



Existence of periodic or nonnegative periodic solutions for totally nonlinear neutral differential equations with infinite delay

Abderrahim Guerfi
University of Annaba, Algeria
and

Abdelouaheb Ardjouni
University of Souk Ahras, Algeria

Received : November 2021. Accepted : January 2022

Abstract

In this work, we investigate the existence of periodic or nonnegative periodic solutions for a totally nonlinear neutral differential equation with infinite delay. In the process, we convert the given neutral differential equation into an equivalent integral equation. Then, we employ Krasnoselskiĭ-Burton's fixed point theorem to prove the existence of periodic or nonnegative periodic solutions. Two examples are provided to illustrate the obtained results.

Subclass [2010]: 34K40, 45J05.

Keywords: *Krasnoselskiĭ-Burton's fixed point, Large contraction, Periodic solutions, Nonnegative periodic solutions, Infinite delay.*

1. Introduction

Delay differential equations have attracted considerable attention in mathematics during recent years since these equations have been showed to be valuable tools in the modeling of many phenomena in various fields of science, physics, chemistry and engineering, etc. In particular, problems concerning qualitative analysis such as periodicity, positivity and stability of solutions for delay differential equations have been studied extensively by many authors, see the references [1]–[15]. In the current paper, we present sufficient conditions for the existence of periodic or nonnegative periodic solutions of the totally nonlinear neutral differential equation with infinite delay

$$\frac{d}{dt}x(t) = -a(t)h(x(t - \tau(t))) + \frac{d}{dt}Q(t, x(t - g(t))) + \int_{-\infty}^t D(t, s)f(x(s))ds, \quad (1.1)$$

where a is a positive continuous function. The functions $h, f : \mathbf{R} \rightarrow \mathbf{R}$ are continuous, $Q : \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$ satisfying the Carathéodory condition. The main purpose of this work is to use Krasnoselskiĭ-Burton's fixed point theorem (see [11]) to prove the existence of periodic or nonnegative periodic solutions for (1.1). During the process, we employ the variation of parameter formula and the integration by parts to transform (1.1) into an equivalent integral equation written as a sum of two mappings; one is a large contraction and the other is compact. After that, we use Krasnoselskiĭ-Burton's fixed point theorem, to prove the existence of periodic or nonnegative periodic solutions. Two examples are given to illustrate the obtained results.

2. Preliminaries

For $T > 0$ define

$$P_T = \{\varphi \in C(\mathbf{R}, \mathbf{R}), \varphi(t + T) = \varphi(t)\},$$

where $C(\mathbf{R}, \mathbf{R})$ is the space of all real valued continuous functions. Then P_T is a Banach space when it is endowed with the supremum norm

$$\|x\| = \sup_{t \in \mathbf{R}} |x(t)| = \sup_{t \in [0, T]} |x(t)|.$$

In this paper, we assume that

$$(2.1) \quad a(t-T) = a(t), \quad D(t-T, s-T) = D(t, s),$$

$$\tau(t-T) = \tau(t) \geq \tau^* > 0, \quad g(t-T) = g(t) \geq g^* > 0.$$

with τ and g are continuously differentiable functions, τ^* and g^* are positive constants, a is a positive function and

$$(2.2) \quad 1 - e^{-\int_{t-T}^t a(k)dk} \equiv \frac{1}{\eta} \neq 0.$$

The function $Q(t, x)$ is periodic in t of period T , that is

$$(2.3) \quad Q(t+T, x) = Q(t, x).$$

Also, there is a positive constant E such that,

$$(2.4) \quad \int_{-\infty}^t |D(t, s)| ds \leq E < \infty.$$

The following lemma is fundamental to our results.

Lemma 1. Suppose (2.1)–(2.3) hold. If $x \in P_T$, then x is a solution of (1.1) if and only if

$$(2.5) \quad \begin{aligned} x(t) &= \eta \int_{t-T}^t a(u) H(x(u)) e^{-\int_u^t a(k)dk} du + Q(t, x(t-g(t))) \\ &+ \int_{t-\tau(t)}^t a(u) h(x(u)) du - \eta \int_{t-T}^t \left[\int_{u-\tau(u)}^u a(s) h(x(s)) ds \right] a(u) e^{-\int_u^t a(k)dk} du \\ &+ \eta \int_{t-T}^t b(u) h(x(u-\tau(u))) e^{-\int_u^t a(k)dk} du \\ &+ \eta \int_{t-T}^t \left[-a(u) Q(u, x(u-g(u))) + \int_{-\infty}^u D(u, s) f(x(s)) ds \right] e^{-\int_u^t a(k)dk} du, \end{aligned}$$

where

$$(2.6) \quad H(x) = x - h(x),$$

and

$$b(u) = (1 - \tau'(u)) a(u - \tau(u)) - a(u).$$

Proof. Let $x \in P_T$ be a solution of (1.1). Rewrite (1.1) as

$$\begin{aligned} & \frac{d}{dt} [x(t) - Q(t, x(t - g(t)))] + a(t) [x(t) - Q(t, x(t - g(t)))] \\ &= a(t) [x(t) - Q(t, x(t - g(t)))] - a(t) h(x(t)) + a(t) h(x(t)) \\ & \quad - a(t) h(x(t - \tau(t))) + \int_{-\infty}^t D(t, s) f(x(s)) ds \\ &= a(t) [x(t) - h(x(t))] + \frac{d}{dt} \int_{t-\tau(t)}^t a(s) h(x(s)) ds \\ & \quad + [(1 - \tau'(t)) a(t - \tau(t)) - a(t)] h(x(t - \tau(t))) \\ & \quad - a(t) Q(t, x(t - g(t))) + \int_{-\infty}^t D(t, s) f(x(s)) ds. \end{aligned}$$

Multiply both sides of the above equation by $\exp\left(\int_0^t a(k) dk\right)$ and then integrate from $t - T$ to t , we get

$$\begin{aligned} & \int_{t-T}^t \left[[x(u) - Q(u, x(u - g(u)))] e^{\int_0^u a(k) dk} \right]' du \\ &= \int_{t-T}^t a(u) [x(u) - h(x(u))] e^{\int_0^u a(k) dk} du \\ & \quad + \int_{t-T}^t \left[\frac{d}{du} \int_{u-\tau(u)}^u a(s) h(x(s)) ds \right] e^{\int_0^u a(k) dk} du \\ & \quad + \int_{t-T}^t b(u) h(x(u - \tau(u))) e^{\int_0^u a(k) dk} du \\ & \quad + \int_{t-T}^t \left[-a(u) Q(u, x(u - g(u))) + \int_{-\infty}^u D(u, s) f(x(s)) ds \right] e^{\int_0^u a(k) dk} du, \end{aligned}$$

with $b(u) = (1 - \tau'(u)) a(u - \tau(u)) - a(u)$. As a consequence, we have

$$\begin{aligned} & [x(t) - Q(t, x(t - g(t)))] e^{\int_0^t a(k) dk} \\ & \quad - [x(t - T) - Q(t - T, x(t - T - g(t - T)))] e^{\int_0^{t-T} a(k) dk} \\ &= \int_{t-T}^t a(u) [x(u) - h(x(u))] e^{\int_0^u a(k) dk} du \\ & \quad + \int_{t-T}^t \left[\frac{d}{du} \int_{u-\tau(u)}^u a(s) h(x(s)) ds \right] e^{\int_0^u a(k) dk} du \\ & \quad + \int_{t-T}^t b(u) h(x(u - \tau(u))) e^{\int_0^u a(k) dk} du \\ & \quad + \int_{t-T}^t \left[-a(u) Q(u, x(u - g(u))) + \int_{-\infty}^u D(u, s) f(x(s)) ds \right] e^{\int_0^u a(k) dk} du. \end{aligned}$$

By dividing both sides of the above equation by $\exp\left(\int_0^t a(u) du\right)$ and using the fact that $x(t) = x(t - T)$, we obtain

$$\begin{aligned} & x(t) - Q(t, x(t - g(t))) \\ &= \eta \int_{t-T}^t a(u) [x(u) - h(x(u))] e^{-\int_u^t a(k) dk} du \\ & \quad + \eta \int_{t-T}^t \left[\frac{d}{du} \int_{u-\tau(u)}^u a(s) h(x(s)) ds \right] e^{-\int_u^t a(k) dk} du \\ & \quad + \eta \int_{t-T}^t b(u) h(x(u - \tau(u))) e^{-\int_u^t a(k) dk} du \\ & \quad + \eta \int_{t-T}^t \left[-a(u) Q(u, x(u - g(u))) + \int_{-\infty}^u D(u, s) f(x(s)) ds \right] e^{-\int_u^t a(k) dk} du. \end{aligned} \tag{2.7}$$

Integration by parts the second integral in the above expression, we get

$$\begin{aligned}
& \int_{t-T}^t \left[\frac{d}{du} \int_{u-\tau(u)}^u a(s) h(x(s)) ds \right] e^{-\int_u^t a(k) dk} du \\
&= \left[\int_{u-\tau(u)}^u a(s) h(x(s)) ds e^{-\int_u^t a(k) dk} \right]_{t-T}^t \\
&\quad - \int_{t-T}^t \left[\int_{u-\tau(u)}^u a(s) h(x(s)) ds \right] a(u) e^{-\int_u^t a(k) dk} du \\
(2.8) \quad &= \left[\int_{t-\tau(t)}^t a(s) h(x(s)) ds - \int_{t-T-\tau(t)}^{t-T} a(s) h(x(s)) ds e^{-\int_{t-T}^t a(k) dk} \right] \\
&\quad - \int_{t-T}^t \left[\int_{u-\tau(u)}^u a(s) h(x(s)) ds \right] a(u) e^{-\int_u^t a(k) dk} du \\
&= - \int_{t-T}^t \left[\int_{u-\tau(u)}^u a(s) h(x(s)) ds \right] a(u) e^{-\int_u^t a(k) dk} du \\
&\quad + \frac{1}{\eta} \int_{t-\tau(t)}^t a(u) h(x(u)) du.
\end{aligned}$$

Then substituting the result of (2.8) into (2.7) to obtain (2.5). The converse implication is easily obtained and the proof is complete. \square

Definition 1. The map $\mathcal{P} : [0, T] \times \mathbf{R}^n \rightarrow \mathbf{R}$ is said to satisfy Carathéodory conditions with respect to $L^1[0, T]$ if the following conditions hold. (i) For each $z \in \mathbf{R}^n$, the mapping $t \mapsto \mathcal{P}(t, z)$ is Lebesgue measurable. (ii) For almost all $t \in [0, T]$, the mapping $t \mapsto \mathcal{P}(t, z)$ is continuous on \mathbf{R}^n . (iii) For each $r > 0$, there exists $\alpha_r \in L^1([0, T], \mathbf{R})$ such that for almost all $t \in [0, T]$ and for all z such that $|z| < r$, we have $|\mathcal{P}(t, z)| \leq \alpha_r(t)$.

Definition 2 ([11]). Let (\mathbf{M}, d) be a metric space and suppose that $B : \mathbf{M} \rightarrow \mathbf{M}$. B is said to be a large contraction, if for $\varphi, \psi \in \mathbf{M}$, with $\varphi \neq \psi$, we have $d(B\varphi, B\psi) \leq d(\varphi, \psi)$ and if $\forall \epsilon > 0$, $\exists \delta < 1$ such that

$$[\varphi, \psi \in \mathbf{M}, d(\varphi, \psi) \geq \epsilon] \Rightarrow d(B\varphi, B\psi) \leq \delta d(\varphi, \psi).$$

Theorem 1 (Krasnoselskiĭ-Burton [11]). Let \mathbf{M} be a closed bounded convex nonempty subset of a Banach space $(\mathbf{B}, \|\cdot\|)$. Suppose that A and B map \mathbf{M} into \mathbf{M} such that

(i) A is completely continuous, (ii) B is large contraction, (iii) $x, y \in \mathbf{M}$, implies $Ax + By \in \mathbf{M}$.

Then there exists $z \in \mathbf{M}$ with $z = Az + Bz$.

Theorem 2 ([1]). Let $\|\cdot\|$ be the supremum norm, $\mathbf{M} = \{\varphi \in P_T : \|\varphi\| \leq L\}$, where L is a positive constant. Suppose that h is satisfying the following conditions

- (H1) $h : \mathbf{R} \rightarrow \mathbf{R}$ is continuous on $[-L, L]$ and differentiable on $(-L, L)$,
 (H2) the function h is strictly increasing on $[-L, L]$,
 (H3) $\sup_{t \in (-L, L)} h'(t) \leq 1$.

Then the mapping H define by (2.6) is a large contraction on the set \mathbf{M} .

3. Existence of periodic solutions

To apply Theorem 1, we need to define a Banach space \mathbf{B} , a closed bounded convex subset \mathbf{M} of \mathbf{B} and construct two mappings; one is a completely continuous and the other is a large contraction. So, we let $(\mathbf{B}, \|\cdot\|) = (P_T, \|\cdot\|)$ and

$$\mathbf{M} = \{\varphi \in P_T, \|\varphi\| \leq L, |\varphi(t_2) - \varphi(t_1)| \leq K|t_2 - t_1|, \forall t_1, t_2 \in [0, T]\}, \quad (3.1)$$

with $L \in (0, 1]$ and $K > 0$. \mathbf{M} is a closed convex and bounded subset of P_T .

Define a mapping $\mathcal{S} : P_T \rightarrow P_T$ by

$$\begin{aligned} (\mathcal{S}\varphi)(t) &= \eta \int_{t-T}^t a(u) H(\varphi(u)) e^{-\int_u^t a(k)dk} du + Q(t, \varphi(t - g(t))) \\ &\quad + \int_{t-\tau(t)}^t a(u) h(\varphi(u)) du - \eta \int_{t-T}^t \left[\int_{u-\tau(u)}^u a(s) h(\varphi(s)) ds \right] \\ &\quad a(u) e^{-\int_u^t a(k)dk} du \\ &\quad + \eta \int_{t-T}^t b(u) h(\varphi(u - \tau(u))) e^{-\int_u^t a(k)dk} du \\ &\quad + \eta \int_{t-T}^t \left[-a(u) Q(u, \varphi(u - g(u))) + \int_{-\infty}^u D(u, s) f(\varphi(s)) ds \right] \\ &\quad e^{-\int_u^t a(k)dk} du. \end{aligned} \quad (3.2)$$

Therefore, we express the above mapping as

$$\mathcal{S}\varphi = A\varphi + B\varphi,$$

where $A, B : P_T \rightarrow P_T$ are given by

$$\begin{aligned}
(A\varphi)(t) &= Q(t, \varphi(t - g(t))) + \int_{t-\tau(t)}^t a(u) h(\varphi(u)) du \\
&\quad - \eta \int_{t-T}^t \left[\int_{u-\tau(u)}^u a(s) h(\varphi(s)) ds \right] a(u) e^{-\int_u^t a(k) dk} du \\
&\quad + \eta \int_{t-T}^t b(u) h(\varphi(u - \tau(u))) e^{-\int_u^t a(k) dk} du \\
&\quad + \eta \int_{t-T}^t \left[-a(u) Q(u, \varphi(u - g(u))) + \int_{-\infty}^u D(u, s) f(\varphi(s)) ds \right] \\
&\quad \quad e^{-\int_u^t a(k) dk} du,
\end{aligned}
\tag{3.3}$$

and

$$(B\varphi)(t) = \eta \int_{t-T}^t a(u) H(\varphi(u)) e^{-\int_u^t a(k) dk} du.$$

We will assume that the following conditions hold.

(H4) $a \in L^1[0, T]$ is bounded.

(H5) h, f, Q are locally Lipschitz continuous, then for $t \geq 0$ and $x, y \in \mathbf{M}$ there exists constants $E_1, E_2, E_3 > 0$, such that

$$\begin{aligned}
|h(x) - h(y)| &\leq E_1 \|x - y\|, \\
|f(x) - f(y)| &\leq E_2 \|x - y\|, \\
|Q(t, x) - Q(t, y)| &\leq E_3 \|x - y\|.
\end{aligned}$$

(H6) Q satisfies Carathéodory condition with respect to $L^1[0, T]$.

(H7) There exist positive periodic functions $q_1, q_2 \in L^1[0, T]$, with period T , such that

$$|Q(t, x)| \leq q_1(t) |x| + q_2(t).$$

(H8) The function $Q(t, x)$ is also supposed locally Lipschitz in t , i.e, there exists $K_Q > 0$ such that

$$|Q(t_2, x) - Q(t_1, x)| \leq K_Q |t_2 - t_1|.$$

Now, we need the following assumptions

$$\beta_1 \beta_2 (E_1 L + |h(0)|) \leq \frac{\gamma_1}{2} L,$$

where $\beta_1 = \max_{t \in [0, T]} |\tau(t)|$ and $\beta_2 = \max_{t \in [0, T]} \{a(t)\}$,

$$q_1(t) L + q_2(t) \leq \frac{\gamma_2}{2} L,$$

$$(3.7) \quad |b(u)|(E_1 L + |h(0)|) \leq \gamma_3 a(u) L,$$

$$(3.8) \quad TE\eta\beta_3(E_2 L + |f(0)|) \leq \gamma_4 L,$$

$$\text{where } \beta_3 = \max_{u \in [t-T, t]} \left\{ e^{-\int_u^t a(k) dk} \right\},$$

$$(3.9) \quad J[\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4] \leq 1.$$

where $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ and J are positive constants with $J \geq 3$. Also, suppose that there are constants $k_1, k_2, k_3 > 0$ such that for $0 \leq t_1 < t_2$

$$(3.10) \quad |\tau(t_2) - \tau(t_1)| \leq k_1 |t_2 - t_1|,$$

$$(3.11) \quad |g(t_2) - g(t_1)| \leq k_2 |t_2 - t_1|,$$

$$(3.12) \quad \int_{t_1}^{t_2} a(u) du \leq k_3 |t_2 - t_1|,$$

and

$$(3.13) \quad \begin{aligned} & K_Q + (1 + k_2) E_3 K + 2\gamma_4 \beta_2 \beta_3 L + [(2 + k_1) E_1 + (1 + 4\eta) \gamma_3 \\ & + \left(\eta + \frac{1}{2}\right) \gamma_2 + \gamma_4 + \left(\eta + \frac{1}{2}\right) \gamma_1] k_3 L \leq \frac{K}{J}. \end{aligned}$$

Lemma 2. For A defined in (3.3), suppose that (2.1)–(2.4), (3.5)–(3.13) and (H4)–(H8) hold. Then $A : \mathbf{M} \rightarrow \mathbf{M}$.

Proof. Let A be defined by (3.3). First by (2.1) and (2.3), a change of variable in (3.3) shows that $(A\varphi)(t+T) = (A\varphi)(t)$. That is, if $\varphi \in P_T$ then $A\varphi$ is periodic with period T . For having $A\varphi \in \mathbf{M}$, we will prove that $\|A\varphi\| \leq L$ and $|(A\varphi)(t_2) - (A\varphi)(t_1)| \leq K |t_2 - t_1|$, $\forall t_1, t_2 \in [0, T]$. By (H5), we have

$$|h(x)| \leq E_1 |x| + |h(0)| \quad \text{and} \quad |f(x)| \leq E_2 |x| + |f(0)|.$$

Then, let $\varphi \in \mathbf{M}$, by (3.5)–(3.9) and (H4)–(H7), we have

$$\begin{aligned}
|(A\varphi)(t)| &\leq |Q(t, \varphi(t-g(t)))| + \int_{t-\tau(t)}^t a(u) |h(\varphi(u))| du \\
&\quad + \eta \int_{t-T}^t a(u) \int_{u-\tau(u)}^u a(s) |h(\varphi(s))| ds e^{-\int_u^t a(k)dk} du \\
&\quad + \eta \int_{t-T}^t |b(u)| |h(\varphi(u-\tau(u)))| e^{-\int_u^t a(k)dk} du \\
&\quad + \eta \int_{t-T}^t \left[a(u) |Q(u, \varphi(u-g(u)))| + \int_{-\infty}^u |D(u, s)| |f(\varphi(s))| ds \right] \\
&\quad \quad e^{-\int_u^t a(k)dk} du \\
&\leq q_1(t) |\varphi(t-g(t))| + q_2(t) + \beta_1 \beta_2 (E_1 L + |h(0)|) \\
&\quad \times \left(1 + \eta \int_{t-T}^t a(u) e^{-\int_u^t a(k)dk} du \right) \\
&\quad + \eta \int_{t-T}^t |b(u)| (E_1 L + |h(0)|) e^{-\int_u^t a(k)dk} du \\
&\quad + \eta \int_{t-T}^t a(u) [q_1(u) |\varphi(u-g(u))| + q_2(u)] e^{-\int_u^t a(k)dk} du \\
&\quad + \eta \int_{t-T}^t E (E_2 L + |f(0)|) e^{-\int_u^t a(k)dk} du \\
&\leq \frac{\gamma_2}{2} L + \gamma_1 L + \gamma_3 L + \frac{\gamma_2}{2} L + \gamma_4 L \leq \frac{L}{f} \leq L.
\end{aligned}$$

Let $t_1, t_2 \in \mathbf{R}$ with $t_1 < t_2$, we get

$$\begin{aligned}
&|(A\varphi)(t_2) - (A\varphi)(t_1)| \\
&\leq |Q(t_2, \varphi(t_2-g(t_2))) - Q(t_1, \varphi(t_1-g(t_1)))| \\
&\quad + \left| \int_{t_2-\tau(t_2)}^{t_2} a(u) h(\varphi(u)) du - \int_{t_1-\tau(t_1)}^{t_1} a(u) h(\varphi(u)) du \right| \\
&\quad + \eta \left| \int_{t_2-T}^{t_2} \left[\int_{u-\tau(u)}^u a(s) h(\varphi(s)) ds \right] a(u) e^{-\int_u^{t_2} a(k)dk} du \right. \\
&\quad \quad \left. - \int_{t_1-T}^{t_1} \left[\int_{u-\tau(u)}^u a(s) h(\varphi(s)) ds \right] a(u) e^{-\int_u^{t_1} a(k)dk} du \right| \\
&\quad + \eta \left| \int_{t_2-T}^{t_2} b(u) h(\varphi(u-\tau(u))) e^{-\int_u^{t_2} a(k)dk} du \right. \\
&\quad \quad \left. - \int_{t_1-T}^{t_1} b(u) h(\varphi(u-\tau(u))) e^{-\int_u^{t_1} a(k)dk} du \right| \\
&\quad + \eta \left| \int_{t_2-T}^{t_2} \left[-a(u) Q(u, \varphi(u-g(u))) + \int_{-\infty}^u D(u, s) f(\varphi(s)) ds \right] e^{-\int_u^{t_2} a(k)dk} du \right. \\
&\quad \quad \left. - \int_{t_1-T}^{t_1} \left[-a(u) Q(u, \varphi(u-g(u))) + \int_{-\infty}^u D(u, s) f(\varphi(s)) ds \right] e^{-\int_u^{t_1} a(k)dk} du \right|.
\end{aligned}
\tag{3.14}$$

By hypotheses (H5) and (3.10)–(3.12), we obtain

$$\begin{aligned}
&\left| \int_{t_2-\tau(t_2)}^{t_2} a(u) h(\varphi(u)) du - \int_{t_1-\tau(t_1)}^{t_1} a(u) h(\varphi(u)) du \right| \\
(3.15) \quad &\leq E_1 L \left(\int_{t_1}^{t_2} a(u) du + \int_{t_1-\tau(t_1)}^{t_2-\tau(t_2)} a(u) du \right) \\
&\leq E_1 L k_3 |t_2 - t_1| + E_1 L k_3 (1 + k_1) |t_2 - t_1| \\
&= (2E_1 L k_3 + E_1 L k_3 k_1) |t_2 - t_1|,
\end{aligned}$$

and

$$\begin{aligned}
(3.16) \quad & |Q(t_2, \varphi(t_2 - g(t_2))) - Q(t_1, \varphi(t_1 - g(t_1)))| \\
& \leq K_Q |t_2 - t_1| + E_3 K |(t_2 - t_1) - (g(t_2) - g(t_1))| \\
& \leq (K_Q + E_3 K + E_3 K k_2) |t_2 - t_1|,
\end{aligned}$$

where K is the Lipschitz constant of φ . By the hypotheses (H5), (3.7) and (3.12), we get

$$\begin{aligned}
& \eta \left| \int_{t_2-T}^{t_2} b(u) h(\varphi(u - \tau(u))) e^{-\int_u^{t_2} a(k) dk} du - \int_{t_1-T}^{t_1} b(u) h(\varphi(u - \tau(u))) \right. \\
& \quad \left. e^{-\int_u^{t_1} a(k) dk} du \right| \\
& \leq \eta \left| \int_{t_1}^{t_2} b(u) h(\varphi(u - \tau(u))) e^{-\int_u^{t_2} a(k) dk} du \right| \\
& \quad + \eta \left| \int_{t_1-T}^{t_1} b(u) h(\varphi(u - \tau(u))) \left(e^{-\int_u^{t_2} a(k) dk} - e^{-\int_u^{t_1} a(k) dk} \right) du \right| \\
& \quad + \eta \left| \int_{t_1-T}^{t_2-T} b(u) h(\varphi(u - \tau(u))) e^{-\int_u^{t_2} a(k) dk} du \right| \\
& \leq 2\eta \left| \int_{t_1}^{t_2} b(u) h(\varphi(u - \tau(u))) e^{-\int_u^{t_2} a(k) dk} du \right| \\
& \quad + \eta \left| \int_{t_1-T}^{t_1} b(u) h(\varphi(u - \tau(u))) e^{-\int_u^{t_1} a(k) dk} \left(e^{-\int_{t_1}^{t_2} a(k) dk} - 1 \right) du \right| \\
& \leq 2\eta (E_1 L + |h(0)|) \int_{t_1}^{t_2} |b(u)| e^{-\int_u^{t_2} a(k) dk} du \\
& \quad + \eta \gamma_3 L \left| e^{-\int_{t_1}^{t_2} a(k) dk} - 1 \right| \int_{t_1-T}^{t_1} a(u) e^{-\int_u^{t_1} a(k) dk} du.
\end{aligned}$$

Consequently,

$$\begin{aligned}
& \eta \left| \int_{t_2-T}^{t_2} b(u) h(\varphi(u - \tau(u))) e^{-\int_u^{t_2} a(k) dk} du - \int_{t_1-T}^{t_1} b(u) h(\varphi(u - \tau(u))) \right. \\
& \quad \left. e^{-\int_u^{t_1} a(k) dk} du \right| \\
& \leq \gamma_3 L \int_{t_1}^{t_2} a(u) du + 2\eta (E_1 L + |h(0)|) \int_{t_1}^{t_2} d \left(\int_{t_1}^u |b(r)| dr \right) e^{-\int_u^{t_2} a(k) dk} du \\
& = \gamma_3 L \int_{t_1}^{t_2} a(u) du + 2\eta (E_1 L + |h(0)|) \left[\int_{t_1}^u |b(r)| dr e^{-\int_u^{t_2} a(k) dk} \right]_{t_1}^{t_2} \\
& \quad + 2\eta (E_1 L + |h(0)|) \int_{t_1}^{t_2} \left(\int_{t_1}^u |b(r)| dr \right) a(u) e^{-\int_u^{t_2} a(k) dk} du \\
& \leq \gamma_3 L \int_{t_1}^{t_2} a(u) du + 2\eta (E_1 L + |h(0)|) \int_{t_1}^{t_2} |b(u)| du \\
& \quad \left(1 + \int_{t_1}^{t_2} a(u) e^{-\int_u^{t_2} a(k) dk} du \right) \\
& \leq \gamma_3 L \int_{t_1}^{t_2} a(u) du + 4\eta \int_{t_1}^{t_2} |b(u)| (E_1 L + |h(0)|) du \\
& \leq \gamma_3 L \int_{t_1}^{t_2} a(u) du + 4\eta \gamma_3 L \int_{t_1}^{t_2} a(u) du \leq (1 + 4\eta) \gamma_3 L k_3 |t_2 - t_1|.
\end{aligned}$$

(3.17)

In the same way, by (3.6)–(3.8) and (3.12), we have

$$\begin{aligned}
 & \eta \left| \int_{t_2-T}^{t_2} \left[-a(u) Q(u, \varphi(u-g(u))) + \int_{-\infty}^u D(u, s) f(\varphi(s)) ds \right] e^{-\int_u^{t_2} a(k) dk} du \right. \\
 & \quad \left. - \int_{t_1-T}^{t_1} \left[-a(u) Q(u, \varphi(u-g(u))) + \int_{-\infty}^u D(u, s) f(\varphi(s)) ds \right] e^{-\int_u^{t_1} a(k) dk} du \right| \\
 & \leq \eta \left| \int_{t_1}^{t_2} \left[-a(u) Q(u, \varphi(u-g(u))) + \int_{-\infty}^u D(u, s) f(\varphi(s)) ds \right] e^{-\int_u^{t_2} a(k) dk} du \right| \\
 & \quad + \eta \left| \int_{t_1-T}^{t_1} \left[-a(u) Q(u, \varphi(u-g(u))) + \int_{-\infty}^u D(u, s) f(\varphi(s)) ds \right] \right. \\
 & \quad \times \left(e^{-\int_u^{t_2} a(k) dk} - e^{-\int_u^{t_1} a(k) dk} \right) du \left| \right. \\
 & \quad + \eta \left| \int_{t_1-T}^{t_2-T} \left[-a(u) Q(u, \varphi(u-g(u))) + \int_{-\infty}^u D(u, s) f(\varphi(s)) ds \right] e^{-\int_u^{t_2} a(k) dk} du \right| \\
 & \leq 2\eta \left| \int_{t_1}^{t_2} \left[-a(u) Q(u, \varphi(u-g(u))) + \int_{-\infty}^u D(u, s) f(\varphi(s)) ds \right] e^{-\int_u^{t_2} a(k) dk} du \right| \\
 & \quad + \eta \left| \int_{t_1-T}^{t_1} \left[-a(u) Q(u, \varphi(u-g(u))) + \int_{-\infty}^u D(u, s) f(\varphi(s)) ds \right] \right. \\
 & \quad \times e^{-\int_u^{t_1} a(k) dk} \left(e^{-\int_{t_1}^{t_2} a(k) dk} - 1 \right) du \left| \right. \\
 & \leq 2\eta \int_{t_1}^{t_2} \left[a(u) \frac{\gamma_2}{2} L + (E_2 L + |f(0)|) \int_{-\infty}^u |D(u, s)| ds \right] e^{-\int_u^{t_2} a(k) dk} du \\
 & \quad + \eta \left| e^{-\int_{t_1}^{t_2} a(k) dk} - 1 \right| \left| \int_{t_1-T}^{t_1} \left[a(u) \frac{\gamma_2}{2} L + (E_2 L + |f(0)|) \int_{-\infty}^u |D(u, s)| ds \right] \right. \\
 & \quad \left. e^{-\int_u^{t_1} a(k) dk} du \right| \\
 & \leq \eta \gamma_2 L \int_{t_1}^{t_2} a(u) du + 2\gamma_4 L \beta_2 \beta_3 |t_2 - t_1| + \left[\frac{\gamma_2}{2} L + \gamma_4 L \right] \int_{t_1}^{t_2} a(u) du \\
 & \leq \left[\left(\eta + \frac{1}{2} \right) \gamma_2 + \gamma_4 \right] k_3 + 2\gamma_4 \beta_2 \beta_3 \Big] L |t_2 - t_1|, \\
 \end{aligned}
 \tag{3.18}$$

and

$$\begin{aligned}
& \eta \left| \int_{t_2-T}^{t_2} \left[\int_{u-\tau(u)}^u a(s) h(\varphi(s)) ds \right] a(u) e^{-\int_u^{t_2} a(k) dk} du \right. \\
& \quad \left. - \int_{t_1-T}^{t_1} \left[\int_{u-\tau(u)}^u a(s) h(\varphi(s)) ds \right] a(u) e^{-\int_u^{t_1} a(k) dk} du \right| \\
& \leq \eta \left| \int_{t_1}^{t_2} \left[\int_{u-\tau(u)}^u a(s) h(\varphi(s)) ds \right] a(u) e^{-\int_u^{t_2} a(k) dk} du \right| \\
& \quad + \eta \left| \int_{t_1-T}^{t_1} \left[\int_{u-\tau(u)}^u a(s) h(\varphi(s)) ds \right] a(u) \left(e^{-\int_u^{t_2} a(k) dk} - e^{-\int_u^{t_1} a(k) dk} \right) du \right| \\
& \quad + \eta \left| \int_{t_1-T}^{t_2-T} \left[\int_{u-\tau(u)}^u a(s) h(\varphi(s)) ds \right] a(u) e^{-\int_u^{t_2} a(k) dk} du \right| \\
& \leq 2\eta \left| \int_{t_1}^{t_2} \left[\int_{u-\tau(u)}^u a(s) h(\varphi(s)) ds \right] a(u) e^{-\int_u^{t_2} a(k) dk} du \right| \\
& \quad + \eta \left| \int_{t_1-T}^{t_1} \left[\int_{u-\tau(u)}^u a(s) h(\varphi(s)) ds \right] a(u) e^{-\int_u^{t_1} a(k) dk} \left(e^{-\int_{t_1}^{t_2} a(k) dk} - 1 \right) du \right| \\
& \leq 2\eta \frac{\gamma_1}{2} L \int_{t_1}^{t_2} a(u) e^{-\int_u^{t_2} a(k) dk} du + \eta \left| e^{-\int_{t_1}^{t_2} a(k) dk} - 1 \right| \frac{\gamma_1}{2} L \int_{t_1-T}^{t_1} a(u) \\
& \quad e^{-\int_u^{t_1} a(k) dk} du \\
& \leq \eta \gamma_1 L \int_{t_1}^{t_2} a(u) du + \frac{\gamma_1}{2} L \int_{t_1}^{t_2} a(u) du \leq \left[\eta + \frac{1}{2} \right] \gamma_1 L k_3 |t_2 - t_1|.
\end{aligned}
\tag{3.19}$$

Thus, by substituting (3.15)–(3.19) in (3.14), we obtain

$$\begin{aligned}
& |(A\varphi)(t_2) - (A\varphi)(t_1)| \\
& \leq (2E_1 L k_3 + E_1 L k_3 k_1) |t_2 - t_1| + (K_Q + E_3 K + E_3 K k_2) |t_2 - t_1| \\
& \quad + (1 + 4\eta) \gamma_3 L k_3 |t_2 - t_1| + \left[\left(\eta + \frac{1}{2} \right) \gamma_2 + \gamma_4 \right] k_3 + 2\gamma_4 \beta_2 \beta_3 \Big] L |t_2 - t_1| \\
& \quad + \left[\eta + \frac{1}{2} \right] \gamma_1 L k_3 |t_2 - t_1| \\
& \leq \frac{K}{3} |t_2 - t_1| \leq K |t_2 - t_1|.
\end{aligned}$$

That is $A\varphi \in \mathbf{M}$. □

Lemma 3. For $A : \mathbf{M} \rightarrow \mathbf{M}$ defined in (3.3), suppose that (2.1)–(2.4), (3.5)–(3.13) and (H4)–(H8) hold. Then A is completely continuous.

Proof. Since \mathbf{M} is a uniformly bounded and equicontinuous subset of the space of continuous functions on the compact $[0, T]$, we can apply the Arzela-Ascoli theorem to confirm that \mathbf{M} is a compact subset from this space. Also, since any continuous operator maps compact sets into compact sets, then to prove that A is a compact operator it suffices to prove that it is continuous.

We prove that A is continuous in the supremum norm, let $\varphi_n \in \mathbf{M}$ where n is a positive integer such that $\varphi_n \rightarrow \varphi$ as $n \rightarrow \infty$. Then

$$\begin{aligned}
& |(A\varphi_n)(t) - (A\varphi)(t)| \\
& \leq |Q(t, \varphi_n(t - g(t))) - Q(t, \varphi(t - g(t)))| \\
& \quad + \int_{t-\tau(t)}^t a(u) |h(\varphi_n(u)) - h(\varphi(u))| du \\
& \quad + \eta \int_{t-T}^t \left[\int_{u-\tau(u)}^u a(s) |h(\varphi_n(s)) - h(\varphi(s))| ds \right] a(u) e^{-\int_u^t a(k) dk} du \\
& \quad + \eta \int_{t-T}^t |b(u)| |h(\varphi_n(u - \tau(u))) - h(\varphi(u - \tau(u)))| e^{-\int_u^t a(k) dk} du \\
& \quad + \eta \int_{t-T}^t [a(u) |Q(u, \varphi_n(u - g(u))) - Q(u, \varphi(u - g(u)))| \\
& \quad + \int_{-\infty}^u |D(u, s)| |f(\varphi_n(s)) - f(\varphi(s))| ds] e^{-\int_u^t a(k) dk} du.
\end{aligned}$$

By the dominated convergence theorem, $\lim_{n \rightarrow \infty} |(A\varphi_n)(t) - (A\varphi)(t)| = 0$. Then A is continuous. Therefore, A is compact. \square

The next result shows the relationship between the mappings H and B in the sense of large contractions. Assume that

$$(3.20) \quad \max\{|H(-L)|, |H(L)|\} \leq \frac{(J-1)L}{J},$$

and

$$(3.21) \quad [2\eta + 1] L k_3 \leq K.$$

Lemma 4. Let B be defined by (3.4), suppose (3.12), (3.20), (3.21) and all conditions of Theorem 2 hold. Then $B : \mathbf{M} \rightarrow \mathbf{M}$ is a large contraction.

Proof. Let B be defined by (3.4). Obviously, B is continuous and it is easy to show that $(B\varphi)(t+T) = (B\varphi)(t)$. For having $B\varphi \in \mathbf{M}$, we will show that $\|B\varphi\| \leq L$ and

$$|(B\varphi)(t_2) - (B\varphi)(t_1)| \leq K |t_2 - t_1|, \quad \forall t_1, t_2 \in [0, T].$$

Let $\varphi \in \mathbf{M}$ by (3.20), we get

$$|(B\varphi)(t)| \leq \eta \int_{t-T}^t a(u) \max\{|H(-L)|, |H(L)|\} e^{-\int_u^t a(k) dk} du \leq \frac{(J-1)L}{J} \leq L.$$

Let $t_1, t_2 \in [0, T]$ with $t_1 < t_2$, by (3.12), (3.20), (3.21), we have

$$\begin{aligned}
& |(B\varphi)(t_1) - (B\varphi)(t_2)| \\
& \leq \eta \left| \int_{t_2-T}^{t_2} a(u) H(\varphi(u)) e^{-\int_u^{t_2} a(k)dk} du - \int_{t_1-T}^{t_1} a(u) H(\varphi(u)) \right. \\
& \quad \left. e^{-\int_u^{t_1} a(k)dk} du \right| \\
& \leq \eta \left| \int_{t_1}^{t_2} a(u) H(\varphi(u)) e^{-\int_u^{t_2} a(k)dk} du \right| \\
& \quad + \eta \left| \int_{t_1-T}^{t_1} a(u) H(\varphi(u)) \left(e^{-\int_u^{t_2} a(k)dk} - e^{-\int_u^{t_1} a(k)dk} \right) du \right| \\
& \quad + \eta \left| \int_{t_1-T}^{t_2-T} a(u) H(\varphi(u)) e^{-\int_u^{t_2} a(k)dk} du \right| \\
& \leq 2\eta \int_{t_1}^{t_2} a(u) |H(\varphi(u))| e^{-\int_u^{t_2} a(k)dk} du \\
& \quad + \eta \left| e^{-\int_{t_1}^{t_2} a(k)dk} - 1 \right| \int_{t_1-T}^{t_1} a(u) |H(\varphi(u))| e^{-\int_u^{t_1} a(k)dk} du \\
& \leq 2 \frac{(J-1)}{J} L \eta \int_{t_1}^{t_2} a(u) du + \frac{(J-1)}{J} L \int_{t_1}^{t_2} a(u) du \\
& \leq [2\eta + 1] \frac{(J-1)}{J} L k_3 |t_2 - t_1| \\
& \leq \frac{(J-1)}{J} K |t_2 - t_1| \leq K |t_2 - t_1|,
\end{aligned}$$

which implies $B : \mathbf{M} \rightarrow \mathbf{M}$.

By Theorem 2, H is large contraction on \mathbf{M} , then for any $\varphi, \psi \in \mathbf{M}$ with $\varphi \neq \psi$, we get

$$\|B\varphi - B\psi\| \leq \|\varphi - \psi\|.$$

Now, let $\varepsilon \in (0, 1)$ be given and let $\varphi, \psi \in \mathbf{M}$, with $\|\varphi - \psi\| \geq \varepsilon$, from the proof of Theorem 2, we have found a $\delta \in (0, 1)$, such that

$$|(H\varphi)(t) - (H\psi)(t)| \leq \delta \|\varphi - \psi\|.$$

Thus,

$$\begin{aligned}
|(B\varphi)(t) - (B\psi)(t)| & \leq \left| \eta \int_{t-T}^t a(u) [H(\varphi(u)) - H(\psi(u))] e^{-\int_u^t a(k)dk} du \right| \\
& \leq \delta \|\varphi - \psi\| \eta \int_{t-T}^t a(u) e^{-\int_u^t a(k)dk} du \leq \delta \|\varphi - \psi\|.
\end{aligned}$$

The proof is complete. \square

Theorem 3. Suppose the hypotheses of Lemmas 2–4 hold. Let \mathbf{M} defined by (3.1), then (1.1) has a T -periodic solution in \mathbf{M} .

Proof. By Lemmas 2 and 3 $A : \mathbf{M} \rightarrow \mathbf{M}$ is continuous and $A(\mathbf{M})$ is contained in a compact set. Also, from Lemma 4, the mapping $B : \mathbf{M} \rightarrow \mathbf{M}$ is a large contraction. Next, we show that if $\varphi, \psi \in \mathbf{M}$, we have $\|A\varphi + B\psi\| \leq L$ and $|(A\varphi + B\psi)(t_2) - (A\varphi + B\psi)(t_1)| \leq K|t_2 - t_1|$, $\forall t_1, t_2 \in [0, T]$. Let $\varphi, \psi \in \mathbf{M}$ with $\|\varphi\|, \|\psi\| \leq L$. By (3.5)–(3.9) and (3.20), we get

$$\|A\varphi + B\psi\| \leq [\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4]L + \frac{(J-1)L}{J} \leq \frac{L}{J} + \frac{(J-1)L}{J} = L.$$

Now, let $\varphi, \psi \in \mathbf{M}$ and $t_1, t_2 \in [0, T]$. By (3.5)–(3.13), (3.20) and (3.21), we have

$$\begin{aligned} & |(A\varphi + B\psi)(t_2) - (A\varphi + B\psi)(t_1)| \\ & \leq |(A\varphi)(t_2) - (A\varphi)(t_1)| + |(B\psi)(t_2) - (B\psi)(t_1)| \\ & \leq \frac{K}{J}|t_2 - t_1| + \frac{(J-1)K}{J}|t_2 - t_1| \\ & = K|t_2 - t_1|. \end{aligned}$$

Clearly, all the hypotheses of Krasnoselskiĭ-Burton's theorem are satisfied. Thus there exists a fixed point $z \in \mathbf{M}$ such that $z = Az + Bz$. By Lemma 1 this fixed point is a solution of (1.1). Hence (1.1) has a T -periodic solution. \square

Example 1. Consider the nonlinear neutral differential equation

$$\frac{d}{dt}[x(t) - Q(t, x(t - g(t)))] = -a(t)h(x(t - \tau(t))) + \int_{-\infty}^t D(t, s)f(x(s))ds, \quad (3.22)$$

where

$$\begin{aligned} T &= 2\pi, \quad a(t) = 2, \quad \tau(t) = \frac{10^{-2}}{\sqrt{3}}, \quad g(t) = 2 \times 10^{-2}e^{-t}, \quad h(x) = x^3, \\ Q(t, x) &= 10^{-4}\sin(x), \quad D(t, s) = e^{s-t}, \quad f(x) = x^2. \end{aligned}$$

Then (3.22) has a 2π -periodic solution.

Proof. We have $h : \mathbf{R} \rightarrow \mathbf{R}$ is continuous on $\left[-\sqrt{3}/3, \sqrt{3}/3\right]$, differentiable on $\left(-\sqrt{3}/3, \sqrt{3}/3\right)$, strictly increasing on $\left[-\sqrt{3}/3, \sqrt{3}/3\right]$ and $\sup_{t \in (-\sqrt{3}/3, \sqrt{3}/3)} h'(t) \leq 1$. By Theorem 2, the mapping $H(x) = x - x^3$ is a large contraction on the set

$$\mathbf{M} = \left\{ \varphi \in P_{2\pi}, \|\varphi\| \leq \sqrt{3}/3, |\varphi(t_2) - \varphi(t_1)| \leq 100|t_2 - t_1|, \forall t_1, t_2 \in [0, 2\pi] \right\},$$

where $L = \sqrt{3}/3$ and $K = 100$. Doing straightforward computations, we obtain

$$\begin{aligned} E &= 1, \beta_1 = \frac{10^{-2}}{\sqrt{3}}, \beta_2 = 2, \beta_3 = e^{-4\pi}, E_1 = 1, E_2 = 2\sqrt{3}/3, E_3 = 10^{-4}, \\ q_1(t) &= 10^{-4}, q_2(t) = 0, \eta = \left(1 - e^{-4\pi}\right)^{-1}, \gamma_1 = \frac{4}{\sqrt{3}}10^{-2}, \gamma_2 = 2 \times 10^{-4}, \\ \gamma_3 &= 0, \gamma_4 = 4\pi \left(1 - e^{-4\pi}\right)^{-1} e^{-4\pi}, J \in [3, 42], k_1 = 0, k_2 = 2 \times 10^{-2}, k_3 = 2. \end{aligned}$$

All hypotheses of Theorem 3 are fulfilled and so (3.22) has a 2π -periodic solution belonging to \mathbf{M} . \square

4. Existence of nonnegative periodic solutions

In this section we obtain the existence of a nonnegative periodic solution of (1.1). By applying Theorem 1, we need to define a closed, convex, and bounded subset \mathcal{M} of P_T . So, let

$$\mathcal{M} = \left\{ \varphi \in P_T : 0 \leq \varphi \leq L, |\varphi(t_2) - \varphi(t_1)| \leq K|t_2 - t_1|, \forall t_1, t_2 \in [0, T] \right\}, \quad (4.1)$$

where L and K are positive constants. To simplify notation, we let

$$F(t, x(t)) = \int_{t-\tau(t)}^t a(u) h(x(u)) du, \quad (4.2)$$

and

$$m = \min_{u \in [t-T, t]} e^{-\int_u^t a(k) dk}, \quad M = \max_{u \in [t-T, t]} e^{-\int_u^t a(k) dk}. \quad (4.3)$$

It is easy to see that for all $(t, u) \in [0, 2T]^2$,

$$(4.4) \quad m \leq e^{-\int_u^t a(k)dk} \leq M.$$

Then, we obtain the existence of a nonnegative periodic solution of (1.1) by considering the two cases

$$(1) \quad F(t, x(t)) \geq 0, \quad \forall t \in [0, T], \quad x \in \mathcal{M}.$$

$$(2) \quad F(t, x(t)) \leq 0, \quad \forall t \in [0, T], \quad x \in \mathcal{M}.$$

In the case one, we assume for all $t \in [0, T]$, $x \in \mathcal{M}$, that there exist positive constants c_1 and c_2 such that

$$(4.5) \quad 0 \leq Q(t, x(t)) \leq c_1 L,$$

$$(4.6) \quad 0 \leq F(t, x(t)) \leq c_2 L,$$

$$(4.7) \quad c_1 + c_2 < 1,$$

$$(4.8) \quad 0 \leq -a(u)F(t, x(t)) + b(t)h(x(t)) - a(t)Q(t, x(t)) + \int_{-\infty}^t D(t, s)f(x(s))ds,$$

$$(4.9) \quad \begin{aligned} & -a(u)F(t, x(t)) + b(t)h(x(t)) + a(t)H(x(t)) \\ & -a(t)Q(t, x(t)) + \int_{-\infty}^t D(t, s)f(x(s))ds \leq \frac{L(1-c_1-c_2)}{M\eta T}. \end{aligned}$$

Lemma 5. Let A, B given by (3.3), (3.4), respectively. Assume (4.5)–(4.9) hold, then $A, B : \mathcal{M} \rightarrow \mathcal{M}$.

Proof. For having $A\varphi, B\varphi \in \mathcal{M}$, we show that $0 \leq A\varphi, B\varphi \leq L$ and $|(A\varphi)(t_2) - (A\varphi)(t_1)| |t_2 - t_1|, |(B\varphi)(t_2) - (B\varphi)(t_1)| \leq K |t_2 - t_1|, \forall t_1, t_2 \in [0, T]$. Let A defined by (3.3). So, for any $\varphi \in \mathcal{M}$, we have

$$\begin{aligned} 0 & \leq (A\varphi)(t) \leq Q(t, \varphi(t - g(t))) + F(t, \varphi(t)) - \eta \int_{t-T}^t F(t, \varphi(u)) a(u) e^{-\int_u^t a(k)dk} du \\ & \quad + \eta \int_{t-T}^t b(u) h(\varphi(u - \tau(u))) e^{-\int_u^t a(k)dk} du \\ & \quad + \eta \int_{t-T}^t \left[-a(u)Q(u, \varphi(u - g(u))) + \int_{-\infty}^u D(u, s)f(\varphi(s))ds \right] e^{-\int_u^t a(k)dk} du \\ & \leq \eta \int_{t-T}^t M \frac{L(1-c_1-c_2)}{M\eta T} du + c_1 L + c_2 L = L. \end{aligned}$$

From Lemma 2, we see that

$$|(A\varphi)(t_2) - (A\varphi)(t_1)| \leq \frac{K}{J} |t_2 - t_1| \leq K |t_2 - t_1|.$$

That is $A\varphi \in \mathcal{M}$.

Now, let B defined by (3.4). So, for any $\varphi \in \mathcal{M}$, we have

$$0 \leq (B\varphi)(t) \leq \eta \int_{t-T}^t M \frac{L(1-c_1-c_2)}{M\eta T} du \leq \eta MT \frac{L}{M\eta T} = L,$$

and from Lemma 4, we see that

$$|(B\varphi)(t_2) - (B\varphi)(t_1)| \leq \frac{(J-1)K}{J} |t_2 - t_1| \leq K |t_2 - t_1|$$

That is $B\varphi \in \mathcal{M}$. □

Theorem 4. Suppose the hypotheses of Lemmas 3–5 hold. Then (1.1) has a nonnegative T -periodic solution x in the subset \mathcal{M} .

Proof. By Lemma 3, A is completely continuous. Also, from Lemma 4, the mapping B is a large contraction. By Lemma 5, $A, B : \mathcal{M} \rightarrow \mathcal{M}$. Next, we show that if $\varphi, \psi \in \mathcal{M}$, we have $0 \leq A\varphi + B\psi \leq L$ and $|(A\varphi + B\psi)(t_2) - (A\varphi + B\psi)(t_1)| \leq K |t_2 - t_1|$, $\forall t_1, t_2 \in [0, T]$. Let $\varphi, \psi \in \mathcal{M}$ with $0 \leq \varphi, \psi \leq L$. By (4.5)–(4.9), we get

$$\begin{aligned} & (A\varphi)(t) + (B\psi)(t) \\ &= \eta \int_{t-T}^t a(u) H(\psi(u)) e^{-\int_u^t a(k)dk} du + Q(t, \varphi(t-g(t))) \\ &+ F(t, \varphi(t)) - \eta \int_{t-T}^t F(t, \varphi(u)) a(u) e^{-\int_u^t a(k)dk} du \\ &+ \eta \int_{t-T}^t b(u) h(\varphi(u-\tau(u))) e^{-\int_u^t a(k)dk} du \\ &+ \eta \int_{t-T}^t \left[-a(u) Q(u, \varphi(u-g(u))) + \int_{-\infty}^u D(u, s) f(\varphi(s)) ds \right] e^{-\int_u^t a(k)dk} du \\ &\leq \eta \int_{t-T}^t M \frac{L(1-c_1-c_2)}{M\eta T} du + c_1 L + c_2 L = L. \end{aligned}$$

On the other hand, we have

$$(A\varphi)(t) + (B\psi)(t) \geq 0.$$

Now, let $\varphi, \psi \in \mathcal{M}$ and $t_1, t_2 \in [0, T]$. By Lemmas 2, 4, we have

$$\begin{aligned}
& |(A\varphi + B\psi)(t_2) - (A\varphi + B\psi)(t_1)| \\
& \leq |(A\varphi)(t_2) - (A\varphi)(t_1)| + |(B\psi)(t_2) - (B\psi)(t_1)| \\
& \leq \frac{K}{J} |t_2 - t_1| + \frac{(J-1)K}{J} |t_2 - t_1| \\
& \leq K |t_2 - t_1|.
\end{aligned}$$

Clearly, all the hypotheses of Krasnoselskiĭ-Burton's theorem are satisfied. Thus there exists a fixed point $z \in \mathcal{M}$ such that $z = Az + Bz$. By Lemma 1 this fixed point is a solution of (1.1) and the proof is complete. \square

Example 2. Consider the equation

$$(4.10) \quad \frac{d}{dt} [x(t) - Q(t, x(t - g(t)))] = -a(t) h(x(t - \tau(t))) + \int_{-\infty}^t D(t, s) f(x(s)) ds,$$

where

$$\begin{aligned}
T &= 2\pi, \quad a(t) = \frac{10^{-2}}{4}, \quad \tau(t) = 2\pi, \quad h(x) = x^3, \quad Q(t, x) = 10^{-4}x, \\
F(t, x(t)) &= \frac{10^{-2}}{4} \int_{t-2\pi}^t x^3(u) du, \quad D(t, s) = e^{s-t}, \quad f(x) = 10^{-4} \left(x + \frac{\pi^4}{4}\right).
\end{aligned}$$

Then (4.10) has a nonnegative 2π -periodic solution.

Proof. By Example 1, the mapping $H(x) = x - x^3$ is a large contraction on the set

$$\mathcal{M} = \left\{ \varphi \in P_{2\pi}, \quad 0 \leq \varphi \leq \sqrt{3}/3, \quad |\varphi(t_2) - \varphi(t_1)| \leq 100 |t_2 - t_1|, \quad \forall t_1, t_2 \in [0, T] \right\}.$$

A simple calculation yields

$$\begin{aligned}
F(t, x(t)) &= \frac{10^{-2}}{4} \int_{t-2\pi}^t x^3(u) du = \frac{1}{4} \int_0^{2\pi} x^3(u) du = \frac{10^{-2}}{4} \left[\frac{x^4}{4} \right]_0^{2\pi} = 10^{-2} \pi^4 \geq 0, \\
m &= e^{-\frac{10^{-2}}{2}\pi}, \quad M = 1, \quad \eta = \left(1 - e^{-\frac{10^{-2}}{2}\pi} \right)^{-1}, \quad c_1 = 10^{-4}, \quad c_2 = \frac{10^{-2}}{6} \pi.
\end{aligned}$$

Then for $x \in [0, \sqrt{3}/3]$, we have

$$0 \leq -a(t) F(t, x(t)) + b(t) h(x(t)) - a(t) Q(t, x(t)) + \int_{-\infty}^t D(t, s) f(x(s)) ds.$$

On the other hand, we have

$$\begin{aligned} & -a(t)F(t, x(t)) + b(t)h(x(t)) + a(t)H(x(t)) \\ & -a(t)Q(t, x(t)) + \int_{-\infty}^t D(t, s)f(x(s))ds \\ & \leq 1.006 \times 10^{-3} < 1.425 \times 10^{-3} \simeq \frac{L(1-c_1-c_2)}{M\eta T}. \end{aligned}$$

All conditions of Theorem 4 hold and so (4.10) has a nonnegative 2π -periodic solution belonging to \mathcal{M} . \square

In the case two, we substitute conditions (4.6)–(4.9) with the following conditions, respectively. We assume that there exist a negative constant c_3 such that

$$(4.11) \quad c_3 L \leq F(t, x(t)) \leq 0,$$

$$(4.12) \quad -c_3 + c_1 < 1,$$

$$(4.13) \quad \begin{aligned} \frac{-c_3 L}{m\eta T} & \leq -a(u)F(t, x(t)) + b(t)h(x(t)) + a(t)H(x(t)) \\ & -a(t)Q(t, x(t)) + \int_{-\infty}^t D(t, s)f(x(s))ds, \end{aligned}$$

and

$$(4.14) \quad \begin{aligned} & -a(u)F(t, x(t)) + b(t)h(x(t)) + a(t)H(x(t)) \\ & -a(t)Q(t, x(t)) + \int_{-\infty}^t D(t, s)f(x(s))ds \leq \frac{L(1-c_1)}{M\eta T}. \end{aligned}$$

Theorem 5. Suppose (4.5), (4.11)–(4.14) and the hypotheses of Lemmas 2–4 hold. Then (1.1) has a nonnegative T -periodic solution x in the subset \mathcal{M} .

Proof. By Lemma 3, A is completely continuous. Also, from Lemma 4, the mapping B is a large contraction. By Lemma 5, $A, B : \mathcal{M} \rightarrow \mathcal{M}$. Next, we show that if $\varphi, \psi \in \mathcal{M}$, we have $0 \leq A\varphi + B\psi \leq L$ and $|(A\varphi + B\psi)(t_2) - (A\varphi + B\psi)(t_1)| \leq K|t_2 - t_1|$, $\forall t_1, t_2 \in [0, T]$. Let $\varphi, \psi \in \mathcal{M}$ with $0 \leq \varphi, \psi \leq L$. By (4.5) and (4.11)–(4.14) we get

$$\begin{aligned}
& (A\varphi)(t) + (B\psi)(t) \\
&= \eta \int_{t-T}^t a(u) H(\psi(u)) e^{-\int_u^t a(k)dk} du + Q(t, \varphi(t-g(t))) \\
&+ F(t, \varphi(t)) - \eta \int_{t-T}^t F(t, \varphi(u)) a(u) e^{-\int_u^t a(k)dk} du \\
&+ \eta \int_{t-T}^t b(u) h(\varphi(u-\tau(u))) e^{-\int_u^t a(k)dk} du \\
&+ \eta \int_{t-T}^t \left[-a(u) Q(u, \varphi(u-g(u))) + \int_{-\infty}^u D(u, s) f(\varphi(s)) ds \right] e^{-\int_u^t a(k)dk} du \\
&\leq \eta \int_{t-T}^t M \frac{L(1-c_1)}{M\eta T} du + c_1 L = L.
\end{aligned}$$

On the other hand, we have

$$(A\varphi)(t) + (B\psi)(t) \geq \eta \int_{t-T}^t m \frac{-c_3 L}{m\eta T} du + c_3 L = 0.$$

Now, let $\varphi, \psi \in \mathcal{M}$ and $t_1, t_2 \in [0, T]$. By Lemmas 2 and 4, we have

$$\begin{aligned}
& |(A\varphi + B\psi)(t_2) - (A\varphi + B\psi)(t_1)| \\
&\leq |(A\varphi)(t_2) - (A\varphi)(t_1)| + |(B\psi)(t_2) - (B\psi)(t_1)| \\
&\leq \frac{K}{J} |t_2 - t_1| + \frac{(J-1)K}{J} |t_2 - t_1| \\
&= K |t_2 - t_1|.
\end{aligned}$$

Clearly, all the hypotheses of Krasnoselskiĭ-Burton's theorem are satisfied. Thus there exists a fixed point $z \in \mathcal{M}$ such that $z = Az + Bz$. By Lemma 1 this fixed point is a solution of (1.1) and the proof is complete. \square

Acknowledgement. The authors would like to thank the anonymous referee for his valuable comments and good advices.

References

- [1] M. Adivar and Y. N. Raffoul, "Existence of periodic solutions in totally nonlinear delay dynamic equations", *Electronic Journal of Qualitative Theory of Differential Equations*, vol. 2009, no. 1, pp. 1-20, 2009. doi: 10.14232/ejqtde.2009.4.1
- [2] S. Althubiti, H. Ali Makhzoum and Y. N. Raffoul, "Periodic solutions and stability in nonlinear neutral system with infinite delay", *Applied Mathematical Sciences*, vol. 7, no. 136, pp. 6749-6764, 2013. doi: 10.12988/ams.2013.38462

- [3] A. Ardjouni, I. Derrardjia and A. Djoudi, "Stability in totally nonlinear neutral differential equations with variable delay", *Acta Mathematica Universitatis Comenianae*, vol. 83, no. 1, pp. 119-134, 2014. [On line]. Available: <https://bit.ly/3NbtrMK>
- [4] A. Ardjouni and A. Djoudi, "Existence of periodic solutions for nonlinear neutral dynamic equations with functional delay on a time scale", *Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica*, vol. 52, no. 1, pp. 5-19, 2013. [On line]. Available: <https://bit.ly/3W7PJTG>
- [5] A. Ardjouni and A. Djoudi, "Stability in neutral nonlinear dynamic equations on time scale with unbounded delay", *Studia Universitatis Babe -Bolyai Mathematica*, vol. 57, no. 4, pp. 481-496, 2012.
- [6] A. Ardjouni and A. Djoudi, "Existence of positive periodic solutions for a nonlinear neutral differential equations with variable delay", *Applied Mathematics E-Notes*, vol. 12, pp. 94-101, 2012. [On line]. Available: <https://bit.ly/3TZYHR4>
- [7] A. Ardjouni and A. Djoudi, "Fixed points and stability in linear neutral differential equations with variable delays", *Nonlinear Analysis: Theory, Methods and Applications*, vol. 74, pp. 2062-2070, 2011. doi: 10.1016/j.na.2010.10.050
- [8] M. Belaid, A. Ardjouni and A. Djoudi, "Global asymptotic stability in nonlinear neutral dynamic equations on time scales", *Journal of Nonlinear Functional Analysis*, vol. 2018, no. 1, pp. 1-10, 2018. doi: 10.23952/jnfa.2018.1
- [9] M. Belaid, A. Ardjouni and A. Djoudi, "Stability in totally nonlinear neutral dynamic equations on time scales", *International Journal of Analysis and Applications*, vol. 11, no. 2, pp. 110-123, 2016. [On line]. Available: <https://bit.ly/3SJCZQn>
- [10] K. Bessioud, A. Ardjouni, A. Djoudi, "Asymptotic stability in nonlinear neutral Levin-Nohel Integro-Differential Equations", *Journal of Nonlinear Functional Analysis*, vol. 2017, no. 19, pp. 1-12, 2017. doi: 10.23952/jnfa.2017.19
- [11] T. A. Burton, *Stability by Fixed Point Theory for Functional Differential Equations*. New York: Dover Publications, 2006.
- [12] I. Derrardjia, A. Ardjouni, and A. Djoudi, "Stability by Krasnoselskii's theorem in totally nonlinear neutral differential equations", *Opuscula Mathematica*, vol. 33, no. 2, pp. 255-272, 2013. doi: 10.7494/opmath.2013.33.2.255
- [13] M. B. Mesmouli, A. Ardjouni, and A. Djoudi, "Study of the periodic or nonnegative periodic solutions of functional differential equations via krasnoselskii-Burton's theorem", *Differential Equations and Dynamical Systems*, vol. 24, no. 4, pp. 391-406, 2015. doi: 10.1007/s12591-014-0235-5

- [14] M. B. Mesmouli, A. Ardjouni, and A. Djoudi, "Existence and uniqueness of periodic solutions for a system of nonlinear neutral functional differential equations with two functional delays", *Rendiconti del Circolo Matematico di Palermo*, vol. 63, no. 3, pp. 409–424, 2014. doi: 10.1007/s12215-014-0162-x
- [15] F. Nouioua, A. Ardjouni, A. Merzougui and A. Djoudi, "Existence of positive periodic solutions for a third-order delay differential equation", *International Journal of Analysis and Applications*, vol. 13, no. 2, pp. 136-143, 2017. [On line]. Available: <https://bit.ly/3Nf0rUl>

Abderrahim Guerfi

Applied Mathematics Lab,
Faculty of Sciences,
Department of Mathematics,
University of Annaba,
P.O. Box 12, Annaba 23000,
Algeria
e-mail: abderrahimg21@gmail.com

and

Abdelouaheb Ardjouni

Department of Mathematics and Informatics,
University of Souk Ahras,
P. O. Box 1553, Souk Ahras, 41000,
Algeria
Applied Mathematics Lab,
Faculty of Sciences,
Department of Mathematics,
University of Annaba,
P.O. Box 12, Annaba 23000,
Algeria
e-mail: abd_ardjouni@yahoo.fr
Corresponding author