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Combination labelings of graphs related to several cycles and paths

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Abstract

Suppose that $G = (V(G), E(G))$ is a graph and $|V(G)| = p$. If there exists a bijective function $f : V(G) \to \{1, 2, 3, ..., p\}$ such that an $f^c : E(G) \to \mathbf{N}$ defined by $f^c(uv) = \binom{f(u)}{f(v)}$ when $f(u) > f(v)$ and $f^c(uv) = \binom{f(v)}{f(u)}$ when $f(v) > f(u)$ is an injection function, then f is called a combination labelings and G is called a combination graph. This article considers a suitable bijective function f and prove that $G(C_n, C_m, P_k)$ which are graphs related to two cycles and one path containing three parameters, are combination graphs

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1. Introduction

For a simple, connected, undirected graph $G = (V(G), E(G))$ several researchers have been studied a mathematical recreation problem, namely graph labeling. Usually graph labeling is a function from $V(G)$ or $E(G)$ to a set of numbers with some special properties. For a complete source of graph labelings, one can see from a dynamic survey by Gallian [1]. Not so long ago, Hedge and Shetty [2] define graph labelings called permutation, combination and strong k-combination labeling. If we can find a bijection $f : V(G) \to \{1, 2, 3, \ldots, |V(G)|\}$ such that, for $uv \in E(G)$,

(i) the induced $f^p(uv) = \begin{cases} {}^{f(u)}P_{f(v)} & \text{if } f(u) > f(v) \\ {}^{f(v)}P_{f(u)} & \text{if } f(v) > f(u) \end{cases}$ is injective, where ${}^aP_b$ denotes the number of permutations of a things taken b at a time, then f is called a *permutation labeling* for G and G is called a *permutation graph*; or

(ii) the induced $f^c(uv) = \begin{cases} \binom{f(u)}{f(v)} & \text{if } f(u) > f(v) \\ \binom{f(v)}{f(u)} & \text{if } f(v) > f(u) \end{cases}$ is injective, then f is called a *combination labeling* for G and G is called a *combination graph*; or

(iii) the induced $f^c(uv) = \begin{cases} \binom{f(u)}{f(v)} & \text{if } f(u) > f(v) \\ \binom{f(v)}{f(u)} & \text{if } f(v) > f(u) \end{cases}$ is injective and $f^c(E(G)) = \{k, k+1, k+2, \ldots, k+|E(G)|-1\}$ for some positive integer k, then f is called a *strong k-combination labeling* for G and G is called a *strong k-combination graph*.

Hedge and Shetty [2] proved that the complete graph K_n is a permutation graph if and only if $n \leq 5$, while, it is a combination graph if and only if $n \leq 2$. In [2], they gave a necessary condition for a graph to be a combination graph and also proved that the cycle C_n admits a combination labeling for all $n > 3$, the complete bipartite graph $K_{r,r}$ is a combination graph if and only if $r \leq 2$ and the wheel graph W_n is not a combination graph for all $n \leq 6$.

In 2012, Li [3] considered a large family of graphs which is a tree. He proved that for a rooted tree T with the property that the depth of any two leaf nodes are the same, T is a combination graph and the complete k-ary tree is also a combinatorial graph. He also explored some conditions on a caterpillar that made it to be a combination graph. For graphs containing cycles, he proved that the generalized Petersen Graph $GP(n,1)$ for $n \geq 4$ and $GP(n,2)$ for $n \geq 5$ are combination graph. He also continued the work of [2] by proving that if $n \geq 7$, then the wheel graph W_n is a combination graph. He gave a condition on n and k that implies the $k \times n$ grid graph to be combination graph. Conditions on the number of elements in each partite set also given to make sure that a complete k partite graph is a combination graph. Finally, he gave some results on the combination graph involving degree, $|V(G)|$ and $E(G)$.

In 2017, Thitiwatthanakan and Leeratanavalee [4] wrote an article in Thai language to prove that the generalized Petersen graphs $GP(n,3)$ and a lollipop graphs $H_{g,l}$ for some cases of g and l are combination graphs.

We can see from the lituratures that [2], [3] and [4] considered only graphs involving at most two parameters. From these motivation, we then try to construct a combination labeling for two families of graphs. The first one is $G(C_n, C_m, P_k)$ with three parameters n, m and k which is a graph containing two cycles with different sizes and a path with arbitrary length. The second one is $G_k(C_n)$ consisting of k cycles of the same size C_n each of which having one vertex incidents to one extra vertex.

Definition 1.1. *Let m, n and k be positive integers such that $m, n \geq 3$. The graph $G(C_n, C_m, P_k)$ consists of two cycles C_n and C_m and a path P_k connecting between these two cycle. That is* $V(G(C_n, C_m, P_k)) = \{u_1, u_2, u_3, \ldots, u_n\} \cup \{v_1, v_2, v_3, \ldots, v_m\} \cup \{s_1, s_2, s_3, \ldots, s_k\} = V(C_n) \cup V(C_m) \cup V(P_k)$ *and*
$E(G(C_n, C_m, P_k)) = \{u_1u_2, u_2u_3, u_3u_4, \ldots, u_nu_1\} \cup \{v_1v_2, v_2v_3, v_3v_4, \ldots, v_mv_1\} \cup \{s_1s_2, s_2s_3, s_3s_4, \ldots, s_{k-1}s_k\} \cup \{u_ns_1, s_kv_m\} = E(C_n) \cup E(C_m) \cup E(P_k) \cup \{u_ns_1, s_kv_m\}$.

Figure 1 shows $G(C_n, C_m, P_k)$ and the way we define each vertex's name.

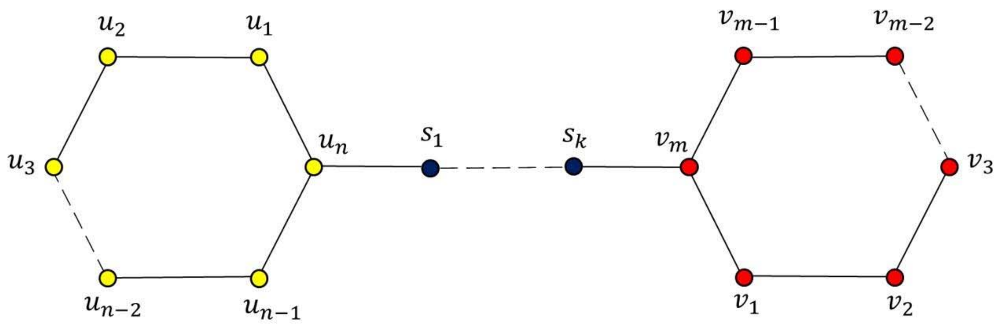

Figure 1: $G(C_n, C_m, P_k)$

Definition 1.2. *Let n and k be positive integers such that $n \geq 3$ and $k \geq 2$. The graph $G_k(C_n)$ consists of k cycles of the same size $C_n^{(i)}$'s each of which having one vertex incidents to one extra vertex. That is* $V(G_k(C_n)) = \left(\bigcup_{i=1}^{k}\{u_{i,1}, u_{i,2}, u_{i,3}, ..., u_{i,n}\}\right) \cup \{s\} = \left(\bigcup_{i=1}^{k} V(C_n^{(i)})\right) \cup \{s\}$ *and* $E(G_k(C_n)) = \left(\bigcup_{i=1}^{k} E(C_n^{(i)})\right) \cup \{u_{1,n}s, u_{2,n}s, u_{3,n}s, \ldots, u_{k,n}s\}$.

Note that $G_2(C_n)$ is $G(C_n, C_n, P_1)$. Figure 2 shows $G_k(C_n)$ and the way we define each vertex's name.

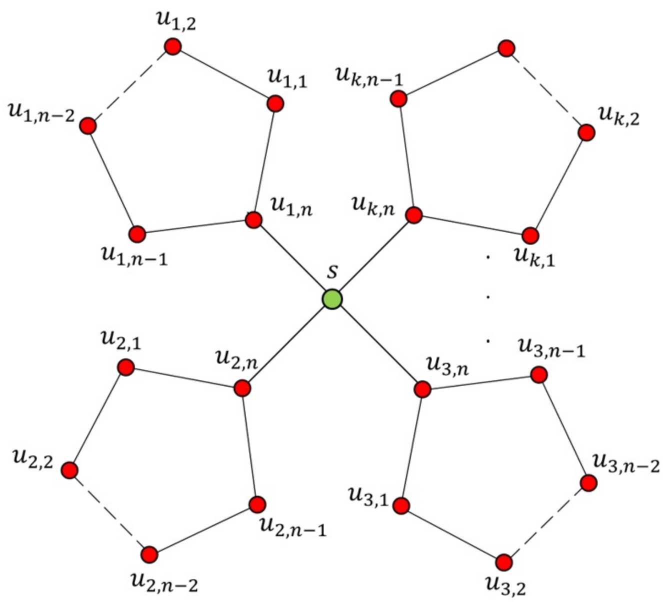

Figure 2: $G_k(C_n)$

In Section 2, some binomial inequalities that will be used to prove that the induced edge labelings are injective. Section 3 shows the vertex labelings which will be proved that they are combination labelings for the two families of graphs that we consider.

2. Some Binomial inequalities

Before we go to the main result, let us prove some inequalities involving binomial coefficients.

Lemma 2.1. *If $\alpha > \beta \geq 1$, then $\binom{\alpha+1}{\beta} > \binom{\alpha}{\beta}$ and $\binom{\alpha+1}{\beta+1} > \binom{\alpha}{\beta}$.*

Proof. Let $\alpha > \beta \geq 1$. Then, $\alpha + 1 > \alpha + 1 - \beta$ and $\alpha + 1 > \beta + 1$. Therefore, $(\alpha+1)!(\alpha-\beta)! > \alpha!(\alpha+1-\beta)!$ and $(\alpha+1)!\beta! > (\beta+1)!\alpha!$. That is $\binom{\alpha+1}{\beta} = \frac{(\alpha+1)!}{(\alpha+1-\beta)!\beta!} > \frac{\alpha!}{(\alpha-\beta)!\beta!} = \binom{\alpha}{\beta}$ and $\binom{\alpha+1}{\beta+1} = \frac{(\alpha+1)!}{(\alpha-\beta)!(\beta+1)!} > \frac{\alpha!}{(\alpha-\beta)!\beta!} = \binom{\alpha}{\beta}$. □

Lemma 2.2. *If $\alpha \geq 4$, then*

(i) $\binom{\alpha}{\beta} > \alpha$ for $2 \leq \beta \leq \alpha - 2$.

(ii) $\binom{\alpha+2}{2} > 3\alpha + 1$.

Proof. (i) Let $\alpha \geq 4$ and $2 \leq \beta \leq \alpha - 2$. Then, $1 < \beta - \gamma + 1 < \alpha - \gamma$ for γ such that $1 \leq \gamma \leq \beta - 1$. Thus,

$$(\alpha-1)(\alpha-2)(\alpha-3)\cdots(\alpha-\gamma-1) > \beta(\beta-1)(\beta-2)\cdots(\beta-\gamma).$$

Let $\gamma = \beta - 2$. Then,

$$(\alpha-1)(\alpha-2)(\alpha-3)\cdots(\alpha-\beta+1) > \beta(\beta-1)(\beta-2)\cdots(2) = \beta!.$$

Therefore, $\alpha(\alpha-1)(\alpha-2)(\alpha-3)\cdots(\alpha-\beta+1) > \alpha\beta!$, i.e., $\binom{\alpha}{\beta} > \alpha$.
(ii) Since $\alpha \geq 4$, direct computation gives $2\binom{\alpha+2}{2} = (\alpha+2)(\alpha+1) = \alpha^2 + 3\alpha + 2 \geq 4\alpha + 3\alpha + 2 > 2(3\alpha+1)$. □

Lemma 2.3. $\binom{\alpha+\beta}{\alpha-1} > \binom{\alpha+\beta-1}{\alpha} > \alpha + \beta + 1$ *for $\alpha \geq 4$ and $3 \leq \beta \leq \alpha - 1$.*

Proof. Let $\alpha \geq 4$ and $3 \leq \beta \leq \alpha - 1$. Then,

$$\binom{\alpha+\beta}{\alpha-1} = \frac{(\alpha+\beta)\alpha}{(\beta+1)\beta}\binom{\alpha+\beta-1}{\alpha} > \frac{(\beta+1)\alpha}{(\beta+1)\beta}\binom{\alpha+\beta-1}{\alpha} > \binom{\alpha+\beta-1}{\alpha}.$$

By Lemma 2.1, we have

$$\binom{\alpha+\beta-1}{\alpha} \geq \binom{\alpha+2}{\alpha} = \frac{(\alpha+2)(\alpha+1)}{2} > \frac{4\alpha+2}{2} = 2\alpha+1$$

Since $\beta \leq \alpha - 1$, $2\alpha + 1 > \alpha + \beta + 1$. Then, $\binom{\alpha+\beta-1}{\alpha} > \alpha + \beta + 1$. □

Lemma 2.4. *If $\alpha \geq 3$ and $\beta \geq 1$, then (i) $\binom{2\alpha+\beta}{2} \neq \binom{\alpha+\beta+1}{\beta+2}$ and (ii) $\binom{\alpha+\beta+1}{\beta+2} > 2\alpha + \beta$.*

Proof. Let $\alpha \geq 3$ and $\beta \geq 1$.
(i) We separate the proof into 9 cases as follows.

Case 1 $3 \leq \alpha \leq 9$ and $\beta = 1$. This case can be proved by direct calculation.

Case 2 $\alpha \geq 10$ and $\beta = 1$. Then,

$$\binom{\alpha+\beta+1}{\beta+2} = \binom{\alpha+2}{3} \geq \frac{12(\alpha+1)\alpha}{6} > \binom{2\alpha+1}{2} = \binom{2\alpha+\beta}{2}.$$

Case 3 $3 \leq \alpha \leq 4$ and $\beta = 2$. This case can be proved by direct calculation.

Case 4 $\alpha \geq 5$ and $\beta = 2$. Then,

$$\binom{\alpha+\beta+1}{\beta+2} = \binom{\alpha+3}{4} \geq \frac{(2\alpha+2)\left(2\alpha+\dfrac{\alpha}{3}\right)}{2} > \frac{(2\alpha+2)(2\alpha+1)}{2} = \binom{2\alpha+2}{2}.$$

Case 5 $\alpha \geq 4$ and $3 \leq \beta < \alpha$. We prove our assertion by using mathematical induction on α.

Basic step $\binom{\alpha+\beta+1}{\beta+2} = \binom{8}{5} = 56 > 55 = \binom{11}{2} = \binom{2\alpha+\beta}{2}$.

Inductive step Let $p, q \in \mathbf{Z}$ such that $p \geq 4$ and $3 \leq q < p$. Assume that $\binom{p+q+1}{q+2} > \binom{2p+q}{2}$. Let $r \in \mathbf{Z}$ such that $3 \leq r < p+1$. We claim in this step that $\binom{p+r+2}{r+2} > \binom{2p+r+2}{2}$.

If $3 \leq r < p$, then $\binom{p+r+2}{r+2} > \frac{p+r+2}{r+2}\binom{2p+r}{2}$. Direct computation gives, for $p \geq 4$ and $3 \leq r < p$, that $(p+r+2)(2p+r-1)(2p+r) > (r+2)(2p+r+1)(2p+r+2)$ which implies $\frac{p+r+2}{r+2} > \frac{(2p+r+1)(2p+r+2)}{(2p+r-1)(2p+r)}$. Thus,

$$\frac{p+r+2}{r+2}\left(\frac{(2p+r)!}{2!(2p+r-2)!}\right) > \frac{(2p+r+1)(2p+r+2)}{(2p+r-1)(2p+r)}\left(\frac{(2p+r)!}{2!(2p+r-2)!}\right)$$

$$= \binom{2p+r+2}{2}.$$

Therefore, $\binom{p+r+2}{r+2} > \binom{2p+r+2}{2}$.

If $r = p$, then we claim that $\binom{p+r+2}{r+2} = \binom{2p+2}{p+2} > \binom{3p+2}{2} = \binom{2p+r+2}{2}$ for $p \geq 4$ by using mathematical induction.

Basic step $\binom{2p+2}{p+2} = \binom{10}{6} = 210 > 91 = \binom{14}{2} = \binom{3p+2}{2}$.

Inductive step Let $t \in \mathbf{Z}$ such that $t \geq 4$. Assume that $\binom{2t+2}{t+2} > \binom{3t+2}{2}$. Then,

$$\binom{2t+4}{t+3} > \frac{(2t+4)(2t+3)}{(t+3)(t+1)}\binom{3t+2}{2}.$$

Direct computation shows, for $t \geq 4$, that $(2t+4)(2t+3)(3t+1)(3t+2) > 9t^4+63t^3+155t^2+161t+60$ which implies $\frac{(2t+4)(2t+3)}{(t+3)(t+1)} > \frac{(3t+4)(3t+5)}{(3t+1)(3t+2)}$. Thus,

$$\frac{(2t+4)(2t+3)}{(t+3)(t+1)}\left(\frac{(3t+2)!}{2!(3t)!}\right) > \frac{(3t+4)(3t+5)}{(3t+1)(3t+2)}\left(\frac{(3t+2)!}{2!(3t)!}\right) = \binom{3t+5}{2}.$$

Therefore, $\binom{2t+4}{t+3} > \binom{3t+5}{2}$.

Case 6 $\alpha = \beta = 3$. This case can be proved by direct calculation.

Case 7 $\alpha = \beta \geq 4$. We show that $\binom{\alpha+\beta+1}{\beta+2} = \binom{2\alpha+1}{\alpha+2} > \binom{3\alpha}{2} = \binom{2\alpha+\beta}{2}$ by using mathematical induction.

Basic step $\binom{2\alpha+1}{\alpha+2} = \binom{9}{6} = 84 > 66 = \binom{12}{2} = \binom{3\alpha}{2}$.

Inductive step Let $p \in \mathbf{Z}$ such that $p \geq 4$. Assume that $\binom{2p+1}{p+2} > \binom{3p}{2}$. Then,

$$\binom{2p+3}{p+3} > \frac{(2p+3)(2p+2)}{(p+3)p}\binom{3p}{2}.$$

Direct computation shows, for $p \geq 4$, that $(2p+3)(2p+2)(3p-1)(3p) > 9p^4 + 42p^3 + 51p^2 + 18p$ which implies $\frac{(2p+3)(2p+2)}{(p+3)p} > \frac{(3p+2)(3p+3)}{(3p-1)3p}$. Thus,

$$\frac{(2p+3)(2p+2)}{(p+3)p}\left(\frac{(3p)!}{2!(3p-2)!}\right) > \frac{(3p+2)(3p+3)}{(3p-1)3p}\left(\frac{(3p)!}{2!(3p-2)!}\right) = \binom{3p+3}{2}.$$

Therefore, $\binom{2p+3}{p+3} > \binom{3p+3}{2}$.

Case 8 $\beta > \alpha = 3$. By Lemma 2.1, we have $\binom{2\alpha+\beta}{2} = \binom{\beta+6}{2} > \binom{\beta+4}{2} = \binom{\beta+4}{\beta+2} = \binom{\alpha+\beta+1}{\beta+2}$.

Case 9 $\beta \geq 5$ and $4 \leq \alpha < \beta$. We prove our assertion by using mathematical induction on β.

Basic step $\binom{\alpha+\beta+1}{\beta+2} = \binom{10}{7} = 120 > 78 = \binom{13}{2} = \binom{2\alpha+\beta}{2}$.

Inductive step Let $p, q \in \mathbf{Z}$ such that $q \geq 5$ and $4 \leq p < q$. Assume that $\binom{p+q+1}{q+2} > \binom{2p+q}{2}$. Let $r \in \mathbf{Z}$ such that $4 \leq r < q+1$.

If $4 \leq r < q$, then $\binom{r+q+2}{q+3} > \frac{r+q+2}{q+3}\binom{2r+q}{2}$. Direct computation gives, for $q \geq 5$ and $4 \leq r < q$, that $(r+q+2)(2r+q-1) > q^2 + (2r+4)q + (6r+3)$ which implies $\frac{r+q+2}{q+3} > \frac{2r+q+1}{2r+q-1}$. Thus,

$$\frac{r+q+2}{q+3}\left(\frac{(2r+q)!}{2!(2r+q-2)!}\right) > \frac{2r+q+1}{2r+q-1}\left(\frac{(2r+q)!}{2!(2r+q-2)!}\right) = \binom{2r+q+1}{2}.$$

Therefore, $\binom{r+q+2}{q+3} > \binom{2r+q+1}{2}$.

If $r = q$, then we cliam that $\binom{r+q+2}{q+3} = \binom{2q+2}{q+3} > \binom{3q+1}{2} = \binom{2r+q+1}{2}$ for $q \geq 5$ by using mathematical induction.

Basic step $\binom{2q+2}{q+3} = \binom{12}{8} = 495 > 120 = \binom{16}{2} = \binom{3q+1}{2}$.

Inductive step Let $t \in \mathbf{Z}$ such that $t \geq 5$. Assume that $\binom{2t+2}{t+3} > \binom{3t+1}{2}$. Then,

$$\binom{2t+4}{t+4} > \frac{(2t+4)(2t+3)}{(t+4)t}\binom{3t+1}{2}.$$

Direct computation shows, for $t \geq 5$, that $(2t+4)(2t+3)(3t)(3t+1) > 9t^4 + 57t^3 + 96t^2 + 48t$ which implies $\frac{(2t+4)(2t+3)}{(t+4)t} > \frac{(3t+3)(3t+4)}{3t(3t+1)}$. Thus,

$$\frac{(2t+4)(2t+3)}{(t+4)t}\left(\frac{(3t+1)!}{2!(3t-1)!}\right) > \frac{(3t+3)(3t+4)}{3t(3t+1)}\left(\frac{(3t+1)!}{2!(3t-1)!}\right) = \binom{3t+4}{2}.$$

Therefore, $\binom{2t+4}{t+4} > \binom{3t+4}{2}$.

(ii) We separate the proof into 2 cases as follows.

Case 1 $\alpha \geq \beta$. By Lemma 2.1, we have $\binom{\alpha+\beta+1}{\beta+2} \geq \binom{\alpha+2}{3} \geq \frac{20\alpha}{6} > 3\alpha \geq 2\alpha + \beta$.

Case 2 $\beta > \alpha$. Then, $\beta \geq 4$. By Lemma 2.1, we have $\binom{\alpha+\beta+1}{\beta+2} \geq \binom{\beta+4}{\beta+2} > \frac{11\beta}{2} > 3\beta > 2\alpha + \beta$. □

Lemma 2.5. *If* $\alpha \geq 4$, $\beta \geq 3$, $\gamma \geq 2$ *and* $\alpha > \beta$, *then (i)* $\binom{\alpha+\gamma+1}{\gamma+2} \neq \binom{\alpha+\beta+\gamma}{2}$ *and (ii)* $\binom{\alpha+\gamma+1}{\gamma+2} > \alpha + \beta + \gamma$.

Proof. Let $\alpha \geq 4$, $\beta \geq 3$, $\gamma \geq 2$ and $\alpha > \beta$.

(i) We separate the proof into 5 cases as follows.

Case 1 $\alpha = \gamma$ and $\alpha > \beta$. Then, $\alpha \geq 4$, $\beta \geq 3$ and $\gamma \geq 4$. By Lemma 2.1, we have

$$\binom{\alpha+\beta+\gamma}{2} = \binom{2\alpha+\beta}{2} < \binom{3\alpha}{2} < \frac{10\alpha^2}{2} = 5\alpha^2.$$

We show that $\binom{\alpha+\gamma+1}{\gamma+2} = \binom{2\alpha+1}{\alpha+2} > 5\alpha^2$ for $\alpha \geq 4$ by using mathematical induction.

Basic step $\binom{2\alpha+1}{\alpha+2} = \binom{9}{6} = 84 > 80 = 5\alpha^2$.

Inductive step Let $p \in \mathbf{Z}$ such that $p \geq 4$. Assume that $\binom{2p+1}{p+2} > 5p^2$. Then,

$$\begin{aligned}\binom{2p+3}{p+3} &= \frac{(2p+3)(2p+2)}{(p+3)p}\binom{2p+1}{p+2} > \frac{(2p+3)(2p+2)(5p^2)}{(p+3)p} > \frac{(2p^2+6p)(5p^2)}{p^2+3p} \\ &= 5(2p^2) > 5(p+1)^2.\end{aligned}$$

Case 2 $\beta = \gamma$ and $\alpha > \beta$. Then, $\alpha \geq 4$, $\beta \geq 3$ and $\gamma \geq 3$. Therefore, $\binom{\alpha+\gamma+1}{\gamma+2} = \binom{\alpha+\beta+1}{\beta+2}$ and $\binom{\alpha+\beta+\gamma}{2} = \binom{\alpha+2\beta}{2}$. By Lemma 2.1, we have

$\binom{\alpha+\beta+1}{\beta+2} \geq \binom{\alpha+4}{5}$ and $\binom{\alpha+2\beta}{2} \leq \binom{3\alpha-2}{2}$. We show that $\binom{\alpha+4}{5} > \binom{3\alpha-2}{2}$, for $\alpha \geq 4$ by using mathematical induction.

Basic step $\binom{\alpha+4}{5} = \binom{8}{5} = 56 > 45 = \binom{10}{2} = \binom{3\alpha-2}{2}$.
Inductive step Let $p \in \mathbf{Z}$ such that $p \geq 4$. Assume that $\binom{p+4}{5} > \binom{3p-2}{2}$. Then,

$$\binom{p+5}{5} = \frac{p+5}{p}\binom{p+4}{5} > \frac{p+5}{p}\binom{3p-2}{2}.$$

Direct computation shows, for $p \geq 4$, that $(p+5)(3p-3)(3p-2) > 9p^3+3p^2$ which implies $\frac{p+5}{p} > \frac{(3p)(3p+1)}{(3p-3)(3p-2)}$. Thus,

$$\frac{p+5}{p}\left(\frac{(3p-2)!}{2!(3p-4)!}\right) > \frac{3p(3p+1)}{(3p-3)(3p-2)}\left(\frac{(3p-2)!}{2!(3p-4)!}\right) = \binom{3p+1}{2}.$$

Therefore, $\binom{p+5}{5} > \binom{3p+1}{2}$.

Case 3 $\alpha > \beta > \gamma$. Then, $\alpha \geq 4$, $\beta \geq 3$ and $\gamma \geq 2$.

Case 3.1 $\alpha = 4$, $\beta = 3$ and $\gamma = 2$. Then, $\binom{\alpha+\gamma+1}{\gamma+2} = \binom{7}{4} = 35 < 36 = \binom{9}{2} = \binom{\alpha+\beta+\gamma}{2}$.

Case 3.2 $\alpha \geq 5$, $\beta \geq 3$ and $\gamma \geq 2$. By Lemma 2.1, we have $\binom{\alpha+\gamma+1}{\gamma+2} \geq \binom{\alpha+3}{4}$ and $\binom{\alpha+\beta+\gamma}{2} \leq \binom{3\alpha-3}{2}$. We show that $\binom{\alpha+3}{4} > \binom{3\alpha-3}{2}$ for $\alpha \geq 5$ by using mathematical induction.

Basic step $\binom{\alpha+3}{4} = \binom{8}{4} = 70 > 66 = \binom{12}{2} = \binom{3\alpha-3}{2}$.

Inductive step Let $p \in \mathbf{Z}$ such that $p \geq 4$. Assume that $\binom{p+3}{4} > \binom{3p-3}{2}$. Then,

$$\binom{p+4}{4} = \frac{p+4}{p}\binom{p+3}{4} > \frac{p+4}{p}\binom{3p-3}{2}.$$

Direct computation gives, for $p \geq 5$, that $(p+4)(3p-4)(3p-3) > 9p^3 - 3p^2$ which implies $\frac{p+4}{p} > \frac{(3p-1)(3p)}{(3p-4)(3p-3)}$. Thus,

$$\frac{p+4}{p}\left(\frac{(3p-3)!}{2!(3p-5)!}\right) > \frac{(3p-1)3p}{(3p-4)(3p-3)}\left(\frac{(3p-3)!}{2!(3p-5)!}\right) = \binom{3p}{2}.$$

Therefore, $\binom{p+4}{4} > \binom{3p}{2}$.

Case 4 $\alpha > \gamma > \beta$. Then, $\alpha \geq 5$, $\beta \geq 3$ and $\gamma \geq 4$. By Lemma 2.1, we have, $\binom{\alpha+\gamma+1}{\gamma+2} \geq \binom{\alpha+5}{6} > \binom{\alpha+3}{4}$ and $\binom{\alpha+\beta+\gamma}{2} \leq \binom{3\alpha-3}{2}$. By using mathematical induction similar to Case 3, we have $\binom{\alpha+3}{4} > \binom{3\alpha-3}{2}$ for $\alpha \geq 5$.

Case 5 $\gamma > \alpha > \beta$. Then, $\alpha \geq 4$, $\beta \geq 3$ and $\gamma \geq 5$. By Lemma 2.1, we have $\binom{\alpha+\gamma+1}{\gamma+2} \geq \binom{\gamma+5}{\gamma+2} = \binom{\gamma+5}{3}$ and $\binom{\alpha+\beta+\gamma}{2} \leq \binom{3\gamma-3}{2}$. We show that $\binom{\gamma+5}{3} > \binom{3\gamma-3}{2}$ for $\gamma \geq 5$ by using mathematical induction.

Basic step For $\gamma \in \{5, 6, 7, 8, 9, 10, 11, 12\}$, direct computation gives $\binom{\gamma+5}{3} > \binom{3\gamma-3}{2}$.

Inductive step Let $p \in \mathbf{Z}$ such that $p \geq 12$. Assume that $\binom{p+5}{3} > \binom{3p-3}{2}$. Then,

$$\binom{p+6}{3} = \frac{p+6}{p+3}\binom{p+5}{3} > \frac{p+6}{p+3}\binom{3p-3}{2}.$$

Direct computation gives, for $p \geq 12$, that $(p+6)(3p-4)(3p-3) > 9p^3 + 24p^2 - 9p$ which implies $\frac{p+6}{p+3} > \frac{(3p-1)3p}{(3p-4)(3p-3)}$. Thus,

$$\frac{p+6}{p+3}\left(\frac{(3p-3)!}{2!(3p-5)!}\right) > \frac{(3p-1)3p}{(3p-4)(3p-3)}\left(\frac{(3p-3)!}{2!(3p-5)!}\right) = \binom{3p}{2}.$$

Therefore, $\binom{p+6}{3} > \binom{3p}{2}$.

(ii) By Lemma 2.1, we have

$$\binom{\alpha+\gamma+1}{\gamma+2} > \binom{\alpha+2}{3} \geq \frac{30\alpha}{6} > 3\alpha - 3 \geq \alpha + \beta + \gamma.$$

$\square$

Lemma 2.6. *If $\alpha \geq 4$, $\beta \geq 3$ and $\alpha \geq \beta$, then $\binom{2\alpha-1}{\alpha-1} > \beta\alpha + 1$.*

Proof. Since $\alpha \geq \beta$, $\alpha^2 + 1 \geq \beta\alpha + 1$. Thus, we will show that $\binom{2\alpha-1}{\alpha-1} > \alpha^2 + 1$ for $\alpha \geq 4$.

Basic step $\binom{2\alpha-1}{\alpha-1} = \binom{7}{3} = 35 > 17 = \alpha^2 + 1$.

Inductive step Let $p \in \mathbf{Z}$ such that $p \geq 4$. Assume that $\binom{2p-1}{p-1} > p^2 + 1$. Then,

$$\binom{2p+1}{p} = \frac{(2p+1)2p}{p(p+1)}\binom{2p-1}{p-1} > 2(p^2+1) > (p+1)^2 + 1.$$

$\square$

Lemma 2.7. *If $\alpha \geq 4$, $\beta \geq 3$ and $\alpha \geq \beta$, then*

(i) $\binom{\beta\alpha+1}{\alpha} < \binom{\beta\alpha+1}{\gamma\alpha}$ *for* $2 \leq \gamma \leq \beta - 1$.

(ii) $\binom{\beta\alpha+1}{\gamma\alpha} \neq \binom{\beta\alpha+1}{\delta\alpha}$ *for* $1 \leq \gamma, \delta \leq \beta - 1$ *and* $\gamma \neq \delta$.

Proof. (i) Since $\gamma \le \beta - 1$, we have $\gamma\alpha \le \beta\alpha - \alpha < \beta\alpha - \alpha + 1$. By Lemma 2.1, $\binom{\beta\alpha-\alpha+1}{\gamma\alpha-\alpha} > \binom{\gamma\alpha}{\gamma\alpha-\alpha}$ and thus, $\frac{\alpha!(\beta\alpha-\alpha+1)!}{(\gamma\alpha)!(\beta\alpha-\gamma\alpha+1)!} > 1$. Therefore,

$$\binom{\beta\alpha+1}{\gamma\alpha} = \left(\frac{\alpha!(\beta\alpha-\alpha+1)!}{(\gamma\alpha)!(\beta\alpha-\gamma\alpha+1)!}\right)\left(\frac{(\beta\alpha+1)!}{\alpha!(\beta\alpha-\alpha+1)!}\right) > \binom{\beta\alpha+1}{\alpha}.$$

(ii) Assume that $\binom{\beta\alpha+1}{\gamma\alpha} = \binom{\beta\alpha+1}{\delta\alpha}$. Without loss of generality, let $\gamma > \delta$.

<u>Case 1</u> $\gamma\alpha = \beta\alpha+1-\delta\alpha$. Then, $(\gamma+\delta-\beta)\alpha = 1$. Since $\alpha \ge 4$ and $\gamma+\delta-\beta$ is an integer, we obtain a contradiction.

<u>Case 2</u> $\gamma\alpha \ne \beta\alpha+1-\delta\alpha$. Without loss of generality, let $\gamma\alpha > \beta\alpha+1-\delta\alpha$. Since $\gamma \le \beta-1 < \beta$, we have $\gamma\alpha-\delta\alpha < \beta\alpha-\delta\alpha < \beta\alpha+1-\delta\alpha$. By Lemma 2.1, $\binom{\gamma\alpha}{\gamma\alpha-\delta\alpha} > \binom{\beta\alpha+1-\delta\alpha}{\gamma\alpha-\delta\alpha}$ and thus, $\dfrac{(\beta\alpha+1-\delta\alpha)!(\delta\alpha)!}{(\beta\alpha+1-\gamma\alpha)!(\gamma\alpha)!} < 1$. Therefore,

$$\binom{\beta\alpha+1}{\gamma\alpha} = \left(\frac{(\delta\alpha)!(\beta\alpha+1-\delta\alpha)!}{(\gamma\alpha)!(\beta\alpha-\gamma\alpha+1)!}\right)\left(\frac{(\beta\alpha+1)!}{(\delta\alpha)!(\beta\alpha+1-\delta\alpha)!}\right) < \binom{\beta\alpha+1}{\delta\alpha},$$

which is a contradiction. □

3. Main results

Now, we are ready to establish the main results for $G(C_n, C_m, P_k)$ and $G_k(C_n)$.

Theorem 3.1. *Let $n \ge 3$, $m \ge 3$ and $k \ge 1$. $G(C_n, C_m, P_k)$ is a combination graph.*

Proof. Let $n \ge 3$, $m \ge 3$, $k \ge 1$ and $G = G(C_n, C_m, P_k)$.

Case 1 $n = m$ and $k \ge 1$. Define $f : V(G) \to \{1, 2, 3, \ldots, 2n+k\}$ by

$$f(u_i) = \begin{cases} n+k+i+1 & \text{if } 1 \le i \le n-3 \\ 2n+k & \text{if } i = n-2 \\ 2n+k-1 & \text{if } i = n-1 \\ 1 & \text{if } i = n \end{cases},$$

$$f(v_i) = \begin{cases} k+i+2 & \text{if } 1 \le i \le n-1 \\ k+2 & \text{if } i = n \end{cases}$$

$$\text{and} \quad f(s_i) = i+1; 1 \le i \le k.$$

It can be seen easily that f is a bijective function. Now, $f^c : E(G) \to \mathbf{N}$ can be written as follows.

$$f^c(u_iu_{i+1}) = \begin{cases} \binom{n+k+i+2}{n+k+i+1} & \text{if } 1 \le i \le n-4 \\ \binom{2n+k}{2n+k-2} & \text{if } i = n-3 \\ \binom{2n+k}{2n+k-1} & \text{if } i = n-2 \\ \binom{2n+k-1}{1} & \text{if } i = n-1 \end{cases},$$

$$f^c(v_iv_{i+1}) = \begin{cases} \binom{k+i+3}{k+i+2} & \text{if } 1 \le i \le n-2 \\ \binom{n+k+1}{k+2} & \text{if } i = n-1 \end{cases},$$

$$f^c(u_1u_n) = \binom{n+k+2}{1}, f^c(v_1v_n) = \binom{k+3}{k+2}, f^c(u_ns_1) = \binom{2}{1}, f^c(v_ns_k)$$
$$= \binom{k+2}{k+1} \quad \text{and} \quad f^c(s_is_{i+1}) = \binom{i+2}{i+1}; 1 \le i \le k-1.$$

Notice that

$$\begin{aligned} \{f^c(u_iu_{i+1})\}_{i=1}^{n-1} &= \{n+k+3, n+k+4, n+k+5, \ldots, \\ &\quad 2n+k-3, 2n+k-2, \tbinom{2n+k}{2}, 2n+k, 2n+k-1\Big\}, \end{aligned}$$

$$\{f^c(v_iv_{i+1})\}_{i=1}^{n-1} = \Big\{k+4, k+5, k+6, \ldots, n+k-1, n+k, n+k+1, \tbinom{n+k+1}{k+2}\Big\},$$
$$f^c(u_1u_n) = n+k+2, f^c(v_1v_n) = k+3, f^c(u_ns_1) = 2, f^c(v_ns_k) = k+2$$

$$\text{and} \quad \Big\{f^c(s_is_{i+1})\}_{i=1}^{k-1} = 3, 4, 5, \ldots, k-1, k, k+1\Big\}.$$

Some of f^c values can be seen that they are distinct and they can even be ordered as follows.

$$2 < 3 < \cdots < k < k{+}1 < k{+}2 < k{+}3 < k{+}4 < \cdots < n{+}k{-}1 < n{+}k < n{+}k{+}1$$

$$< n{+}k{+}2 < n{+}k{+}3 < \cdots < 2n{+}k{-}3 < 2n{+}k{-}2 < 2n{+}k{-}1 < 2n{+}k.$$

In addition, by Lemmas 2.2 and 2.4, we can conclude that

$$2n+k < \binom{2n+k}{2}, 2n+k < \binom{n+k+1}{k+2} \text{ and } \binom{2n+k}{2} \neq \binom{n+k+1}{k+2}.$$

Therefore, f^c is an injective function and G is a combination graph.

Case 2 $n \neq m$ and $k = 1$. Without loss of generality, let $n > m$. Define $f : V(G) \to \{1, 2, 3, ..., n+m+1\}$ by

$$f(u_i) = \begin{cases} i+1 & \text{if } 1 \leq i \leq n-2 \\ n+m & \text{if } i = n-1 \\ 1 & \text{if } i = n \end{cases}, f(v_i) = \begin{cases} n+i & \text{if } 1 \leq i \leq m-1 \\ n & \text{if } i = m \end{cases}$$

$$\text{and } f(s_1) = n+m+1.$$

It can be seen easily that f is a bijective function. Now, $f^c : E(G) \to \mathbf{N}$ can be written as follows.

$$f^c(u_iu_{i+1}) = \begin{cases} \binom{i+2}{i+1} & \text{if } 1 \leq i \leq n-3 \\ \binom{n+m}{n-1} & \text{if } i = n-2 \\ \binom{n+m}{1} & \text{if } i = n-1 \end{cases},$$

$$f^c(v_iv_{i+1}) = \begin{cases} \binom{n+i+1}{n+i} & \text{if } 1 \leq i \leq m-2 \\ \binom{n+m-1}{n} & \text{if } i = m-1 \end{cases}, f^c(u_1u_n) = \binom{2}{1},$$

$$f^c(v_1v_m) = \binom{n+1}{n}, f^c(u_ns_1) = \binom{n+m+1}{1}$$

$$\text{and } f^c(v_ms_1) = \binom{n+m+1}{n}.$$

Notice that

$$\begin{aligned} \{f^c(u_iu_{i+1})\}_{i=1}^{n-1} &= \left\{3, 4, 5, ..., n-2, n-1, \tbinom{n+m}{n-1}, n+m\right\}, \\ \{f^c(v_iv_{i+1})\}_{i=1}^{m-1} &= \left\{n+2, n+3, ..., n+m-2, n+m-1, \tbinom{n+m-1}{n}\right\}, \\ f^c(u_1u_n) &= 2, f^c(v_1v_m) = n+1, f^c(u_ns_1) = n+m+1 \\ &\text{ and } f^c(v_ms_1) = \tbinom{n+m+1}{n}. \end{aligned}$$

Since $n \geq 3$, $m \geq 3$ and $n > m$, $3 \leq m \leq n-1$ and $n \geq 4$. By Lemmas 2.2 and 2.3, the values of f^c are all distinct and actually can be ordered as follows.

$$2 < 3 < \cdots < n-1 < n+1 < \cdots < n+m-1 < n+m < n+m+1$$

$$< \binom{n+m-1}{n} < \binom{n+m}{n-1} < \binom{n+m+1}{n}.$$

Therefore, f^c is an injective function and G is a combination graph.

Case 3 $n \neq m$ and $k \geq 2$. Without loss of generality, let $m > n$. Define $f : V(G) \to \{1, 2, 3, ..., n+m+k\}$ by

$$f(u_i) = \begin{cases} m+k+i+1 & \text{if } 1 \leq i \leq n-3 \\ n+m+k & \text{if } i = n-2 \\ n+m+k-1 & \text{if } i = n-1 \\ 1 & \text{if } i = n \end{cases},$$

$$f(v_i) = \begin{cases} k+i+2 & \text{if } 1 \leq i \leq m-1 \\ k+2 & \text{if } i = m \end{cases}$$

$$\text{and } f(s_i) = i+1, 1 \leq i \leq k.$$

It can be seen easily that f is a bijective function. Now, $f^c : E(G) \to \mathbf{N}$ can be written as follows.

$$f^c(u_iu_{i+1}) = \begin{cases} \binom{m+k+i+2}{m+k+i+1} & \text{if } 1 \leq i \leq n-4 \\ \binom{n+m+k}{n+m+k-2} & \text{if } i = n-3 \\ \binom{n+m+k}{n+m+k-1} & \text{if } i = n-2 \\ \binom{n+m+k-1}{1} & \text{if } i = n-1 \end{cases},$$

$$f^c(v_iv_{i+1}) = \begin{cases} \binom{k+i+3}{k+i+2} & \text{if } 1 \leq i \leq m-2 \\ \binom{m+k+1}{k+2} & \text{if } i = m-1 \end{cases},$$

$$f^c(u_1u_n) = \binom{m+k+2}{1}, f^c(v_1v_m) = \binom{k+3}{k+2}, f^c(u_ns_1) = \binom{2}{1},$$

$$f^c(v_ms_k) = \binom{k+2}{k+1} \text{ and } f^c(s_is_{i+1}) = \binom{i+2}{i+1}; 1 \leq i \leq k-1.$$

Notice that

$$\begin{aligned} \{f^c(u_iu_{i+1})\}_{i=1}^{n-1} &= \Big\{m+k+3, m+k+4, ..., n+m+k-4, \\ &\quad n+m+k-3, n+m+k-2, \\ &\quad \tbinom{n+m+k}{2}, n+m+k, n+m+k-1\Big\}, \\ \{f^c(v_iv_{i+1})\}_{i=1}^{m-1} &= \Big\{k+4, k+5, ..., m+k-1, m+k, m+k+1, \tbinom{m+k+1}{k+2}\Big\}, \\ f^c(u_1u_n) &= m+k+2, f^c(v_1v_m) = k+3, f^c(u_ns_1) = 2, \\ &\quad f^c(v_ms_k) = k+2 \text{ and} \\ \{f^c(s_is_{i+1})\}_{i=1}^{k-1} &= \{3, 4, 5, ..., k-1, k, k+1\}. \end{aligned}$$

Some of f^c values can be seen that they are distinct and they can even be ordered as follows.

$$2 < 3 < \cdots < k < k+1 < k+2 < k+3 < k+4 < \cdots < m+k-1 < m+k$$
$$< m+k+1 < m+k+2$$
$$< m+k+3 < \cdots < n+m+k-4 < n+m+k-3 < n+m+k-2 < n+m+k-1$$
$$< n+m+k.$$

In addition, since, in this case, we assume that $m > n \geq 3$, we have $m \geq 4$. By Lemmas 2.2 and 2.5, we can conclude that

$$n+m+k < \binom{m+k+1}{k+2}, n+m+k < \binom{n+m+k}{2} \text{ and } \binom{m+k+1}{k+2}$$
$$\neq \binom{n+m+k}{2}.$$

Therefore, f^c is an injective function and G is a combination graph. □

Theorem 3.2. *If (i) $n = k = 3$ or (ii) $n \geq 4$ and $n \geq k \geq 3$ or (iii) $n = 3$ and $k \geq 4$, then $G_k(C_n)$ is a combination graph.*

Proof. Let $n \geq 3$, $k \geq 2$ and $G = G_k(C_n)$.

Case 1 $n = k = 3$. We show that G is a combination graph by illustrating the vertex and edge labelings in the following Figure 3.

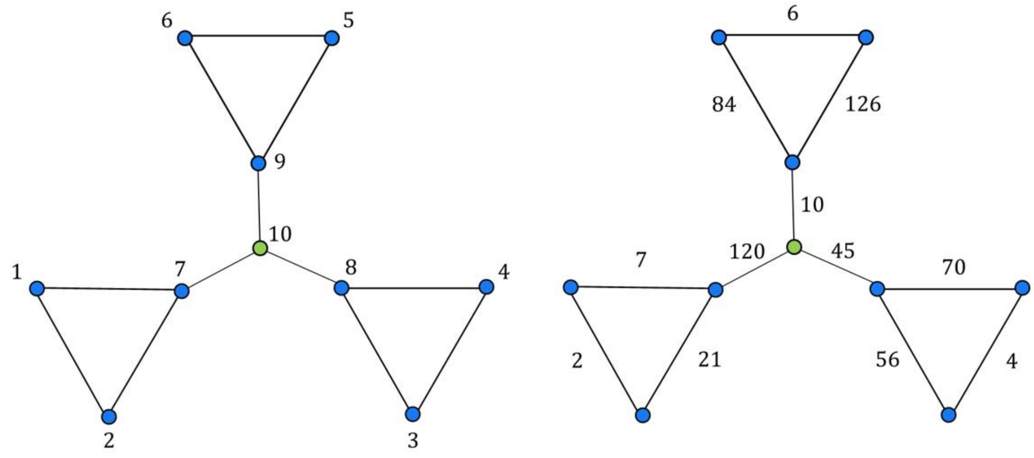

Figure 3: vertex labeling and edge labeling for $G_3(C_3)$

Case 2 $n \geq 4$, $k \geq 3$ and $n \geq k$. Define $f : V(G) \to \{1, 2, 3, ..., kn + 1\}$ by

$$f(u_{i,j}) = \begin{cases} in + j - n & \text{if } 1 \leq i \leq k \text{ and } j \neq n \\ kn & \text{if } i = 1 \text{ and } j = n \\ in - n & \text{if } 2 \leq i \leq k \text{ and } j = n \end{cases} \quad \text{and } f(s) = kn + 1.$$

It can be seen easily that f is a bijective function. Now, $f^c : E(G) \to \mathbf{N}$ can be written as follows.

$$f^c(u_{i,j}u_{i,j+1}) = \begin{cases} \binom{in+j-n+1}{in+j-n} & \text{if } 1 \leq i \leq k, 1 \leq j \leq n-2 \\ \binom{kn}{n-1} & \text{if } i = 1, j = n-1 \\ \binom{in-1}{in-n} & \text{if } 2 \leq i \leq k, j = n-1 \end{cases},$$

$$f^c(u_{i,1}u_{i,n}) = \begin{cases} \binom{kn}{1} & \text{if } i = 1 \\ \binom{in-n+1}{in-n} & \text{if } 2 \leq i \leq k \end{cases} \text{ and}$$

$$f^c(u_{i,n}s) = \begin{cases} \binom{kn+1}{kn} & \text{if } i = 1 \\ \binom{kn+1}{in-n} & \text{if } 2 \leq i \leq k \end{cases}.$$

Notice that

$$\begin{aligned}\left(\bigcup_{i=1}^{n}\{f^c(u_{i,j}u_{i,j+1})\}_{j=1}^{n-2}\right) & \cup\{f^c(u_{i,1}u_{i,n})\}_{i=1}^{k}\cup\{f^c(u_{1,n}s)\}\\ &=\{2,3,...,n-1,n+1,n+2,...,\\ &\quad 2n-1,2n+1,2n+2,...,kn-n+1,\\ &\quad kn-n+2,...,kn-1,kn,kn+1\},\\ f^c(u_{i,n-1}u_{i,n})\}_{i=1}^{k} &=\left\{\tbinom{2n-1}{n-1},\tbinom{3n-1}{n-1},...,\tbinom{kn-1}{n-1},\tbinom{kn}{n-1}\right\}\ \text{ and}\\ f^c(u_{i,n}s)\}_{i=2}^{k} &=\left\{\tbinom{kn+1}{n},\tbinom{kn+1}{2n},...,\tbinom{kn+1}{kn-n}\right\}.\end{aligned}$$

Some of f^c values can be seen that they are distinct and they can even be ordered as follows.

$$2<3<\cdots<n-1<n+1<n+2<\cdots<2n-1<2n+1<2n+2<\cdots$$

$$<kn-n+1<kn-n+2<\cdots<kn-1<kn<kn+1$$

In addition, by using Lemma 2.6 we obtain the first inequality and by Lemma 2.1 we obtain the followings inequalities

$$kn+1<\binom{2n-1}{n-1}<\binom{3n-1}{n-1}<\cdots<\binom{kn-1}{n-1}<\binom{kn}{n-1}<\binom{kn+1}{n}.$$

Finally, by Lemma 2.7, we have $\binom{kn+1}{n}<\binom{kn+1}{in}$ for $2\leq i\leq k-1$ and $\binom{kn+1}{in}\neq\binom{kn+1}{ln}$ for $1\leq i,l\leq k-1$ and $i\neq l$. Therefore, f^c is an injective function and G is a combination graph.

Case 3 $n=3$ and $k\geq 4$. Define $f:V(G)\to\{1,2,3,...,3k+1\}$ by

$$f(u_{i,j})=\begin{cases}k+2i+j-1 & \text{if} \leq i\leq k \text{ and } j\neq 3\\ i+1 & \text{if } 1\leq i\leq k \text{ and } j=3\end{cases}\quad\text{and}\quad f(s)=1.$$

It can be seen easily that f is a bijective function. Now, $f^c:E(G)\to\mathbf{N}$ can be written as follows. For $1\leq i\leq k$,

$$f^c(u_{i,1}u_{i,2})=\binom{k+2i+1}{k+2i}, f^c(u_{i,1}u_{i,3})=\binom{k+2i}{i+1},$$

$$f^c(u_{i,2}u_{i,3})=\binom{k+2i+1}{i+1}\text{ and } f^c(u_{i,3}s)=\binom{i+1}{1}.$$

Notice that

$f^c(u_{i,1}u_{i,2})_{i=1}^k \cup f^c(u_{i,3}s)\}_{i=1}^k$
$= \{2, 3, ...i, ..., k+1, k+3, k+5, ..., k+2(i-1)+1, ..., 3k+1\}$ and
$f^c(u_{i,1}u_{i,3})_{i=1}^k \cup f^c(u_{i,2}u_{i,3})_{i=1}^k$
$= \left\{ \binom{k+2}{2}, \binom{k+3}{2}, \binom{k+4}{3}, \binom{k+5}{3}, ..., \binom{k+2(i-1)}{i}, \ \binom{k+2(i-1)+1}{i}, ..., \binom{3k}{k+1}, \binom{3k+1}{k+1} \right\}.$

Some of f^c values can be seen that they are distinct and they can even be ordered as follows.

$2 < 3 < \cdots < i < \cdots < k+1 < k+3 < k+5 < \cdots < k+2(i-1)+1 < \cdots < 3k+1.$

In addition, by using Lemmas 2.1 and 2.2, we obtain

$$\begin{aligned} 3k+1 < \binom{k+2}{2} < \binom{k+3}{2} < \binom{k+4}{3} < \binom{k+5}{3} \\ < \cdots < \binom{k+2(i-1)}{i} < \binom{k+2(i-1)+1}{i} \\ < \cdots < \binom{3k}{k+1} < \binom{3k+1}{k+1}. \end{aligned}$$

Therefore, f^c is an injective function and G is a combination graph. □

4. Conclusion and discussion

We construct vertex labelings for $\mathrm{G}_(C_n, C_m, P_k)$ and prove that they are combination labeling. For $G(C_n)$, under some conditions of n and k, we can prove that it is a combination graph. We expect in the future that we may be able to construct a combination labeling for $G_k(C_n)$ in the case that $k > n \geq 4$ as well as generalize the result of $G_k(C_n)$ so that it can be a combination labeling for the graph that consists of cycles of different sizes or may contain a longer path.

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