le\int_0^t|z(\zeta)|\,|h(t-\zeta)|\,d\zeta \tag{4.5}$$

for all $t\ge 0$.

Now assume that $x(t)$ satisfies the inequality (2.5), then inequality (4.5) becomes

$$|x(t)-y(t)|\le\varepsilon\int_0^t|h(t-\zeta)|\,d\zeta\le\varepsilon\int_0^t\left|\frac{1}{\mu}e^{-\frac{\alpha}{2}(t-\zeta)}\sin\mu\,(t-\zeta)\right|d\zeta\le\varepsilon K_1,$$

for $\alpha>0$, where K_1 is given in (3.4).

If $x(t)$ satisfies inequality (2.6), then inequality (4.5) becomes

$$|x(t)-y(t)|\le\varepsilon\phi\,(t)\int_0^t|h(t-\zeta)|\,d\zeta\le\varepsilon\phi\,(t)\,K_1$$

for $\alpha>0$.

If $x(t)$ satisfies inequality (2.7), then inequality (4.5) becomes

$$|x(t)-y(t)|\le\varepsilon E_\xi\,(t)\int_0^t|h(t-\zeta)|\,d\zeta\le\varepsilon E_\xi\,(t)\,K_1$$

for $\alpha>0$.

If $x(t)$ satisfies inequality (2.8), then inequality (4.5) becomes

$$|x(t)-y(t)|\le\varepsilon\phi\,(t)\,E_\xi\,(t)\int_0^t|h(t-\zeta)|\,d\zeta\le\varepsilon\phi\,(t)\,E_\xi\,(t)\,K_1$$

for $\alpha>0$.

In this case, using the Definition 2.2 and the results obtained in Section 3, we obtain the following stability results for equation (1.2) for $4\beta>\alpha^2$.

Theorem 4.1. *Let $4\beta>\alpha^2$ in the equation (1.2), then*
(i)Assume that $\alpha>0$ is a constant. Then the differential equation (1.2) is stable in the sense of Hyers-Ulam with Hyers-Ulam constant K_1, which is

given in (3.4) in the class of continuously differentiable functions of exponential order.
(ii) Assume that $\alpha > 0$ *is a constant and* $\phi(t)$ *is a positive definitely increasing function. Then the differential equation (1.2) is stable in the sense of Hyers-Ulam-Rassias with Hyers-Ulam-Rassias constant* K_1*, which is given in (3.4) in the class of continuously differentiable functions of exponential order.*
(iii)Assume that $\alpha > 0$ *is a constant and* $E_\xi(t)$*,* $\xi > 0$ *is Mittag-Leffler function. Then the differential equation (1.2) is stable in the sense of Mittag-Leffler-Hyers-Ulam with Mittag-Leffler-Hyers-Ulam constant* K_1*, which is given in (3.4) in the class of continuously differentiable functions of exponential order.*
(iv)Assume that $\alpha > 0$ *is a constant,* $\phi(t)$ *is a positive definitely increasing function and* $E_\xi(t)$*,* $\xi > 0$ *is Mittag-Leffler function. Then the differential equation (1.2) is stable in the sense of Mittag-Leffler-Hyers-Ulam-Rassias with Mittag-Leffler-Hyers-Ulam-Rassias constant* K_1*, which is given in (3.4) in the class of continuously differentiable functions of exponential order.*

If we use the arguments used in Theorem 4.1 in cases $4\beta < \alpha^2$ and $4\beta = \alpha^2$, we can easily obtain the following theorems respectively.

Theorem 4.2. *Let* $4\beta < \alpha^2$ *in the equation (1.2), then*
(i)Assume that $\beta > 0$ *and* $2\mu + \alpha > 0$*. Then the differential equation (1.2) is stable in the sense of Hyers-Ulam with Hyers-Ulam constant* K_2*, which is given in (3.7) in the class of continuously differentiable functions of exponential order.*
(ii) Assume that $\beta > 0$*,* $2\mu+\alpha > 0$ *and* $\phi(t)$ *is a positive definitely increasing function. Then the differential equation (1.2) is stable in the sense of Hyers-Ulam-Rassias with Hyers-Ulam-Rassias constant* K_2*, which is given in (3.7) in the class of continuously differentiable functions of exponential order.*
(iii)Assume that $\beta > 0$*,* $2\mu + \alpha > 0$ *and* $E_\xi(t)$*,* $\xi > 0$ *is Mittag-Leffler function. The differential equation (1.2) is stable in the sense of Mittag-Leffler-Hyers-Ulam with Mittag-Leffler-Hyers-Ulam constant* K_2*, which is given in (3.7) in the class of continuously differentiable functions of exponential order.*
(iv)Assume that $\beta > 0$*,* $2\mu + \alpha > 0$*,* $\phi(t)$ *is a positive definitely increasing function and* $E_\xi(t)$*,* $\xi > 0$ *is Mittag-Leffler function. Then the differential equation (1.2) is stable in the sense of Mittag-Leffler-Hyers-Ulam-Rassias with Mittag-Leffler-Hyers-Ulam-Rassias constant* K_2*, which is given in (3.7)*

in the class of continuously differentiable functions of exponential order.

Theorem 4.3. *Let $4\beta = \alpha^2$ in the equation (1.2), then*
(i)Assume that $\alpha > 0$ is a constant. Then the differential equation (1.2) is stable in the sense of Hyers-Ulam with Hyers-Ulam constant K_3, which is given in (3.9) in the class of continuously differentiable functions of exponential order.
(ii) Assume that $\alpha > 0$ is a constant and $\phi(t)$ is a positive definitely increasing function. Then the differential equation (1.2) is stable in the sense of Hyers-Ulam-Rassias with Hyers-Ulam-Rassias constant K_3, which is given in (3.9) in the class of continuously differentiable functions of exponential order.
(iii)Assume that $\alpha > 0$ is a constant and $E_\xi(t)$, $\xi > 0$ is Mittag-Leffler function. The differential equation (1.2) is stable in the sense of Mittag-Leffler-Hyers-Ulam with Mittag-Leffler-Hyers-Ulam constant K_3, which is given in (3.9) in the class of continuously differentiable functions of exponential order.
(iv)Assume that $\alpha > 0$ is a constant, $\phi(t)$ is a positive definitely increasing function and $E_\xi(t)$, $\xi > 0$ is Mittag-Leffler function. Then the differential equation (1.2) is stable in the sense of Mittag-Leffler-Hyers-Ulam-Rassias with Mittag-Leffler-Hyers-Ulam-Rassias constant K_3, which is given in (3.9) in the class of continuously differentiable functions of exponential order.

5. Examples and remarks

In this section, we will introduce some examples to make it easier to understand the main results of this paper.

Example 5.1. *We consider the following non-homogeneous linear differential equation*

$$x''(t) + x'(t) + 2x(t) = 2\cos t. \tag{5.1}$$

We know that $f(t) = 2\cos t$ is a function of exponential order and $\alpha = 1$ and $\beta = 2$. Thus $\lambda = \beta - \frac{\alpha^2}{4} = \frac{7}{4} = \mu^2 > 0$.

If a continuously differentiable function $z : [0,\infty) \to \mathcal{F}$ of exponential order satisfies

$$|z''(t) + 2z'(t) + z(t) - 2\cos t| \le \varepsilon$$

for all $t \ge 0$ and for some $\varepsilon > 0$, then Theorem 4.1-(i) implies that there exists a solution $y : [0,\infty) \to \mathbf{K}$ of the differential equation (5.1) such that

$$|z(t) - y(t)| \le K\varepsilon$$

for all $t \geq 0$, *where* $K = \frac{2}{\alpha\mu} = \frac{4}{\sqrt{7}}$.

Example 5.2. *We consider the following non-homogeneous linear differential equation*

$$x''(t) - 4x'(t) + 3x(t) = \sin t, \tag{5.2}$$

where $f(t) = \sin t$ *is a function of exponential order and* $\alpha = -4, \beta = 3$. *Thus* $\lambda = \beta - \frac{\alpha^2}{4} = -1$. *and* $\mu = 1$

If a continuously differentiable function $z : [0, \infty) \to \mathcal{F}$ *of exponential order satisfies*

$$|z''(t) - 4z'(t) + 3z(t) - \sin t| \leq \varepsilon$$

for all $t \geq 0$ *and for some* $\varepsilon > 0$, *then Theorem 4.2-(i) implies that there exists a solution* $y : [0, \infty) \to \mathbf{K}$ *of the differential equation (5.2) such that*

$$|z(t) - y(t)| \leq K\varepsilon$$

for all $t \geq 0$, *where we set* $K = \frac{1}{4\beta} = \frac{1}{12}$.

Example 5.3. *We consider the following non-homogeneous linear differential equation*

$$x''(t) + 2x'(t) + x(t) = t, \tag{5.3}$$

where $f(t) = t$ *is a function of exponential order and* $\alpha = 2, \beta = 1$. *Thus* $\lambda = \beta - \frac{\alpha^2}{4} = 0$.

If a continuously differentiable function $z : [0, \infty) \to \mathcal{F}$ *of exponential order satisfies*

$$|z''(t) + 2z'(t) + z(t) - t| \leq \varepsilon$$

for all $t \geq 0$ *and for some* $\varepsilon > 0$, *then Theorem 4.3-(i) implies that there exists a solution* $y : [0, \infty) \to \mathbf{K}$ *of the differential equation (5.3) such that*

$$|z(t) - y(t)| \leq K\varepsilon$$

for all $t \geq 0$, *where we set* $K = \frac{4}{\alpha^2} = 1$.

Remark 5.4. *The above examples are also true when we replace* ε *and* $K\varepsilon$ *with* $\phi(t)\varepsilon$ *and* $K\phi(t)\varepsilon$, *respectively, where* $\phi(t)$ *is an increasing function. In this case, we see that the corresponding differential equations have the Hyers-Ulam Rassias-stability.*

Remark 5.5. *The differential equations (5.1), (5.2) and (5.3) have the Mittag-Leffler-Hyers-Ulam stability if $\xi > 0$. In particular, they also have the Mittag-Leffler-Hyers-Ulam–Rassias stability when $\phi(t)$ is an increasing function and $\xi > 0$.*

Remark 5.6. *If $\alpha = 0$ and $\beta > 0$, then $4\beta > \alpha^2$ satisfies and there exists a positive constant γ such that $\beta = \gamma^2$. Then equation (3.1) coincides whit the equation (3.2) in [55]. Thus if $u^2 + \gamma^2 = (u - l)(u - m)$ with $l + m = 0$ and $lm = \gamma^2$ then the results of this paper coincides whit the results of [55]. If $\alpha = 0$ and $\beta < 0$, then $4\beta < \alpha^2$ satisfies and in this case we can use the Theorem 3.2 for equation (1.1) and Theorem 4.2 for equation (1.2) but as far as we know, no study has been done in the literature that responds to this situation. Furthermore, we also removed the conditions of the characteristic equation have two different positive roots which is given in [37].*

6. Conclusion

In this paper, we proved the Hyers-Ulam stability, Hyers-Ulam–Rassias stability, Mittag-Leffler-Hyers-Ulam stability, and Mittag-Leffler-Hyers-Ulam–Rassias stability of the linear differential equations of second order with constant coefficients having damping term using the Aboodh transform method. In other words, we established sufficient criteria for the Hyers-Ulam stability of second-order linear differential equations with constant coefficients having damping term using the Aboodh transform method.

Moreover, we provided a new method to investigate the Hyers-Ulam stability of differential equations. This is the first attempt to use the Aboodh transformation to prove the Hyers–Ulam stability for second-order linear differential equations with constant coefficients having damping term. Furthermore, we showed that the Aboodh transform method is more convenient for investigating the stability problems for linear differential equations with constant coefficients. In the present paper we take α and β are real constant. Readers can also apply this terminology to various problems on differential equations for α and β any scalars in the equations (1.1) and (1.2).

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