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Invariant bilinear forms under the operator group of order p^3 with odd prime p

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Abstract

For an odd prime p, we formulate the number of all degree n representations of a group of order p^3. And calculating the dimension of space of invariant bilinear forms corresponding to degree n representation over a field $\mathbf{F}$ *which contains a primitive p^3 root of unity. Here we also explicitly discussed the existence of a non-degenerate invariant bilinear form of the same space.*

Keyword: *Bilinear forms, Representation theory, Vector spaces, Direct sums, Semi direct product.*

MSC [2020]: *15A04, 15A63, 20C15.*

1. Introduction

One of the important roles of Representation theory is the study of a group as operators on certain vector spaces. These operators behave as an orthogonal (unitary) group with respect to a corresponding bilinear form. Generating new non-degenerate invariant bilinear forms attracted many researcher due to its important in quantum mechanics and other branches of physical sciences.

Let G be a group and $\mathbf{V}$ a vector space over a field $\mathbf{F}$, then we have following.

Definition 1.1. *A homomorphism $\rho : G \to GL(\mathbf{V})$ is called a representation of the group G. $\mathbf{V}$ is also called a representing space of G. The dimension of $\mathbf{V}$ over $\mathbf{F}$ is called degree of the representation ρ.*

Definition 1.2. *A class function is a map $f : G \to \mathbf{F}$ so that $f(g) = f(h)$ if g is a conjugate to h in G.*

Definition 1.3. *A bilinear form on a finite dimensional vector space $\mathbf{V}(\mathbf{F})$ is said to be invariant under the representation ρ of a finite group G if*

$$\mathbf{B}(\rho(g)x, \rho(g)y) = \mathbf{B}(x, y), \ \forall\ g \in G \text{ and } x, y \in \mathbf{V}(\mathbf{F}).$$

For the basic properties of a bilinear form, we may refer to [9].
Let Ξ denotes the space of bilinear forms on the vector space $\mathbf{V}$ over $\mathbf{F}$ and $\mathcal{C}_{\mathbf{F}}(G)$, the set of all class functions on G.

Definition 1.4. *The set of invariant bilinear forms under the representation ρ is given by*

$$\Xi_G = \{\mathbf{B} \in \Xi \mid \mathbf{B}(\rho(g)x, \rho(g)y) = \mathbf{B}(x, y), \ \forall\ g \in G \ \text{ and } \ x, y \in \mathbf{V}\}.$$

Assume that $\mathbf{F}$ consists of a primitive $|G|^{th}$ root of unity. The representation (ρ, $\mathbf{V}$) [13], is irreducible of degree n if and only if $\{0\}$ and $\mathbf{V}$ are the only invariant sub spaces of $\mathbf{V}$ under the representation ρ. The class function $\mathcal{C}_{\mathbf{F}}(G)$ is a vector space over $\mathbf{F}$ with dimension r, where r is the number of conjugacy classes of G. By the Frobenius ([1], Theorem 5.9, p. 318) there are r irreducible representations ρ_i (say), $1 \leq i \leq r$ of G and χ_i (say) the character corresponding to ρ_i. Also by Maschke's theorem 2.1, we have a degree n representation $\rho = \oplus_{i=1}^{r} k_i\rho_i$ of G, the coefficient k_i of ρ_i, $1 \leq i \leq r$, so that $\sum_{i=1}^{r} d_i k_i = n$, and $\sum_{i=1}^{r} d_i^2 = |G|$, where d_i is the degree of ρ_i and $d_i||G|$ with $d_j \geq d_i$ when $j > i$. It is already well understood in the literature that the invariant space Ξ_G under ρ can be expressed by the set $\Xi'_G = \{X \in \mathbf{M}_n(\mathbf{F}) \,|\, C^t_{\rho(g)} X C_{\rho(g)} = X, \forall g \in G\}$ with respect to an ordered basis $\underline{e}$ of $\mathbf{V}(\mathbf{F})$, where $\mathbf{M}_n(\mathbf{F})$ is the set of square matrices of order n with entries from $\mathbf{F}$ and $C_{\rho(g)}$ is the matrix representation of the linear transformation $\rho(g)$ with respect to $\underline{e}$.

If we consider G to be a group of order p^3 with p an odd prime, $\mathbf{F}$ a field with char $(\mathbf{F}) \neq p$, which consists of a primitive p^3th root of unity and (ρ, $\mathbf{V}$), degree n representation of G over $\mathbf{F}$. Then obviously the set Ξ_G, space of invariant bilinear forms on $\mathbf{V}$ corresponding to ρ, forms a subspace of Ξ. In the view of considered group and bilinear space, in this paper, our main focus to investigate and answers the following three questions:

1. How many degree n representations (up to isomorphism) of G can be there?

2. What is the dimension of Ξ_G for every degree n representation?

3. What are the necessary and sufficient conditions for the existence of a non-degenerate invariant bilinear form?

The existence of a non-degenerate invariant bilinear form over the field of complex numbers is well stabilised results. Also it is well known that every maximal (proper) subgroup of G with index p is normal (For finite p, groups are nilpotent and any proper subgroup of a nilpotent group is properly contained in its normalizer). So it is evident that there are epimorphisms from G to the cyclic group $\mathbf{Z}_p$.

Now, fixing a generator c of $\mathbf{Z}_p$ and $1 \neq \zeta$ a primitive pth-root of unity. Let U and V be the one-dimensional representations of $\mathbf{Z}_p$ on which c acts respectively by ζ and $\bar{\zeta}$. We claim that $U \oplus V$ admits a $\mathbf{Z}_p$-invariant non-degenerate bilinear form. Using some suitable epimorphism to $\mathbf{Z}_p$ one can pull-back these representations and the forms to G.

For proving the claim: considering the vectors $0 \neq u \in U$ and $0 \neq v \in V$. Using these vector spaces, we define a bilinear form B on $U \oplus V$ as follows: $B(u,u) = 0 = B(v,v)$, $B(u,v) = 1 = B(v,u)$ so that $B(\lambda_1 u + \mu_1 v, \lambda'_1 u + \mu'_1 v) = \lambda_1 \mu'_1 + \lambda'_1 \mu_1$.

Now we may easily check the $\mathbf{Z}_p$ invariance, as follows: $B(c(\lambda_1 u + \mu_1 v), c(\lambda'_1 u + \mu'_1 v)) = B(\zeta(\lambda_1 u + \mu_1 v), \bar{\zeta}(\lambda'_1 u + \mu'_1 v)) = \zeta\bar{\zeta}\lambda_1\mu'_1 + \zeta\bar{\zeta}\lambda'_1\mu_1 = B(\lambda_1 u + \mu_1 v, \lambda'_1 u + \mu'_1 v)$.

The questions in concern have been studied in several literatures regards to distinct contexts. Gongopadhyay and Kulkarni [6], studied the existence of T-invariant non-degenerate symmetric (resp. skew-symmetric) bilinear forms. Gongopadhyay, Mazumder and Sardar [8], investigated for an invertible linear map T on V, admit a T -invariant non-degenerate c-hermitian form. Chen [2], discussed the all matrix representation of the real numbers. Kulkarni and Tanti [10], formulated the dimension of space of T-invariant bilinear forms. Sergeichuk [14], studied systems of forms and linear mappings by associating with them self-adjoint representations of a category with involution. Frobenius [5], proved that every endomorphism of a finite dimensional vector space V is self-adjoint for at least one non-degenerate symmetric bilinear form on V. Later, Stenzel [12], determined when an endomorphism could be skew- self adjoint for a non-degenerate quadratic form, or self-adjoint or skew-self adjoint for a symplectic form on complex vector spaces. However his results were later generalized to an arbitrary field [7]. Pazzis [11], tackled the case of the automorphisms of a finite dimensional vector space that are orthogonal (resp. symplectic) for at least one non-degenerate quadratic form (resp. symplectic form) over an arbitrary field of characteristics 2.

We present our study and outcomes based on above mentioned three questions in the following three main theorems.

Theorem 1.1. *The number of degree n representations (up to isomorphism) of a group G of order p^3, with p an odd prime is $\binom{n+p^3-1}{p^3-1}$ when G is abelian and $\sum_{\mu=0}^{\lfloor \frac{n}{p} \rfloor} \binom{\mu+p-2}{p-2}\binom{n-\mu p+p^2-1}{p^2-1}$ otherwise.*

Theorem 1.2. *The space Ξ_G of invariant bilinear forms of a group G of order p^3 (p an odd prime), under degree n representation $(\rho, \mathbf{V}(\mathbf{F}))$ is isomorphic to the direct sum of the sub spaces $\mathbf{W}_{(i,j)}$, $(i,j) \in A_G$ of $\mathbf{M}_n(\mathbf{F})$, i.e., $\Xi'_G = \bigoplus_{(i,j)\in A_G} \mathbf{W}_{(i,j)}$, where $A_G = \{(i,j) \mid \rho_i$ and ρ_j are dual to each other$\}$ and for every $(i,j) \in A_G$, $\mathbf{W}_{(i,j)} = \{X \in \mathbf{M}_n(\mathbf{F}) \mid X^{ij}_{d_ik_i\times d_jk_j} = C^t_{k_i\rho_i(g)} X^{ij}_{d_ik_i\times d_jk_j} C_{k_j\rho_j(g)}, \forall g \in G$ and rest blocks are zeros$\}$. Also for $(i,j) \in A_G$, the dimension of $\mathbf{W}_{(i,j)} = k_ik_j$.*

Theorem 1.3. *If G is group of order p^3, with p an odd prime, then degree n representation of G consists of a non-degenerate invariant bilinear form if and only if every irreducible representation and its dual have the same multiplicity.*

Remark 1.1. *Thus we get the necessary and sufficient condition for the existence of a non-degenerate invariant bilinear form under degree n representation.*

2. Preliminaries

The classification of groups of order p^3, with p an odd prime has been well documented in the several literatures. Due to the structure theorem of finite abelian groups, there are only three abelian groups (up to isomorphism) of this order viz, $\mathbf{Z}_{p^3}$, $\mathbf{Z}_{p^2} \times \mathbf{Z}_p$ and $\mathbf{Z}_p \times \mathbf{Z}_p \times \mathbf{Z}_p$. Amongst non-abelian groups of this order, Heisenberg group [3] is well known and named after a German theoretical physicist Werner Heisenberg. In this group every non identity element is of order p. The elements of this group are usually seen in the form of 3×3 upper triangular matrices whose diagonal entries consist of 1 and other three entries are chosen from the finite field $\mathbf{Z}_p$. If there exists any other non-abelian group of this order then it must have a non identity element of order p^2. Let us consider the 2×2 upper triangular matrices with $a_{11} = 1 + pm, (m \in \mathbf{Z}_p)$, $a_{12} = a \in \mathbf{Z}_{p^2}$ and $a_{22} = 1$. Here the element with entries $a_{11} = a_{12} = a_{22} = 1$ has order p^2 making it non-isomorphic to the Heisenberg group. We denote this group by G_p. Thus up to isomorphism there are five groups of order p^3 with an odd prime p [3]. For an abelian group of order p^3, there are p^3 number of irreducible representations each having degree 1 and for non-abelian cases, the number of trivial conjugacy classes is $|Z(G)| = p$. To find a non-trivial conjugacy class we refer to the theory of group action and class equation

$$|G| = |Z(G)| + \sum_{g \notin Z(G) \text{ varying over distinct conjugacy classes}} |C_g|$$

with $|C_g| = \frac{|G|}{|C(g)|}$, where C_g and $C(g)$ are the conjugacy class and the centralizer respectively of g in G. If $g \notin Z(G)$ then $|C(g)| = p^2$. Therefore there are $p^2 - 1$ non trivial conjugacy classes of order p. Thus total number of conjugacy classes for a non-abelian group is $r = p^2 - 1 + p$, which is same as the number of irreducible representations with degree d_i. We also have $d_i||G|$ and $\sum_{i=1}^{r} d_i^2 = |G|$, therefore $d_i = 1$ or p. So we conclude that there are p^2 number of degree 1 representations and $p - 1$ number of degree p representations. We here formulate every irreducible representation ρ_i in such a way that the entries of $C_{\rho_i(g)}$ are either 0 or p^3th primitive roots of unity.

Definition 2.1. *The character of ρ is a function $\chi : G \rightarrow \mathbf{F}$, $\chi(g) = tr(\rho(g))$ and is also called character of the group G.*

Theorem 2.1. *(Maschke's Theorem): If char($\mathbf{F}$) does not divide $|G|$, then every representation of G is a direct sum of irreducible representations.*

Proof. See ([1], Corollary 4.9, p. 316). □

Theorem 2.2. *Two representations $(\rho, \mathbf{V}(\mathbf{F}))$ and $(\rho', \mathbf{V}(\mathbf{F}))$ of G are isomorphic if and only if their character values are same i.e, $\chi(g) = \chi'(g)$ for all $g \in G$.*

Proof. See ([1], Corollary 5.13, p. 319). □

3. Irreducible representation (irrep.) of group of order p^3 with an odd prime p.

In this section, G is a group of order p^3 with p an odd prime, $(\rho_i, \mathbf{W}_{\rho_i})$ stands for an irreducible representation ρ_i of degree d_i of G over a field $\mathbf{F}$ with char($\mathbf{F}$) $\neq p$, which consists of $\omega \in \mathbf{F}$, a primitive p^3th root of unity. Let $\sigma_s = \rho_{p^2+s}$, $1 \leq s \leq p-1$ denote the irreducible representations of degree p when G is non-abelian. Since σ_s is a homomorphism from G to $GL(\mathbf{W}_{\sigma_s}) \cong GL(p, \mathbf{F})$, by the fundamental theorem of homomorphism $\frac{G}{Ker(\sigma_s)} \cong \sigma_s(G)$ and the possible value of $|Ker(\sigma_s)|$ is 1 or p. If $|Ker(\sigma_s)| = p$ then Hermitian inner product $\langle \chi_s, \chi_s \rangle \geq 1$, therefore $\langle \chi_s, \chi_s \rangle = 1$ only when $g \notin Ker(\sigma_s)$, so we have $\chi_s(g) = 0$. Also we have trivial character $\chi_i(g) = 1, \forall g \in G$ and $\langle \chi_i, \chi_s \rangle = \frac{p^2}{p^3} \neq 0$, which fails the orthonormality property of the irreducible characters. Thus $|Ker(\sigma_s)| = 1$ and hence $\sigma_s(G) \cong \frac{G}{\{e_G\}}$ which is isomorphic to a non-abelian group of order p^3. Now $\sigma_s(G)$ has subgroups H and K of order p and p^2 respectively ([4], Exercise 29, p. 132). Since K is a maximal subgroup of $\sigma_s(G)$ so K must be a normal subgroup and there exists a subgroup H_p of order p which is not normal (not contained in $Z(\sigma_s(G))$) ([4], Theorem 1, p. 188), thus we have, $\sigma_s(G) = KH_p$ and every element of $\sigma_s(G)$ can be expressed uniquely in the form of kh for some $k \in K$ and $h \in H_p$ (this uniqueness follows from the condition $K \cap H_p = \{\sigma_s(e_G)\}$), therefore $\sigma_s(Heis(\mathbf{Z}_p))$ is isomorphic to semidirect product of $(\mathbf{Z}_p \times \mathbf{Z}_p)$ and $\mathbf{Z}_p$ and G_p is isomorphic to semidirect product of $\mathbf{Z}_{p^2}$ and $\mathbf{Z}_p$. As $|Z(G)| = p$, we can choose a subgroup of order p from $GL(p, \mathbf{F})$ and say that it is the Image of $Z(G)$

(subset of a normal subgroup of order p^2 of G) denoted by $\sigma_s(Z(G)) = \{\omega^{sp^2} I_p \mid 1 \leq s \leq p\}$ under the irreducible representation σ_s (since center elements commute and are scalar matrices). Thus $\sigma_s(Z(G))(\subsetneq \sigma_s(K)) \cong \mathbf{Z}_p$, each non-identity element of $Z(G)$ have $p-1$ choices in $\mathbf{Z}_p$ under σ_s and rest $p^3 - p$ elements of G map to rest $p^3 - p$ elements of $\sigma_s(G)$. We decide all degree p representations by the elements of $Z(G)$ and elements of $G - Z(G)$ by mapping to the set $\sigma_s(G) - \sigma_s(Z(G)) \subseteq GL(p, \mathbf{F})$, which consists of those elements whose trace is zero and order of every element is either p or p^2. We will depict all irreducible representations of G in the next subsections.

3.1. Heisenberg group.

$$G = Heis(\mathbf{Z}_p) = \left\{ ((\alpha, \beta), \gamma) = \begin{bmatrix} 1 & \alpha & \beta \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{bmatrix} \mid \alpha, \beta, \gamma \in \mathbf{Z}_p \right\}.$$

Another presentation of Heisenberg group is ([4], p. 179)

$$G = \langle x, a \mid x^p = a^p = 1,\ x(a^{-1}xa)x^{-1} = a^{-1}xa,\ a(xax^{-1})a^{-1} = xax^{-1} \rangle.$$

For the Heisenberg group, center $Z(G) = \langle xax^{-1}a^{-1} \rangle = \langle ((0,1),0) \rangle$. Each of the irreducible representations σ_s, $1 \leq s \leq p-1$ of degree p maps the center of G to the center of $GL(p, \mathbf{F})$, i.e., if $z \in Z(G)$, $\sigma_s(z)$ is a scalar matrix say $c_s I_p$, $c_s \in \mathbf{F}$, and order of z is p so $c_s^p = 1$. Hence $c_s \in \{1,\ \omega^{p^2} = \epsilon,\ \epsilon^2,\ \cdots,\ \epsilon^{(p-1)}\}$. Thus each of the $p-1$ irreducible representations of degree p maps $Z(G)$ into the $Z(GL(p, \mathbf{F}))$ even maps to the $Z(\sigma_s(G))$, it has been recorded in the following table.

Table 3.1: All degree p irreducible representations of $Heis(\mathbf{Z}_p)$

All irrep. $\rightarrow$	$\sigma_{2\eta-1}$	$\sigma_{2\eta}$
Runing variable $\rightarrow$	$1 \leq \eta \leq \frac{p-1}{2}$,	
$g \in Z(G)$	$\epsilon^{\eta} I_p$	$\epsilon^{(p-\eta)} I_p$
$g \notin Z(G)$	See Note 3.1	See Note 3.1
Dual irrep.	$\sigma_{2\eta-1}^{*} = \sigma_{2\eta}$	
# irrep.	$p-1$	

Note 3.1. *For degree p representations σ_s of the Heisenberg group G, if $xax^{-1}a^{-1} \in Z(G)$ then $\sigma_s(xax^{-1}a^{-1}) = \rho_{p^2+s}(xax^{-1}a^{-1}) = \epsilon^m I_p$, for some m, $1 \leq m < p$ and the elements of $G - Z(G)$ get mapped bijectively to the following set $\sigma_s(G) - \sigma_s\Big(Z(G)\Big)$*

$$= \Big\{A_\nu \in GL(p, \mathbf{F}) \mid Tr(A_\nu) = 0 \text{ and } A_\nu^p = I_p \text{ for } 1 \leq \nu \leq p^3 - p\Big\}.$$

Thus the degree p irreducible representations σ_s can be expressed as below

$$\sigma_{2\eta-1}(x) = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}^{\eta} \text{ and } \sigma_{2\eta-1}(a) = \begin{bmatrix} \epsilon^{\eta} & 0 & 0 & 0 & \cdots & 0 \\ 0 & \epsilon^{\eta+1} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \epsilon^{\eta+2} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \epsilon^{\eta+3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \epsilon^{\eta+p-1} \end{bmatrix},$$

$$\sigma_{2\eta}(x) = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}^{-\eta} \text{ and } \sigma_{2\eta}(a) = \begin{bmatrix} \epsilon^{p+1-\eta} & 0 & 0 & 0 & \cdots & 0 \\ 0 & \epsilon^{p+2-\eta} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \epsilon^{p+3-\eta} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \epsilon^{p+4-\eta} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \epsilon^{p-\eta} \end{bmatrix}.$$

Non-abelian groups of order p^3 have p^2 representations of degree one. Let $\sigma_{(s-1,t-1)}$, $1 \leq s, t \leq p$, denote representations of degree 1. Here we present all degree one representations for the Heisenberg group.

Table 3.2: All irreducible representations of degree 1 for the Heisenberg group.

All irrep. $\rightarrow$	$\rho_{(t-1)p+2s-2} = \sigma_{(s-1,t-1)}$	$\rho_{(t-1)p+2s-1} = \sigma_{(p-s+1,p-t+1)}$	$\rho_{(t-1)p+2s-1} = \sigma_{(s-1,t-1)}$	$\rho_{(t-1)p+2s} = \sigma_{(p-s+1,p-t+1)}$
Running Variable $\rightarrow$	$1 \leq s \leq \frac{p+1}{2}, t = 1$		$1 \leq s \leq p,\ 2 \leq t \leq \frac{p+1}{2}$	
$\begin{bmatrix} 1 & 0 & \beta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\epsilon^p = 1$	1	1	1
$\begin{bmatrix} 1 & \alpha & 0 \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{bmatrix}$	$\epsilon^{\alpha(s-1)}$	$\epsilon^{\alpha(p-s+1)}$	$\epsilon^{\{\alpha(s-1)+\gamma(t-1)\}}$	$\epsilon^{\{\alpha(p-s+1)+\gamma(p-t+1)\}}$
Dual irrep.	$\rho^*_{2s-2} = \rho_{2s-1}$ & $\rho_0 = \rho_1$		$\rho^*_{(t-1)p+2s-1} = \rho_{(t-1)p+2s}$	
# irrep.	p		$p^2 - p$	

Since $1 \leq s, t \leq p$, and $\alpha, \gamma \in \mathbf{Z}_p$, so $\epsilon^{(\alpha(s-1)+\gamma(t-1))}$ is generated by ϵ, thus the representation $\sigma_{(s-1,t-1)}$ maps an element $g \in G$ to ϵ^m, for some m, $1 \leq m \leq p$.

3.2. The non-abelian group G_p

$$G_p = \left\{ (p\gamma, \delta) = \begin{bmatrix} 1 + p\gamma & \delta \\ 0 & 1 \end{bmatrix}, \mid \ \gamma \in \mathbf{Z}_p,\ \delta \in \mathbf{Z}_{p^2} \right\}.$$

Another presentation of this group is ([4], p. 180)

$$G_p = \langle x, y \mid x^p = y^{p^2} = 1, xy = y^{p+1}x \rangle.$$

The center of G_p is $Z(G_p) = \left\langle y^p = \begin{bmatrix} 1+p & 0 \\ 0 & 1 \end{bmatrix} \right\rangle$.

Note 3.2. *For degree p representations σ_s of group G_p, if $y^p \in Z(G_p)$ then $\sigma_{2\eta-1}(y^p) = \rho_{p^2+2\eta-1}(y^p) = \omega^{\eta p^2} I_p$ and its dual $\sigma_{2\eta}(y^p) = \rho_{p^2+2\eta}(y^p) = \omega^{(p-\eta)p^2} I_p$, $1 \leq \eta \leq \frac{p-1}{2}$. Also the elements of $G_p - Z(G_p)$ map bijectively to the set*

$$\sigma_s(G_p) - \sigma_s\Big(Z(G_p)\Big) = \{A_\nu \in GL(p, \mathbf{F}) Tr(A_\nu) = 0 \text{ and } A_\nu^p = I_p \text{or } A_\nu^{p^2} = I_p \text{ for } 1 \leq \nu \leq p^3 - p\}.$$

We have recorded all degree p irreducible representations of G_p in the following table.

Table 3.3: All irreducible representations of degree p for the group G_p.

All irrep. $\to$	$\sigma_{2\eta-1}$	$\sigma_{2\eta}$
Runing variable $\to$	$1 \le \eta \le \frac{p-1}{2}$,	
$g \in Z(G_p)$	$\omega^{\eta p^2} I_p$	$\omega^{(p-\eta)p^2} I_p$
$g \notin Z(G_p)$	See Note 3.2	See Note 3.2
Dual irrep.	$\sigma^*_{2\eta-1} = \sigma_{2\eta}$	
# irrep.	$p-1$	

Thus the degree p irreducible representations σ_s is defined as below

$$\sigma_{2\eta-1}(x) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}^{\eta} \quad and \sigma_{2\eta-1}(y) = \begin{bmatrix} \omega^{\eta p} & & 0 & \cdots & 0 \\ 0 & \omega^{p^2+\eta p} & 0 & \cdots & 0 \\ 0 & 0 & \omega^{2p^2+\eta p} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \omega^{(p-1)p^2+\eta p} \end{bmatrix},$$

$$\sigma_{2\eta}(x) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}^{-\eta} \quad and \, \sigma_{2\eta}(y) = \begin{bmatrix} \omega^{p^2-\eta p} & & 0 & \cdots & 0 \\ 0 & \omega^{2p^2-\eta p} & 0 & \cdots & 0 \\ 0 & 0 & \omega^{3p^2-\eta p} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \omega^{p^3-\eta p} \end{bmatrix}.$$

There are p^2 representations $\sigma_{(s-1,t-1)}, 1 \le s, t \le p$ of degree one. As $g \in G_p - Z(G_p)$, we have $g = (p\gamma, \delta) = \begin{bmatrix} 1+p\gamma & \delta \\ 0 & 1 \end{bmatrix}$, $\delta \in \mathbf{Z}^*_{p^2}$, $|G_p - Z(G_p)| = (p^2-1)p$. In the table 3.4 we have recorded all degree one representations of G_p.

Table 3.4: All irreducible representations of degree 1 for the group G_p.

All irrep. $\rightarrow$	$\rho_{(t-1)p+2s-2} = \sigma_{(s-1,t-1)}$	$\rho_{(t-1)p+2s-1} = \sigma_{(p-s+1,p-t+1)}$	$\rho_{(t-1)p+2s-1} = \sigma_{(s-1,t-1)}$	$\rho_{(t-1)p+2s} = \sigma_{(p-s+1,p-t+1)}$
Running Variable $\rightarrow$	$1 \leq s \leq \frac{p+1}{2}, t = 1$		$1 \leq s \leq p,\ 2 \leq t \leq \frac{p+1}{2}$	
$y^{p\gamma} = \begin{bmatrix} 1+p\gamma & 0 \\ 0 & 1 \end{bmatrix}$	$\omega^{p^3} = 1$	1	1	1
$\begin{bmatrix} 1+p\gamma & \delta \\ 0 & 1 \end{bmatrix}, \delta \neq 0$	$\omega^{p^2\gamma(s-1)}$	$\omega^{p^2\gamma(p-s+1)}$	$\omega^{p^2\{\gamma(s-1)+\delta'(t-1)\}}$ where $\delta' = \delta \bmod p$	$\omega^{p^2\{\gamma(p-s+1)+\delta'(p-t+1)\}}$ Where $\delta' = \delta \bmod p$
Dual irrep.	$\rho^*_{2s-2} = \rho_{2s-1}$ & $\rho_0 = \rho_1$		$\rho^*_{(t-1)p+2s-1} = \rho_{(t-1)p+2s}$	
# irrep.	p		$p^2 - p$	

Here $\omega^{p^2(\gamma(s-1)+\delta'(t-1))}$ is generated by ω^{p^2}, thus the representation $\sigma_{(s-1,t-1)}$ maps an element $g \in G_p$ to ω^{mp^2}, for some m, $1 \leq m \leq p$.

Note 3.3. *As a finite abelian group is finitely generated, here for the abelian groups $\mathbf{Z}_{p^3}$, $\mathbf{Z}_{p^2} \times \mathbf{Z}_p$ and $\mathbf{Z}_p \times \mathbf{Z}_p \times \mathbf{Z}_p$ there exist the finite generating subsets $\{a\}$, $\{b, c\}$ and $\{d, e, f\}$ respectively. For example we may take $a = 1 + p^3\mathbf{Z}$, $b = (1 + p^2\mathbf{Z}, 0 + p\mathbf{Z})$, $c = (0 + p^2\mathbf{Z}, 1 + p\mathbf{Z})$, $d = (1 + p\mathbf{Z}, 0 + p\mathbf{Z}, 0 + p\mathbf{Z})$, $e = (0 + p\mathbf{Z}, 1 + p\mathbf{Z}, 0 + p\mathbf{Z})$ and $f = (0 + p\mathbf{Z}, 0 + p\mathbf{Z}, 1 + p\mathbf{Z})$. Further order of an element g under $\rho_i, 1 \leq i \leq r$ is $|\rho_i(g)|$.*

3.3. The cyclic group $\mathbf{Z}_{p^3}$.

Here $\mathbf{Z}_{p^3} = \langle a \,|\, a^{p^3} = 1 \rangle$. The representation tables are given as below.

Table 3.5: All irreducible representations of the group $\mathbf{Z}_{p^3}$, with an odd prime p.

$T_{\mathbf{Z}_{p^3}} =$

All irrep. $\rightarrow$	ρ_t	ρ_{p^s-1+2t}	$\rho_{p^s-1+2t+1}$
Running variable $\rightarrow$	$t=1$ $i=1$	$\left\{ i \mid gcd(p^3, ip^{3-s}) = p^{3-s},\ 1 \le i \le \frac{p(p^s-p^{s-1}-2)}{2(p-1)} + 1 \right\}$ $t=$ position of i (ordered in acending order) in above set	
$\lvert\rho_i(a)\rvert$	1	$p^s,\ s=1,2,3$	
a	$\omega^{p^3}=1$	$\omega^{ip^{3-s}}$,	$\omega^{p^3-ip^{3-s}}$
Dual irrep.	self	$\rho^*_{p^s-1+2t} = \rho_{p^s-1+2t+1}$	
# irrep.	1	$p^s - p^{s-1}$	

,

Table 3.6: All irreducible representations of the group $\mathbf{Z}_8$.

$T_{\mathbf{Z}_8} =$

	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	ρ_6	ρ_7	ρ_8
a	ω^8	ω^4	ω^2	ω^6	ω	ω^7	ω^3	ω^5

.

3.4. The group $\mathbf{Z}_{p^2} \times \mathbf{Z}_p$.

Here $\mathbf{Z}_{p^2} \times \mathbf{Z}_p = \langle a, b \mid a^{p^2} = b^p = 1, ab = ba \rangle$.

$$T_{\mathbf{Z}_{p^2}\times\mathbf{Z}_p} =$$

Table 3.7: All irreducible representations of group $\mathbf{Z}_{p^2} \times \mathbf{Z}_p$, with an odd prime p.

All irrep.$\to$	$\sigma_{(s,t)}$	$\sigma_{(p^2,t)}$	$\sigma_{(p^2,p-t)}$	$\sigma_{(s,p^2)}$	$\sigma_{(p-s,p^2)}$	$\sigma_{p^2(s,t)}$	$\sigma_{p^2(p-s,p-t)}$	$\sigma_{(s,t)}$	$\sigma_{(p^2-s,p-t)}$	$\sigma_{(s,p)}$	$\sigma_{(p^2-s,p)}$
Running variable $\to$	$s = t = p$	$1 \leq t \leq \frac{p-1}{2}$		$1 \leq s \leq \frac{p-1}{2}$		$t \in \{1, \cdots, p-1\}$ & $1 \leq s \leq \frac{p-1}{2}$ or $s \in \{1, \cdots, p-1\}$ & $1 \leq t \leq \frac{p-1}{2}$		$t \in \{1, \cdots, p-1\}$ & $1 \leq s \leq \frac{p^2-p}{2}$ or $s \in \{1, \cdots, p^2-p\}$ & $1 \leq t \leq \frac{p-1}{2}$		$1 \leq s \leq \frac{p^2-p}{2}$	
$\lvert\rho_i(a)\rvert$	1	1		p		p		p^2		p^2	
$\lvert\rho_i(b)\rvert$	1	p		1		p		p		1	
a	1	1	1	ω^{sp^2}	$\omega^{(p-s)p^2}$	ω^{sp^2}	$\omega^{(p-s)p^2}$	ω^{sp}	$\omega^{(p^2-s)p}$	ω^{sp}	$\omega^{(p^2-s)p}$
b	1	ω^{tp^2}	$\omega^{(p-t)p^2}$	1	1	ω^{tp^2}	$\omega^{(p-t)p^2}$	ω^{tp^2}	$\omega^{(p-t)p^2}$	1	1
Dual irrep.	self	$\sigma^*_{(p^2,t)} = \sigma_{(p^2,p-t)}$		$\sigma^*_{(s,p^2)} = \sigma_{(p-s,p^2)}$		$\sigma^*_{p^2(s,t)} = \sigma_{p^2(p-s,p-t)}$		$\sigma^*_{(s,t)} = \sigma_{(p^2-s,p-t)}$		$\sigma^*_{(s,p)} = \sigma_{(p^2-s,p)}$	
# irrep.	1	$p-1$		$p-1$		$(p-1)(p-1)$		$(p^2-p)(p-1)$		p^2-p	

Table 3.8: All irreducible representations of group $\mathbf{Z}_4 \times \mathbf{Z}_2$.

$T_{\mathbf{Z}_4\times\mathbf{Z}_2} =$

	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	ρ_6	ρ_7	ρ_8
a	ω^8	ω^8	ω^4	ω^4	ω^2	ω^6	ω^2	ω^6
b	ω^8	ω^4	ω^8	ω^4	ω^4	ω^4	ω^8	ω^8

3.5. The group $\mathbf{Z}_p \times \mathbf{Z}_p \times \mathbf{Z}_p$.

Here $\mathbf{Z}_p \times \mathbf{Z}_p \times \mathbf{Z}_p = \langle a, b, c \,|\, a^p = b^p = c^p = 1, ab = ba, ac = ca, bc = cb\rangle$. The corresponding tables are given by

$T_{\mathbf{Z}_p\times\mathbf{Z}_p\times\mathbf{Z}_p} =$

Table 3.9: All irreducible representations of $\mathbf{Z}_p \times \mathbf{Z}_p \times \mathbf{Z}_p$, with an odd prime p.

3-tuples → (s,t,m)	All places are same (s,s,s)		Exactly two places are same $(s,s,m),(s,m,s),(m,s,s), m \neq s$		All places are distinct $(s,t,m),\ s \neq t \neq m \neq s$	
All irrep. →	$\sigma_{(s,s,s)}$	$\sigma_{(p-s,p-s,p-s)}$	$\sigma_{(s,s,m)}$	$\sigma_{(p-s,p-s,p-m)}$	$\sigma_{(s,t,m)}$	$\sigma_{(p-s,p-t,p-m)}$
Running variable →	$\frac{p+1}{2} \leq s \leq p$		$s = p\ \&\ 1 \leq m \leq \frac{p-1}{2}$ or $1 \leq s \leq \frac{p-1}{2}\ \&\ m \in \{1,2,\cdots,p\}$		$1 \leq s,t,m \leq p,$	
a	ω^{sp^2}	$\omega^{(p-s)p^2}$	ω^{sp^2}	$\omega^{(p-s)p^2}$	ω^{sp^2}	$\omega^{(p-s)p^2}$
b	ω^{sp^2}	$\omega^{(p-s)p^2}$	ω^{sp^2}	$\omega^{(p-s)p^2}$	ω^{tp^2}	$\omega^{(p-t)p^2}$
c	ω^{sp^2}	$\omega^{(p-s)p^2}$	ω^{mp^2}	$\omega^{(p-m)p^2}$	ω^{mp^2}	$\omega^{(p-m)p^2}$
Twice repeated irrep. →	$\sigma_{(p,p,p)} = \sigma_{(0,0,0)}$ Only trivial irrep. repeated twice		No irrep. repeated twice		$\sigma_{(s,t,m)} = \sigma_{(p-s_1,p-t_1,p-m_1)}$ if $s+s_1 = t+t_1 = m+m_1 = p$ or $2p$, Each irrep. repeated twice	
Dual irrep.	$\sigma^*_{(s,s,s)} = \sigma_{(p-s,p-s,p-s)}$		$\sigma^*_{(s,s,m)} = \sigma_{(p-s,p-s,p-m)}$		$\sigma^*_{(s,t,m)} = \sigma_{(p-s,p-t,p-m)}$	
Distinct # irrep.	p		$3p(p-1)$		$p(p-1)(p-2)$	

Table 3.10 All irreducible representations of $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$.

$T_{\mathbf{Z}_2\times\mathbf{Z}_2\times\mathbf{Z}_2} =$

	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	ρ_6	ρ_7	ρ_8
a	ω^8	ω^4	ω^8	ω^8	ω^8	ω^4	ω^4	ω^4
b	ω^8	ω^4	ω^4	ω^4	ω^8	ω^4	ω^8	ω^8
c	ω^8	ω^4	ω^8	ω^4	ω^4	ω^8	ω^8	ω^4

.

Note 3.4. *In a degree n representation ρ, the presence of an irreducible representation ρ_{2i} and its dual $\rho^*_{2i} = \rho_{2i+1}$, with multiplicities are adjacent to each other. Also the notation $\sigma_{(S,T)}$ (irreducible representation) used in Table 3.7 (resp. $\sigma_{(s,t,m)}$ in Table 3.9) is corresponds to ρ_{2i}, for some unique i, $1 \leq i \leq \frac{r-1}{2}$ and so the dual $\sigma^*_{(S,T)}$ (resp. $\sigma^*_{(s,t,m)}$) corresponds to ρ_{2i+1}. The trivial representation corresponds to ρ_1.*

Now

$$\rho = k_1\rho_1 \oplus k_2\rho_2 \oplus \cdots \oplus k_r\rho_r, \tag{3.1}$$

where for every $1 \leq i \leq r$, $k_i\rho_i$ stands for the direct sum of k_i copies of the irreducible representation ρ_i.
Let χ be the character corresponding to the representation ρ, then

$$\chi = k_1\chi_1 + k_2\chi_2 + \cdots + k_r\chi_r,$$

where χ_i is the character of ρ_i, $\forall\ 1 \leq i \leq r$. Degree of the character χ is being calculated at the identity element of the group. i.e,

$$deg(\rho) = \chi(1) = tr(\rho(1)).$$

$$\Longrightarrow d_1k_1 + d_2k_2 + \cdots + d_rk_r = n. \tag{3.2}$$

Note 3.5. *Equation (3.2) holds in more general situation, which helps us to find all possible distinct r-tuples $(k_1, k_2, \cdots, k_r)$, which correspond to the distinct degree n representations (up to isomorphism) of a given finite group.*

Theorem 3.1. *Let G be a group of order p^3 with p an odd prime. If σ is an irreducible representation of G of degree p, then σ is a faithful representation.*

Proof. For non-abelian groups proof is completes from the Tables 3.1 and 3.3 in the subsections 3.1 and 3.2. For an abelian group there is no irreducible representation of degree p. $\square$

4. Existence of non-degenerate invariant forms

It is known fact that an element in the space of invariant bilinear forms under representation of a finite group is either non-degenerate or degenerate. When $k_{2i} \neq k_{2i+1}$ then all the elements of space are degenerate and such a space is called a degenerate invariant space. It is a matter of investigation that how many such representations exist out of total representations. As some of the spaces contain both non-degenerate and degenerate invariant bilinear forms under a particular representation. In this section, we compute the number of such representations of the group G of order p^3, with p an odd prime.

The next remark will be use consistently to prove rest of the lemmas and theorems. The results of the Remark 4.1, is obtained from Section 3, Frobenius- Schur indicator and Schur lemma [[1] Theorem 9.6, p. 326, [13], p. 13, 106-108].

Remark 4.1. *The space Ξ'_G of invariant bilinear forms under degree n representation ρ contains only those $X \in \mathbf{M}_n(\mathbf{F})$ whose $(i,j)^{th}$ block is a $\mathbf{0}$ (zero) sub-matrix of order $d_ik_i \times d_jk_j$ when $(i,j) \notin A_G = \{(i,j) \mid \rho_i$ and ρ_j are dual to each other$\}$ whereas for $(i,j) \in A_G$ with $d_i = d_j = 1$, the block matrix $X^{ij}_{d_ik_i\times d_jk_j}$ is given by*

$$X^{ij}_{d_ik_i\times d_jk_j} = X^{ij}_{k_i\times k_j} = \begin{bmatrix} x^{ij}_{1,1} & x^{ij}_{1,2} & \cdots & x^{ij}_{1,k_j} \\ x^{ij}_{2,1} & x^{ij}_{2,2} & \cdots & x^{ij}_{2,k_j} \\ \vdots & \vdots & \ddots & \vdots \\ x^{ij}_{k_i,1} & x^{ij}_{k_i,2} & \cdots & x^{ij}_{k_i,k_j} \end{bmatrix}.$$

And for $(i,j) \in A_{G_p}$ with $d_i = d_j = p$ it is,

$$X^{ij}_{pk_i\times pk_j} = \begin{bmatrix} x^{ij}_{1,p}J & x^{ij}_{1,2p}J & \cdots & x^{ij}_{1,pk_j}J \\ x^{ij}_{p+1,p}J & x^{ij}_{p+1,2p}J & \cdots & x^{ij}_{p+1,pk_j}J \\ \vdots & \vdots & \ddots & \vdots \\ x^{ij}_{(k_i-1)p+1,p}J & x^{ij}_{(k_i-1)p+1,2p}J & \cdots & x^{ij}_{(k_i-1)p+1,pk_j}J \end{bmatrix},$$

where $J = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \end{bmatrix}_{p\times p}.$

Lemma 4.1. *If $X \in \Xi'_G$, and $k_{2i} \neq k_{2i+1}$, then X must be singular.*

Proof. With reference to the above remark and Note 3.4, for every $X \in \Xi'_G$, we have $X = [X^{ij}_{d_ik_i \times d_jk_j}]_{(i,j)\in A_G}$. I.e.

$$X = \begin{bmatrix} X^{11}_{d_1k_1\times d_1k_1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \begin{bmatrix} \mathbf{0} & X^{23}_{d_2k_2\times d_3k_3} \\ X^{32}_{d_3k_3\times d_2k_2} & \mathbf{0} \end{bmatrix} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \begin{bmatrix} \mathbf{0} & X^{r-1\,r}_{d_{r-1}k_{r-1}\times d_rk_r} \\ X^{r\,r-1}_{d_rk_r\times d_{r-1}k_{r-1}} & \mathbf{0} \end{bmatrix} \end{bmatrix},$$

with $X^{ij}_{d_ik_i\times d_jk_j} = C^t_{k_i\rho_i(g)} X^{ij}_{d_ik_i\times d_jk_j} C_{k_i\rho_i(g)}$, for $(i,j) \in A_G$. If $k_{2i} \neq k_{2i+1}$ and since $d_{2i} = d_{2i+1}$, so the number of rows and columns of $X^{ij}_{d_ik_i\times d_jk_j}$ are differ hence either rows (or columns) are linearly dependent. This completes the proof. □

In the next lemma we characterize the representations of G each of which admits a non-degenerate invariant bilinear form. To prove the next lemma we will choose only those $X \in \mathbf{M}_n(\mathbf{F})$ whose $(i,j)^{th}$ block is zero sub-matrix for $(i,j) \notin A_G$, whereas $(i,j) \in A_G$ and whenever $k_i = k_j$, the block sub-matrices $X^{ij}_{d_ik_i\times d_jk_j}$, is non-singular.

Lemma 4.2. *For $n \in \mathbf{Z}^+$, degree n representation of G has a non-degenerate invariant bilinear form if and only if $k_{2i} = k_{2i+1}, 1 \leq i \leq \frac{r-1}{2}$.*

$$X = \begin{bmatrix} X^{11}_{d_1k_1\times d_1k_1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \begin{bmatrix} \mathbf{0} & X^{23}_{d_2k_2\times d_3k_3} \\ X^{32}_{d_3k_3\times d_2k_2} & \mathbf{0} \end{bmatrix} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \begin{bmatrix} \mathbf{0} & X^{r-1\,r}_{d_{r-1}k_{r-1}\times d_rk_r} \\ X^{r\,r-1}_{d_rk_r\times d_{r-1}k_{r-1}} & \mathbf{0} \end{bmatrix} \end{bmatrix}.$$

Suppose $k_{2i} = k_{2i+1}$, and for $(i,j) \in A_G$, the chosen block sub-matrices $X^{ij}_{d_ik_i \times d_jk_j}$ is non-singular. Thus the rows (or columns) of the X is linearly independent. Also from Remark 4.1, we have
$X^{ij}_{d_ik_i \times d_jk_j} = C^t_{k_i\rho_i(g)} X^{ij}_{d_ik_i \times d_jk_j} C_{k_i\rho_i(g)}$ so $C^t_{\rho(g)} X C_{\rho(g)} = X$. Therefore $X \in \Xi'_G$.
On other hand $X \in \Xi'_G$ non-singular implies that $X^{ij}_{d_ik_i \times d_jk_j}$ is non-singular for $(i,j) \in A_G$, therefore the block sub-matrix is square, which is possible only when $k_{2i} = k_{2i+1}, \forall\, i \in \{1, 2, ..., \frac{r-1}{2}\}$. □

Note that the Lemmas 4.1 and 4.2, can be covered in a more general situation by stating as no non-trivial irreducible representation of a finite p-group can be self dual. For if L a finite p-group and V a non-trivial irreducible representation of L, replacing L by its image in the matrix group (the general linear group), we may assume that the representation is faithful. Being a non-trivial finite p-group, L has non-trivial center. Let g be a non-trivial central element in L. The action of g on V is by multiplication by a root of unity $\zeta \neq \pm 1$. Its action on the dual of V is by multiplication by $\bar{\zeta}(= \zeta^{-1})$. Since $\zeta \neq \bar{\zeta}$, it follows that V is not self-dual.

Corollary 4.1. *For $n \in \mathbf{Z}^+$, degree n representation of a group of order p^3 has a non-degenerate invariant bilinear form if and only if every irreducible representation and its dual have same multiplicity in the representation.*

Proof. Follows from the proof of the Lemma 4.2. □

Remark 4.2. *If $\mathbf{F}$ is algebraically closed, it has infinitely many non zero elements, hence if there is one non-degenerate invariant bilinear form in the space Ξ_G, it has infinitely many.*

Lemma 4.3. *Let G be a group of order p^3 and $n \in \mathbf{N}$. Then the number of degree n representations of G each of which admits a non-degenerate invariant bilinear form is* $\displaystyle\sum_{\ell=0}^{\lfloor \frac{n}{2p} \rfloor} \left[\binom{\ell + \frac{p-3}{2}}{\frac{p-3}{2}} \sum_{s=0}^{\lfloor \frac{n-2p\ell}{2} \rfloor} \binom{s + \frac{p^2-3}{2}}{\frac{p^2-3}{2}} \right]$ *when G is non-abelian whereas it is* $\displaystyle\sum_{\ell=0}^{\lfloor \frac{n}{2} \rfloor} \binom{\ell + \frac{p^3-3}{2}}{\frac{p^3-3}{2}}$ *when G is abelian.*

Proof. Let $\rho = \oplus_{i=1}^r k_i \rho_i$ be degree n representation of G which admits a non-degenerate bilinear form. So, we have $k_{2i} = k_{2i+1}$, $1 \leq i \leq \frac{r-1}{2}$. In the non-abelian case, G is either G_p or $Heis(\mathbf{Z}_p)$ and for each of these two, we have $r = p^2+p-1$, d_i =1 for $1 \leq i \leq p^2$ and $d_i = p$ for $p^2+1 \leq i \leq p^2+p-1$. Now from equation (3.2), we have

$$k_1 + 2(\underbrace{k_2 + k_4 + \cdots + k_{p^2-1}}_{\frac{p^2-1}{2}}) + 2p(\underbrace{k_{p^2+1} + k_{p^2+3} + \cdots + k_{p^2+p-2}}_{\frac{p-1}{2}}) = n.$$

$$\Longrightarrow k_1 + 2(\underbrace{k_2 + k_4 + \cdots + k_{p^2-1}}_{\frac{p^2-1}{2}}) = n - 2p(\underbrace{k_{p^2+1} + k_{p^2+3} + \cdots + k_{p^2+p-2}}_{\frac{p-1}{2}}).$$

$$\Longrightarrow k_1 + 2(\underbrace{k_2 + k_4 + \cdots + k_{p^2-1}}_{\frac{p^2-1}{2}}) = n - 2p\ell. \tag{4.1}$$

To solve the above equation we have $(\underbrace{k_{p^2+1} + k_{p^2+3} + \cdots + k_{p^2+p-2}}_{\frac{p-1}{2}}) = \ell, 0 \leq \ell \leq \lfloor \frac{n}{2p} \rfloor$. i.e, we have $\lfloor \frac{n}{2p} \rfloor + 1$ equations. The ℓ^{th} equation is

$$\underbrace{k_{p^2+1} + k_{p^2+3} + \cdots + k_{p^2+p-2}}_{\frac{p-1}{2}} = \ell. \tag{4.2}$$

The number of distinct solution to above equation 4.2 is $\binom{\ell + \frac{p-1}{2} - 1}{\frac{p-1}{2} - 1}$, $0 \leq \ell \leq \lfloor \frac{n}{2p} \rfloor$.

Further from equation 4.1 we have

$$k_1 = n - 2p\ell - 2(\underbrace{k_2 + k_4 + \cdots + k_{p^2-1}}_{\frac{p^2-1}{2}}).$$

$$\Longrightarrow k_1 = n - 2p\ell - 2\lambda,$$

where $\underbrace{k_2 + k_4 + \cdots + k_{p^2-1}}_{\frac{p^2-1}{2}} = \lambda$,$0 \leq \lambda \leq \lfloor \frac{n-2p\ell}{2} \rfloor$. i.e, we have $\lfloor \frac{n-2p\ell}{2} \rfloor + 1$ equations and the number of solutions for every λ to the equation is $\binom{\lambda + \frac{p^2-1}{2} - 1}{\frac{p^2-1}{2} - 1}$.

Thus the number of all distinct p^2+p-1 tuples $(k_1, k_2, k_3, \cdots, k_{p^2+p-2}, k_{p^2+p-1})$ with $k_{2i} = k_{2i+1}$ is

$$\sum_{\ell=0}^{\lfloor \frac{n}{2p} \rfloor} \left[\binom{\ell + \frac{p-1}{2} - 1}{\frac{p-1}{2} - 1} \sum_{\lambda=0}^{\lfloor \frac{n-2p\ell}{2} \rfloor} \binom{\lambda + \frac{p^2-1}{2} - 1}{\frac{p^2-1}{2} - 1} \right].$$

Now in the abelian case G is either of $\mathbf{Z}_p \times \mathbf{Z}_p \times \mathbf{Z}_p, \mathbf{Z}_{p^2} \times \mathbf{Z}_p$ and $\mathbf{Z}_{p^3}$ for each of which $r = p^3$ and $d_i = 1$ for $1 \leq i \leq p^3$. Now from (3.2), we have

$$k_1 + 2(\underbrace{k_2 + k_4 + \cdots + k_{p^3-1}}_{\frac{p^3-1}{2}}) = n.$$

So the number of all distinct p^3-tuples $(k_1, k_2, k_3, \cdots, k_{p^3})$ with $k_{2i} = k_{2i+1}$ is $\sum_{\ell=0}^{\lfloor \frac{n}{2} \rfloor} \binom{\ell+\frac{p^3-1}{2}-1}{\frac{p^3-1}{2}-1}$.

Thus from equation (3.2) and Theorem 2.2 the number of degree n representations (upto isomorphism) of a group G consisting non-degenerate invariant bilinear form is $\sum_{\ell=0}^{\lfloor \frac{n}{2p} \rfloor} \left[\binom{\ell+\frac{p-1}{2}-1}{\frac{p-1}{2}-1} \sum_{\lambda=0}^{\lfloor \frac{n-2p\ell}{2} \rfloor} \binom{\lambda+\frac{p^2-1}{2}-1}{\frac{p^2-1}{2}-1} \right]$ for non-abelian groups and $\sum_{\ell=0}^{\lfloor \frac{n}{2} \rfloor} \binom{l+\frac{p^3-1}{2}-1}{\frac{p^3-1}{2}-1}$ for abelian groups of order p^3, with p an odd prime. $\square$

5. Dimensions of spaces of invariant bilinear forms under the representations of groups of order p^3 with prime $p > 2$.

The space of invariant bilinear forms under degree n representation is generated by finitely many vectors, so its dimension is finite along with its symmetric and the skew-symmetric sub spaces. In this section, we formulate the dimension of the space of invariant bilinear forms under a representation of a group of order p^3, with p an odd prime.

Theorem 5.1. *If Ξ_G is the space of invariant bilinear forms under degree n representation $\rho = \oplus_{i=1}^{r} k_i \rho_i$ of a group G of order p^3, then dim$(\Xi_G) = \sum_{(i,j) \in A_G} k_i k_j$.*

Proof. For every $X \in \Xi'_G$, we have

$$X = \begin{bmatrix} X^{11}_{d_1k_1\times d_1k_1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \begin{bmatrix} \mathbf{0} & X^{23}_{d_2k_2\times d_3k_3} \\ X^{32}_{d_3k_3\times d_2k_2} & \mathbf{0} \end{bmatrix} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \begin{bmatrix} \mathbf{0} & X^{r-1\ r}_{d_{r-1}k_{r-1}\times d_rk_r} \\ X^{r\ r-1}_{d_rk_r\times d_{r-1}k_{r-1}} & \mathbf{0} \end{bmatrix} \end{bmatrix},$$

with $X^{ij}_{d_ik_i\times d_jk_j} = C^t_{k_i\rho_i(g)} X^{ij}_{d_ik_i\times d_jk_j} C_{k_j\rho_j(g)}$, for $(i,j) \in A_G$ and to generate each of these sub-matrices of X it needs k_ik_j vectors from Ξ'_G. $\square$

Corollary 5.1. *The space of invariant symmetric bilinear forms under degree n representation $\rho = \oplus_{i=1}^r k_i\rho_i$ of a group G of order p^3 has dimension $= \frac{k_1(k_1+1)}{2} + \sum_{\substack{(i,j)\in A_G \\ i\neq j}} \frac{k_ik_j}{2}$.*

Proof. Completes from the proof of Theorem 5.1. $\square$

Corollary 5.2. *The space of invariant skew-symmetric bilinear forms under degree n representation $\rho = \oplus_{i=1}^r k_i\rho_i$ of a group G of order p^3 has dimension $= \frac{k_1(k_1-1)}{2} + \sum_{\substack{(i,j)\in A_G \\ i\neq j}} \frac{k_ik_j}{2}$.*

Proof. Completes from the proof of Theorem 5.1. $\square$

6. Main results

Here we publish the proofs of our main theorems stated in the Introduction section.

Proof of Theorem 1.1 Since G is the group of order p^3, with an odd prime p and ρ is degree n representation, if G is either G_p or $Heis(\mathbf{Z}_p)$, we have $r = p^2 + p - 1$, d_i =1 for $1 \leq i \leq p^2$ and $d_i = p$ for $p^2 + 1 \leq i \leq p^2 + p - 1$. Now from equation (3.2), we have

$$k_1 + k_2 + \cdots + k_{p^2} + pk_{p^2+1} + \cdots + pk_{p^2+p-1} = n.$$

$$\implies \qquad k_1 + k_2 + \cdots + k_{p^2} = n - p(k_{p^2+1} + \cdots + k_{p^2+p-1}).$$

$$\text{(6.1)} \qquad \implies \qquad k_1 + k_2 + \cdots + k_{p^2} = n - p\mu,$$

where $\mu = k_{p^2+1} + \cdots + k_{p^2+p-1}$, $0 \le \mu \le \lfloor \frac{n}{p} \rfloor$. i.e, we have $\lfloor \frac{n}{p} \rfloor + 1$ equations placed in the chronological order and the μ^{th} equation is given by

$$\text{(6.2)} \qquad k_{p^2+1} + k_{p^2+2} + \cdots + k_{p^2+p-1} = \mu.$$

The number of distinct solutions to equation 6.2 is $\binom{\mu+p-2}{p-2}$, $0 \le \mu \le \lfloor \frac{n}{p} \rfloor$. Thus the number of all distinct $p^2 + p - 1$ tuples $(k_1, k_2, \cdots, k_{p^2+p-1})$ is $\sum_{\mu=0}^{\lfloor \frac{n}{p} \rfloor} \binom{\mu+p-2}{p-2} \binom{n-\mu p+p^2-1}{p^2-1}$.

On the other hand if G is either of $\mathbf{Z}_p \times \mathbf{Z}_p \times \mathbf{Z}_p$, $\mathbf{Z}_{p^2} \times \mathbf{Z}_p$ and $\mathbf{Z}_{p^3}$ then $r = p^3$ and d_i =1 for $1 \le i \le p^3$. Now from (3.2), we have

$$k_1 + k_2 + \cdots + k_p + \cdots + k_{p^2} + \cdots + k_{p^3} = n.$$

Thus the number of all distinct p^3-tuples $(k_1, k_2, k_3, \cdots, k_{p^3})$ is $\binom{n+p^3-1}{p^3-1}$.

Therefore from equation (3.2) and Theorem 2.2 the number of degree n representations (up to isomorphism) of a group G of order p^3 is $\sum_{\mu=0}^{\lfloor \frac{n}{p} \rfloor} \binom{\mu+p-2}{p-2} \binom{n-\mu p+p^2-1}{p^2-1}$, when G is non-abelian, whereas it is $\binom{n+p^3-1}{p^3-1}$, when G is abelian.

6.1. Degenerate invariant spaces

From Lemma 4.1, if $k_{2i} \neq k_{2i+1}$ then all the elements of the space are degenerate. Thus by the Theorem 1.1 and Lemma 4.2, the number of degree n representations which admit only degenerate invariant bilinear forms is $\sum_{\mu=0}^{\lfloor \frac{n}{p} \rfloor} \binom{\mu+p-2}{p-2} \binom{n-\mu p+p^2-1}{p^2-1} - \sum_{\ell=0}^{\lfloor \frac{n}{2p} \rfloor} \left[\binom{\ell+\frac{p-3}{2}}{\frac{p-3}{2}} \sum_{\lambda=0}^{\lfloor \frac{n-2p\ell}{2} \rfloor} \binom{\lambda+\frac{p^2-3}{2}}{\frac{p^2-3}{2}} \right]$ in the non-abelian case and it is $\binom{n+p^3-1}{p^3-1} - \sum_{\ell=0}^{\lfloor \frac{n}{2} \rfloor} \binom{\ell+\frac{p^3-3}{2}}{\frac{p^3-3}{2}}$ in the abelian case.

Proof of Theorem 1.2 Let $A_G = \{(i,j) \mid \rho_i$ and ρ_j are dual to each other$\}$ and for every $(i,j) \in A_G$, $\mathbf{W}_{(i,j)} = \{X \in \mathbf{M}_n(\mathbf{F}) \mid X^{ij}_{d_i k_i \times d_j k_j} =$

$C^t_{k_i\rho_i(g)}X^{ij}_{d_ik_i\times d_jk_j}C_{k_j\rho_j(g)}, \forall g \in G$ and rest blocks are zero$\}$. Then for $(i,j) \in A_G$, $\mathbf{W}_{(i,j)}$ is a subspace of $\mathbf{M}_n(\mathbf{F})$. Let X be an element of Ξ'_G, then

$$C^t_{\rho(g)}XC_{\rho(g)} = X \text{ and } X = [X^{ij}_{d_ik_i\times d_jk_j}]_{(i,j)\in A_G}.$$

Existence:
Let $X \in \Xi'_G$ then for every $(i,j) \in A_G$, there exists at least one $X_{(i,j)} \in \mathbf{W}_{(i,j)}$, such that $\sum_{(i,j)\in A_G} X_{(i,j)} = X$.

Uniqueness:
For every $(i,j) \in A_G$, suppose there are $Y_{(i,j)}$ and $X_{(i,j)} \in \mathbf{W}_{(i,j)}$, such that $\sum_{(i,j)\in A_G} Y_{(i,j)} = X = \sum_{(i,j)\in A_G} X_{(i,j)}$, then $\sum_{(i,j)\in A_G} X_{(i,j)} = \sum_{(i,j)\in A_G} Y_{(i,j)}$ i.e., $Y_{(i',j')} - X_{(i',j')} = \sum_{\substack{(i,j)\in A_G \\ (i,j)\neq(i',j)}} \neq \Big((X_{(i,j)} - Y_{(i,j)}\Big)$. Therefore $Y_{(i',j')} - X_{(i',j')} \in \sum_{\substack{(i,j)\in A_G \\ (i,j)\neq(i',j)}} \neq \Big(\mathbf{W}_{(i,j)}$, hence $Y_{(i',j')} - X_{(i',j')} = 0 \implies Y_{(i',j')} = X_{(i',j')}$ for all $(i',j') \in A_G$. Thus we have

$$\Xi'_G = \oplus_{(i,j)\in A_G}\mathbf{W}_{(i,j)} \text{ and } \dim(\Xi'_G) = \sum_{(i,j)\in A_G} \dim(\mathbf{W}_{(i,j)}). \tag{6.3}$$

Now as for $(i,j) \in A_G$, $\mathbf{W}_{(i,j)} = \{X \in \mathbf{M}_n(\mathbf{F}) \mid (i,j)^{th}$ block X^{ij} is a sub-matrix of order $d_ik_i \times d_jk_j$ satisfying $X^{ij} = C^t_{k_i\rho_i(g)}X^{ij}C_{k_j\rho_j(g)}$, $\forall g \in G$ and rest blocks are zero$\}$. So by the Remark 4.1, we see that for $(i,j) \in A_G$, the sub-matrices X^{ij} in $\mathbf{W}_{(i,j)}$ have k_ik_j free variables & $\mathbf{W}_{(i,j)} \cong \mathbf{M}_{k_i\times k_j}(\mathbf{F})$. Thus $\Xi'_G \cong \oplus_{(i,j)\in A_G}\mathbf{M}_{k_i\times k_j}(\mathbf{F})$ and $dim(\mathbf{W}_{(i,j)}) = k_ik_j$.

Thus substituting these in equation (6.3) we get the dimension of Ξ'_G.

Proof of Theorem 1.3 Follows immediately from Lemmas 4.1 and 4.2.

In this way, we have completely characterized the representations of a group of order p^3, each of which admits a non-degenerate invariant bilinear form over a field of characteristic different from p consisting of a primitive p^3th root of unity.

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