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A note on m-Zumkeller cordial labeling of graphs

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Abstract

Let $G=(V,E)$ be a graph. An m-Zumkeller cordial labeling of the graph G is defined by an injective function $f:V\rightarrow\mathbf{N}$ such that there exists an induced function $f^*:E\rightarrow\{0,1\}$ defined by $f^*(uv)=f(u)\cdot f(v)$ that satisfies the following conditions-

i. *For every $uv\in E$,*

$$f^*(uv)=\begin{cases}1, & \text{if } f(u)\cdot f(v) \text{ is an } m\text{-Zumkeller number}\\ 0, & \text{otherwise}\end{cases}$$

ii. $|e_{f^*}(0)-e_{f^*}(1)|\leq 1$

where $e_{f^*}(0)$ and $e_{f^*}(1)$ denote the number of edges of the graph G having label 0 and 1 respectively under f^*.

In this paper we prove that there exist an m-Zumkeller cordial labeling of graphs in paths, cycles, comb graphs, ladder graphs, twig graphs, helm graphs, wheel graphs, crown graphs and star graphs.

Key-Words: *m-Zumkeller numbers, comb graphs, ladder graph,; twig graphs, helm graphs.*

2010 Mathematics Subject Classification*: 11Axx; 97F60; 05Cxx.*

1. Introduction

A positive integer n is perfect if $\sum_{d|n} d = 2n$. $6, 28, 496$ are the first few perfect numbers. Till now there are 51 known perfect numbers [9], all of which are even. It is unknown till now whether the odd perfect numbers exist or not.

Many mathematician generalized the concept of perfect number time to time. One of such generalization is Zumkeller numbers. A positive integer n is Zumkeller [8] if we can partition the set of all the positive divisors of an integer n into two disjoint subsets such that sum of each partition subset is $\frac{\sigma(n)}{2}$, where $\sigma(n)$ gives the sum of all the positive divisors of n.

In graph theory, a graph labeling is an assignment of labels by integers to the vertices or edges, or both of a graph subject to certain conditions. The first attempt of labeling of graphs was seen in [1] in mid 1960s. Various types of Zumkeller labeling of graphs are seen in the literature [2, 3, 6, 7].

Generalizing the concept of Zumkeller number H. Patodia and H. K. Saikia defined a new type of number as m-Zumkeller number in [5].

Definition 1.1. *A positive integer n is an m-Zumkeller number if we can partition the set of all the positive divisors of n into two disjoint subsets of equal product.*

In [4] m-Zumkeller labeling techniques of complete graphs and bipartite graphs are seen. In this paper, we discuss the cordial labeling of various graphs by m-Zumkeller numbers.

2. Properties of m-Zumkeller numbers

Various properties of m-Zumkeller numbers discussed in [5] are given below:

1. If n is an m-Zumkeller number, then $\tau(n) \geq 4$, where $\tau(n)$ gives the number of positive divisors of n.
2. The integer $n = \prod_{i=1}^{r} p_i^{\alpha_i}$ (where $p_i^{'s}$ are distinct primes) is an m-Zumkeller number if and only if $4|\alpha_i \tau(n)$, $\forall\, i = 1, 2, ..., r$.
3. The product of distinct prime numbers i.e. $\prod_{i=1}^{r} p_i$ (where $p_i^{'s}$ are distinct primes, $r \geq 2$) are m-Zumkeller numbers.

4. The integers of the form $2^k \prod_{i=1}^{r} p_i$ where k is any positive integer, $r \geq 2$ and $p_i^{\prime s}$ are distinct odd primes are m-Zumkeller numbers.

3. Main Results

Definition 3.1. *Let $G = (V, E)$ be a graph. An m-Zumkeller cordial labeling of the graph G is defined by an injective function $f : V \to \mathbf{N}$ such that there exists an induced function $f^* : E \to \{0, 1\}$ defined by*
$f^*(uv) = f(u) \cdot f(v)$ that satisfies the following conditions-

i. For every $uv \in E$,

$$f^*(uv) = \begin{cases} 1, & \text{if } f(u) \cdot f(v) \text{ is an } m\text{-Zumkeller number} \\ 0, & \text{otherwise} \end{cases}$$

ii. $|e_{f^*}(0) - e_{f^*}(1)| \leq 1$

where $e_{f^*}(0)$ and $e_{f^*}(1)$ denote the number of edges of the graph G having label 0 and 1 respectively under f^*.

Definition 3.2. *If a graph $G(V, E)$ admits an-m-Zumkeller cordial labeling then the graph G is known as m-Zumkeller cordial graph.*

Example 3.1. *Figure 1 gives an example of m-Zumkeller cordial graph.*

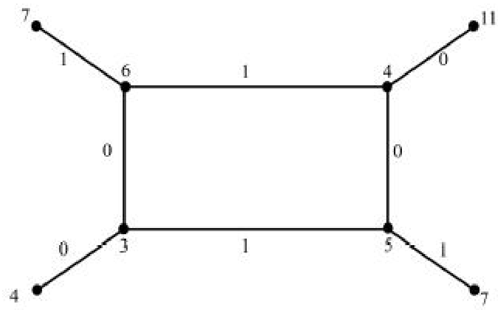

Figure 1: *m-Zumkeller cordial graph.*

Proposition 3.1. *A subgraph of an m-Zumkeller cordial graph need not be an m-Zumkeller cordial graph.*

Proof. The proof is obvious. □

Proposition 3.2. *The path P_n with n vertices admits an m-Zumkeller cordial labeling.*

Proof. Let, $V = \{v_i | 1 \leq i \leq n\}$ be the vertex set and $E = \{v_i v_{i+1} | 1 \leq i \leq n-1\}$ be the edge set of the path P_n.

Define a $1-1$ function $f : V \rightarrow \mathbf{N}$ such that for $i = 1, 3, 5, ...$

$$f(v_i) = 2^{\frac{i+1}{2}}$$
$$f(v_{i+1}) = 2^{\frac{i+3}{2}} p$$

where p is an odd prime and is less than 10.

Now the edge labels of P_n are given below,

$$f^*(v_i v_{i+1}) = f(v_i) \cdot f(v_{i+1}) = 2^{\frac{i+1}{2}}.2^{\frac{i+3}{2}} p = 2^{i+2} p, \text{ an } m\text{-Zumkeller number.}$$

$$f^*(v_{i+1} v_{i+2}) = f(v_{i+1}) \cdot f(v_{i+2}) = 2^{\frac{i+3}{2}} p.2^{\frac{i+3}{2}} = 2^{i+3} p, \text{ not an } m\text{-Zumkeller number.}$$

Thus alternate edges of the path P_n have m-Zumkeller numbers.

Therefore, $|e_{f^*}(0) - e_{f^*}(1)| = 0$ if n is odd.

and $|e_{f^*}(0) - e_{f^*}(1)| = 1$ if n is even.

Hence, we can conclude that P_n admits an m-Zumkeller cordial labeling. □

Example 3.2. *The m-Zumkeller cordial labeling of path P_8 and P_9 for $p = 5$ are shown in figure 2 and figure 3 respectively.*

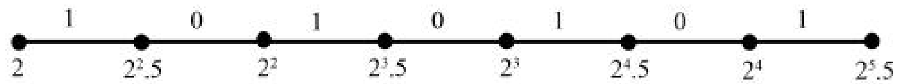

Figure 2: *2-m-Zumkeller labeling of P_8.*

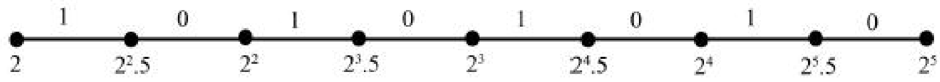

Figure 3: *2-m-Zumkeller labeling of P_9.*

Proposition 3.3. *The odd cycle C_n admits an m-Zumkeller cordial labeling.*

Proof. Let, $V = \{v_i | 1 \leq i \leq n\}$ be the vertex set and $E = \{v_i v_{i+1} | 1 \leq i \leq n-1\} \cup \{v_n v_1\}$ be the edge set of the odd cycle C_n.

Now define an injective function $f : V \to \mathbf{N}$ similarly as proposition 3.2 we get

$$f^* (v_i v_{i+1}) = 2^{i+2} p$$
$$f^* (v_{i+1} v_{i+2}) = 2^{i+3} p$$

Also

$$f^* (v_n v_1) = f (v_n) \cdot f (v_1) = 2^{\frac{n+1}{2}}.2 = 2^{\frac{n+3}{2}},$$

which may or may not be an m-Zumkeller number.

Hence, we can conclude that

$$|e_{f^*} (0) - e_{f^*} (1)| = 1.$$

Thus, the odd cycle C_n admits an m-Zumkeller cordial labeling. □

Example 3.3. *The m-Zumkeller cordial labeling of the odd cycle C_9 for $p = 5$ is shown in figure 4.*

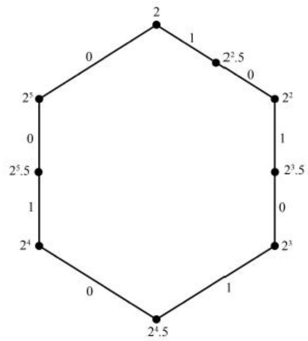

Figure 4: *m-Zumkeller cordial labeling of the odd cycle C_9.*

Proposition 3.4. *The even cycle C_n admits an m-Zumkeller cordial labeling for $n \equiv 0$ (mod 4).*

Proof. Let, $V = \{v_i | 1 \leq i \leq n\}$ be the vertex set and $E = \{v_i v_{i+1} | 1 \leq i \leq n-1\} \cup \{v_n v_1\}$ be the edge set of the even cycle C_n.

Now define an injective function $f : V \to \mathbf{N}$ similarly as proposition 3.2 we get,

$$f^*(v_i v_{i+1}) = 2^{i+2} p$$
$$f^*(v_{i+1} v_{i+2}) = 2^{i+3} p.$$

Thus till now we get $\frac{n}{2}$ number of m-Zumkeller numbers and $\frac{n}{2} - 1$ number of non m-Zumkeller numbers on the edges of C_n.

Now

$$f^*(v_n v_1) = f(v_n) \cdot f(v_1) = 2^{\frac{n+2}{2}} p.2 = 2^{\frac{n+4}{2}} p.$$

Therefore C_n is an m-Zumkeller cordial graph if $2^{\frac{n+4}{2}} p$ is a non m-Zumkeller number i.e. if $n \equiv 0$ (mod 4).

Thus, we can conclude that even cycle C_n admits an m-Zumkeller cordial labeling if $n \equiv 0$ (mod 4) . □

Example 3.4. *The m-Zumkeller cordial labeling of the even cycle C_8 is shown in figure 5.*

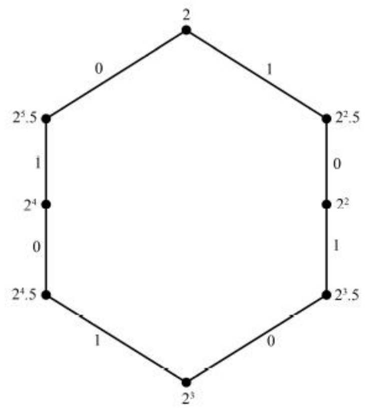

Figure 5: *m-Zumkeller cordial labeling of the even cycle C_8.*

Definition 3.3. *A comb graph is obtained by joining a single pendent edge to each vertex of a path. It is denoted by $P_n \odot K_1$ and it contains $2n$ number of vertices and $2n-1$ number edges.*

Proposition 3.5. *The comb graph $P_n \odot K_1$ admits an m-Zumkeller cordial labeling.*

Proof. Let, $V = \{v_i, u_i | 1 \le i \le n\}$ be the vertex set and $E = \{v_i v_{i+1} | 1 \le i \le n-1\} \cup \{v_i u_i | 1 \le i \le n\}$ be the edge set of the comb graph $P_n \odot K_1$.

Define a 1-1 function $f : V \to \mathbf{N}$ such that for $i = 1, 3, 5, ...$

$$f(v_i) = 2^{\frac{i+1}{2}}$$
$$f(v_{i+1}) = 2^{\frac{i+3}{2}} p_1$$
$$f(u_i) = 2^{\frac{i+1}{2}} p_1$$
$$f(u_{i+1}) = p_2$$

where p_1, p_2 are distinct odd primes and are less than 10.

Now

$f^*(v_i v_{i+1}) = f(v_i) \cdot f(v_{i+1}) = 2^{\frac{i+1}{2}}.2^{\frac{i+3}{2}} p_1 = 2^{i+2} p_1$, an m-Zumkeller number.

$f^*(v_{i+1} v_{i+2}) = f(v_{i+1}) \cdot f(v_{i+2}) = 2^{\frac{i+3}{2}} p_1 . 2^{\frac{i+3}{2}} = 2^{i+3} p_1$, not an m-Zumkeller number.

$f^*(v_i u_i) = f(v_i) \cdot f(u_i) = 2^{\frac{i+1}{2}}.2^{\frac{i+1}{2}} p_1 = 2^{i+1} p_1$, not an m-Zumkeller number.

$f^*(v_{i+1} u_{i+1}) = f(v_{i+1}) \cdot f(u_{i+1}) = 2^{\frac{i+3}{2}} p_1 . p_2 = 2^{\frac{i+3}{2}} p_1 p_2$, an m-Zumkeller number.

Now, if n is even then we get total n number of m-Zumkeller numbers on the edges of $P_n \odot K_1$ and $n-1$ number of non-m-Zumkeller numbers on the edges of $P_n \odot K_1$.

Hence, in this case $|e_{f^*}(0) - e_{f^*}(1)| = 1$.

Again if n is odd then we get total $n-1$ number of m-Zumkeller numbers on the edges of $P_n \odot K_1$ and n number of non-m-Zumkeller numbers on the edges of $P_n \odot K_1$.

Thus, in this case also $|e_{f^*}(0) - e_{f^*}(1)| = 1$.

Hence, $P_n \odot K_1$ admits an m-Zumkeller cordial labeling. □

Example 3.5. *The m-Zumkeller cordial labeling of the comb graphs $P_7 \odot K_1$ and $P_8 \odot K_1$ for $p_1 = 3$, $p_2 = 7$ are shown in figure 6 and figure 7 respectively.*

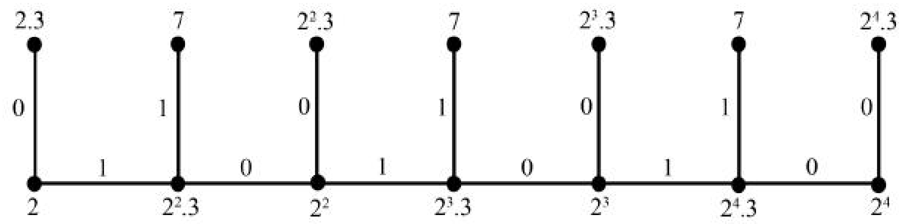

Figure 6: *m-Zumkeller cordial labeling of $P_7 \odot K_1$.*

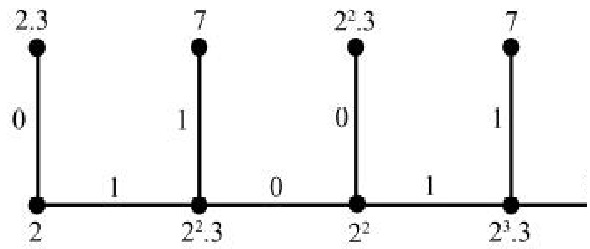

Figure 7: *m-Zumkeller cordial labeling of $P_8 \odot K_1$.*

Definition 3.4. *The Ladder graph L_n $(n \geq 2)$ is defined as the product of two path graphs P_n and P_2 i.e. $L_n = P_n \times P_2$. Ladder graph L_n contains $2n$ number of vertices and $3n-2$ number of edges.*

Proposition 3.6. *The ladder graph L_n admits an m-Zumkeller cordial labeling.*

Proof. Let, $V = \{v_i, u_i | 1 \leq i \leq n\}$ be the vertex set and $E = \{v_i v_{i+1} | 1 \leq i \leq n-1\} \cup \{v_i u_i | 1 \leq i \leq n\} \cup \{u_j u_{j+1} | 1 \leq j \leq n-1\}$ be the edge set of the ladder graph L_n.

Define a 1-1 function $f : V \to \mathbf{N}$ such that for $i = 1, 3, 5, ...$

$$f(v_i) = 2^{\frac{i+3}{2}}$$
$$f(v_{i+1}) = 2^{\frac{i+3}{2}} p_1$$
$$f(u_i) = 2^{\frac{i+1}{2}} p_2$$
$$f(u_{i+1}) = 2^{\frac{i+3}{2}}$$

where p_1, p_2 are distinct odd primes and are less than 10.

Now

$f^*(v_i v_{i+1}) = f(v_i) \cdot f(v_{i+1}) = 2^{\frac{i+3}{2}}.2^{\frac{i+3}{2}} p_1 = 2^{i+3} p_1$, not an m-Zumkeller number.

$f^*(v_{i+1} v_{i+2}) = f(v_{i+1}) \cdot f(v_{i+2}) = 2^{\frac{i+3}{2}} p_1.2^{\frac{i+5}{2}} = 2^{i+4} p_1$, an m-Zumkeller number.

$f^*(v_i u_i) = f(v_i) \cdot f(u_i) = 2^{\frac{i+3}{2}}.2^{\frac{i+1}{2}} p_2 = 2^{i+2} p_2$, an m-Zumkeller number.

$f^*(v_{i+1} u_{i+1}) = f(v_{i+1}) \cdot f(u_{i+1}) = 2^{\frac{i+3}{2}} p_1.2^{\frac{i+3}{2}} = 2^{i+3} p_1$, not an m-Zumkeller number.

$f^*(u_i u_{i+1}) = f(u_i) \cdot f(u_{i+1}) = 2^{\frac{i+1}{2}} p_2.2^{\frac{i+3}{2}} = 2^{i+2} p_2$, an m-Zumkeller number.

$f^*(u_{i+1} u_{i+2}) = f(u_{i+1}) \cdot f(u_{i+2}) = 2^{\frac{i+3}{2}}.2^{\frac{i+3}{2}} p_2 = 2^{i+3} p_2$, not an m-Zumkeller number.

Now, if n is even then we get total $\left(\frac{n}{2} - 1\right) + \frac{n}{2} + \frac{n}{2} = \frac{3n}{2} - 1$ number of m-Zumkeller numbers on the edges of L_n and $\frac{n}{2} + \frac{n}{2} + \left(\frac{n}{2} - 1\right) = \frac{3n}{2} - 1$ number of non m-Zumkeller numbers on the edges of L_n.

Hence, in this case $|e_{f^*}(0) - e_{f^*}(1)| = 0$.

Again, if n is odd then we get total $\frac{n-1}{2} + \left(\frac{n-1}{2} + 1\right) + \frac{n-1}{2} = \frac{3n-1}{2}$ number of m-Zumkeller numbers on the edges of L_n and $\frac{n-1}{2} + \frac{n-1}{2} + \frac{n-1}{2} = \frac{3(n-1)}{2}$ number of non m-Zumkeller numbers on the edges of L_n.

Thus, in this case we have $|e_{f^*}(0) - e_{f^*}(1)| = 1$.

Hence, we can conclude that ladder graph L_n admits an m-Zumkeller cordial labeling. □

Example 3.6. *The m-Zumkeller cordial labeling of the ladder graph L_7 and L_8 for $p_1 = 3$, $p_2 = 5$ are shown in figure 8 and figure 9 respectively.*

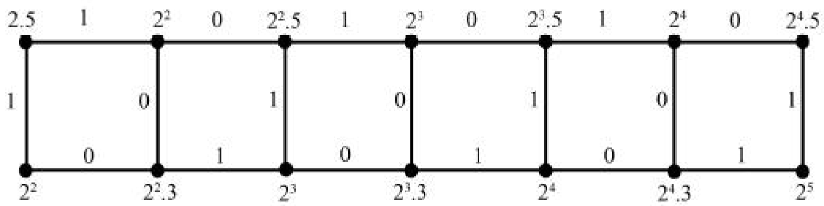

Figure 8: *m-Zumkeller cordial labeling of L_7.*

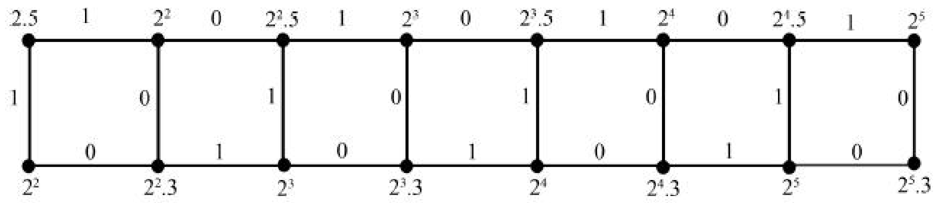

Figure 9: *m-Zumkeller cordial labeling of L_8.*

Definition 3.5. *A twig graph [2] is a graph obtained from a path of n vertices P_n by attaching exactly two pendent edges to each internal vertex of P_n. If the number of vertices in the path P_n is even then the twig is called an even twig, otherwise its called an odd twig.*

Proposition 3.7. *The twig graph admits an m-Zumkeller cordial labeling.*

Proof. Let, $V = \{u_j | 1 \leq j \leq n-2\} \cup \{v_i | 1 \leq i \leq n\} \cup \{w_k | 1 \leq k \leq n-2\}$ be the vertex set and

$$\begin{aligned} E = \ & \{v_i v_{i+1} | 1 \leq i \leq n-1\} \cup \{v_2 u_1\} \cup \{v_2 w_1\} \\ & \cup \{v_i u_{j+1} | 3 \leq i \leq n-1, 1 \leq j \leq n-3, i = j+2\} \\ & \cup \{v_i w_{k+1} | 3 \leq i \leq n-1, 1 \leq k \leq n-3, i = k+2\} \end{aligned}$$

be the edge set of the twig graph.

Now define an injective function $f : V \to \mathbf{N}$ such that for $i, j, k = 1, 3, 5, ...$

$$f(u_j) = p_1$$
$$f(u_{j+1}) = 2^{\frac{j+3}{2}} p_2,$$
$$f(v_i) = 2^{\frac{i+1}{2}},$$
$$f(v_{i+1}) = 2^{\frac{i+3}{2}} p_2,$$
$$f(w_k) = 2^{\frac{k+3}{2}},$$
$$f(w_{k+1}) = 2^{\frac{k+5}{2}} p_1$$

where p_1, p_2 are distinct odd primes and are less than 10.

Now,

$$f^*(v_i v_{i+1}) = f(v_i) \cdot f(v_{i+1}) = 2^{\frac{i+1}{2}}.2^{\frac{i+3}{2}} p_2 = 2^{i+2} p_2, \text{ an } m\text{-Zumkeller number.}$$

$$f^*(v_{i+1} v_{i+2}) = f(v_{i+1}) \cdot f(v_{i+2}) = 2^{\frac{i+3}{2}} p_2.2^{\frac{i+3}{2}} = 2^{i+3} p_2, \text{ not an } m\text{-Zumkeller number.}$$

$$f^*(v_2 u_1) = f(v_2) \cdot f(u_1) = 2^2 p_2.p_1 = 2^2 p_1 p_2, \text{ an } m\text{-Zumkeller number.}$$

$$f^*(v_2 w_1) = f(v_2) \cdot f(w_1) = 2^2 p_2.2^2 = 2^4 p_2, \text{ not an } m\text{-Zumkeller number.}$$

$$f^*(v_i u_{j+1}) = f(v_i) \cdot f(u_{j+1}) = 2^{\frac{i+1}{2}}.2^{\frac{j+3}{2}} p_2 \text{ for } i \geq 3,\ j \geq 1 \text{ and } i = j+2$$
$$= 2^{j+3} p_2, \text{ not an } m\text{-Zumkeller number.}$$

$$f^*(v_i w_{k+1}) = f(v_i) \cdot f(w_{k+1}) = 2^{\frac{i+1}{2}}.2^{\frac{k+5}{2}} p_1 \text{ for } i \geq 3,\ k \geq 1 \text{ and } i = k+2$$
$$= 2^{k+4} p_1, \text{ an } m\text{-Zumkeller number.}$$

$$f^*(v_{i+1} u_j) = f(v_{i+1}) \cdot f(u_j) = 2^{\frac{i+3}{2}} p_2.p_1 \text{ for } i \geq 3,\ j \geq 3 \text{ and } i = j$$
$$= 2^{\frac{i+3}{2}} p_1 p_2, \text{ an } m\text{-Zumkeller number.}$$

$$f^*(v_{i+1} w_k) = f(v_{i+1}) \cdot f(w_k) = 2^{\frac{i+3}{2}} p_2.2^{\frac{k+3}{2}} \text{ for } i \geq 3,\ k \geq 3 \text{ and } i = k$$
$$= 2^{i+3} p_2, \text{ not an } m\text{-Zumkeller number.}$$

Hence, we can see that if n is even the difference between the total number of Zumkeller labeling edges and non-Zumkeller labeling edges is 1 whereas if n is odd the difference is 0.

Thus, we can conclude that the twig graph admits an m-Zumkeller cordial labeling. □

Example 3.7. *The m-Zumkeller cordial labeling of the even twig graph for $n = 8$, $p_1 = 3$, $p_2 = 7$ is shown in figure 10.*

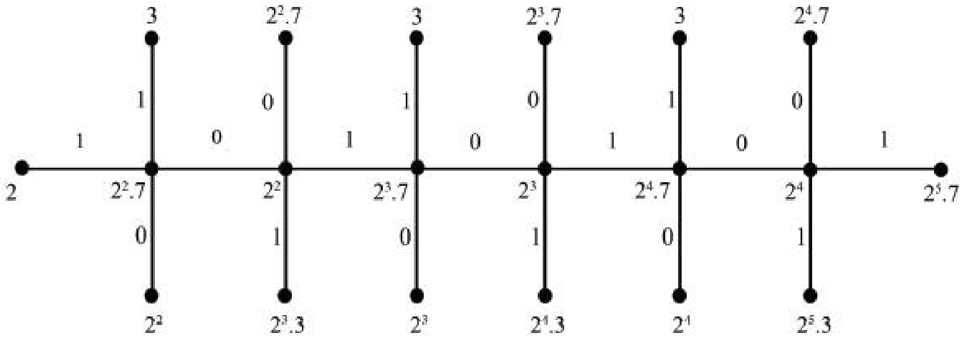

Figure 10: *m-Zumkeller cordial labeling of an even twig graph.*

Example 3.8. *The m-Zumkeller cordial labeling of the odd twig graph for $n = 9$, $p_1 = 3$, $p_2 = 7$ is shown in figure 11.*

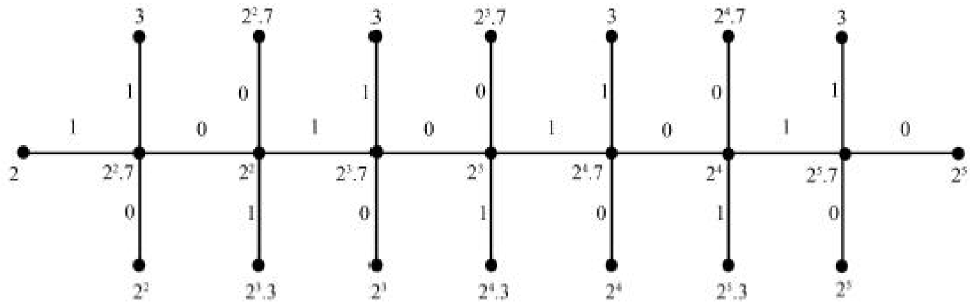

Figure 11: *m-Zumkeller cordial labeling of an odd twig graph.*

Definition 3.6. *A wheel graph denoted by W_n is a graph formed by connecting a single universal vertex v_0 to all the vertices of a cycle C_n i.e. $W_n = K_1 + C_n$.*

Definition 3.7. *The helm graph H_n is obtained by joining a pendent edge to each vertex of the wheel graph W_n.*

Proposition 3.8. *The helm graph H_n admits an m-Zumkeller cordial labeling.*

Proof. Let, $V = \{v_0\} \cup \{v_i | 1 \leq i \leq n\} \cup \{u_j | 1 \leq j \leq n\}$ be the vertex set and

$$\begin{aligned} E = \{ \quad & v_0 v_i | 1 \leq i \leq n\} \cup \{v_i v_{i+1} | 1 \leq i \leq n-1\} \cup \{v_n v_1\} \\ & \cup \{v_i u_j | 1 \leq i \leq n, 1 \leq j \leq n\} \end{aligned}$$

be the edge set of the helm graph H_n.

Now define an injective function $f : V \to \mathbf{N}$ such that for $i, j = 3, 5, \ldots$

$$f(v_0) = 6, \qquad f(v_1) = 9, \qquad f(v_2) = 5, \qquad f(u_1) = 3, \qquad f(u_2) = 4$$

$$\begin{aligned} f(u_j) &= 2^j p_1 \\ f(u_{j+1}) &= 2^{j+1} \\ f(v_i) &= 2^{i+1} \\ f(v_{i+1}) &= 2^{i+1} p_2 \end{aligned}$$

where p_1, p_2 are distinct odd primes and $p_2 > 3$.

Case 1. For the spokes $f^*(v_0 v_i)$ of H_n we have,

$f^*(v_0 v_1) = f(v_0) \cdot f(v_1) = 6 \times 9 = 54$, an m-Zumkeller number.

$f^*(v_0 v_2) = f(v_0) \cdot f(v_2) = 6 \times 5 = 30$, an m-Zumkeller number.

$f^*(v_0 v_i) = f(v_0) \cdot f(v_i)$ for $i \geq 3 = 6 \times 2^{i+1} = 2^{i+2} \times 3$, an m-Zumkeller number.

$f^*(v_0 v_{i+1}) = f(v_0) \cdot f(v_{i+1})$ for $i \geq 3 = 6 \times 2^{i+1} p_2 = 2^{i+2} \times 3p_2$, an m-Zumkeller number.

Hence, all the edges of the spokes of H_n have m-Zumkeller numbers. Thus we get n number of m-Zumkeller numbers for the spokes of H_n.

Case 2. For the edges on the cycle C_n of H_n we have,

$$f^*(v_1v_2) = f(v_1) \cdot f(v_2) = 9 \times 5 = 45, \text{ not an } m\text{-Zumkeller number.}$$

$$f^*(v_2v_3) = f(v_2) \cdot f(v_3) = 5 \times 2^4, \text{ not an } m\text{-Zumkeller number.}$$

$$f^*(v_iv_{i+1}) = f(v_i) \cdot f(v_{i+1}) \text{ for } i \geq 3 = 2^{i+1}.2^{i+1}p_2 = 2^{2(i+1)}p_2, \text{ not an } m\text{-Zumkeller number.}$$

$$f^*(v_{i+1}v_{i+2}) = f(v_{i+1}) \cdot f(v_{i+2}) \text{ for } i \geq 3 = 2^{i+1}p_2.2^{i+3} = 2^{2(i+2)}p_2, \text{ not an } m\text{-Zumkeller number.}$$

If n is odd then

$$f^*(v_nv_1) = f(v_n) \cdot f(v_1) = 2^{n+1} \times 9, \text{ not an } m\text{-Zumkeller number.}$$

If n is even then

$$f^*(v_nv_1) = f(v_n) \cdot f(v_1) = 2^n p_2 \times 9, \text{ not an } m\text{-Zumkeller number.}$$

So, we get total n number of non m-Zumkeller numbers on the edges of the cycle C_n of H_n.

Case 3. For the pendent edges $f^*(v_iu_j)$ of H_n we have,

$$f^*(v_1u_1) = f(v_1) \cdot f(u_1) = 9 \times 3 = 27, \text{ an } m\text{-Zumkeller number.}$$

$$f^*(v_2u_2) = f(v_2) \cdot f(u_2) = 5 \times 4 = 20, \text{ not an } m\text{-Zumkeller number.}$$

$$f^*(v_iu_j) = f(v_i) \cdot f(u_j) = 2^{i+1}.2^j p_1, \text{ for } i, j \geq 3,\ i = j$$

$$= 2^{2i+1}p_1, \text{ an } m\text{-Zumkeller number.}$$

$$f^*(v_{i+1}u_{j+1}) = f(v_{i+1}) \cdot f(u_{j+1}) = 2^{i+1}p_2.2^{j+1}, \text{ for } i, j \geq 3,\ i = j$$

$$= 2^{2(i+1)}p_2, \text{ not an } m\text{-Zumkeller number.}$$

Hence, for the pendent edges of H_n if n is odd we get total $\frac{n-1}{2}$ number of non m-Zumkeller numbers and $\frac{n-1}{2} + 1$ number of m-Zumkeller numbers. And if n is even then we get total $\frac{n}{2}$ number of m-Zumkeller numbers and $\frac{n}{2}$ number of non m-Zumkeller numbers on the pendent edges of H_n.

Thus, from the above three cases we can conclude that if n is odd then

$$|e_{f^*}(0) - e_{f^*}(1)| = 1$$

and if n even then

$$|e_{f^*}(0) - e_{f^*}(1)| = 0.$$

Hence, the helm graph H_n admits an m-Zumkeller cordial labeling. □

Example 3.9. *The m-Zumkeller cordial labeling of the helm graph H_8 and H_9 for $p_1 = 5$ and $p_2 = 7$ are shown in figure 12 and figure 13 respectively.*

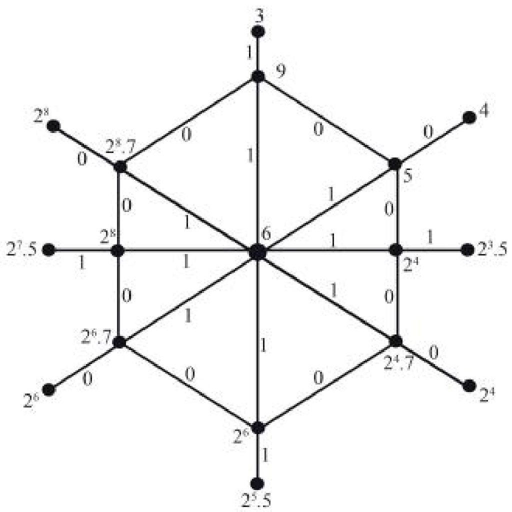

Figure 12: *m-Zumkeller cordial labeling of the helm graph H_8.*

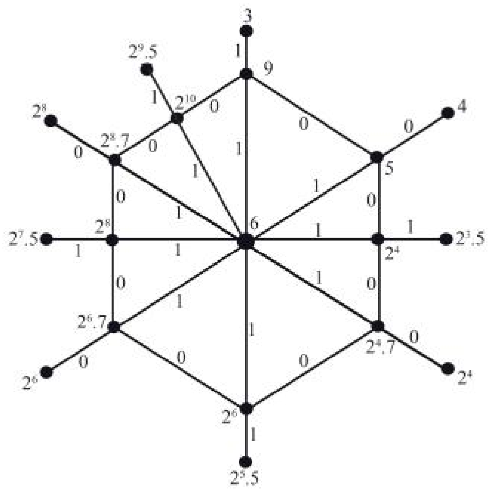

Figure 13: *m-Zumkeller cordial labeling of the helm graph H_9.*

Proposition 3.9. *The wheel graph W_n admits an m-Zumkeller cordial labeling.*

Proof. If we remove the pendent edges from the helm graph H_n, then the proof follows from the proposition 3.8 . □

Example 3.10. *The m-Zumkeller cordial labeling of the wheel graph W_7 is shown in figure 14.*

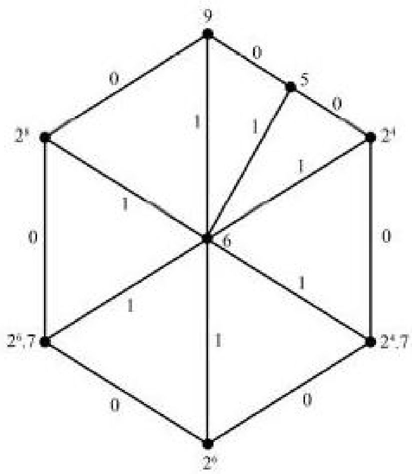

Figure 14: *m-Zumkeller cordial labeling of the wheel graph W_7.*

Definition 3.8. *The crown graph $C_n \odot K_1$ is obtained by joining a pendent edge to each vertex of cycle C_n.*

Proposition 3.10. *The crown graph $C_n \odot K_1$ admits an m-Zumkeller cordial labeling.*

Proof. Let, $V = \{v_i | 1 \leq i \leq n\} \cup \{u_j | 1 \leq j \leq n\}$ be the vertex set and $= \{v_i v_{i+1} | 1 \leq i \leq n-1\} \cup \{v_n v_1\} \cup \{v_i u_j | 1 \leq i \leq n, 1 \leq j \leq n\}$ be the edge set of the crown graph $C_n \odot K_1$.

Now define an injective function $f : V \to \mathbf{N}$ such that for $i, j = 3, 5, ...$

$$f(v_1) = 9, \qquad f(v_2) = 5, \qquad f(u_1) = 3, \qquad f(u_2) = 6$$

$$f(u_j) = 2^j p_1$$

$$f(u_{j+1}) = p_1$$

$$f(v_i) = 2^{i+1}$$

$$f(v_{i+1}) = 2^{i+1} p_2$$

where p_1, p_2 are distinct odd primes and $p_2 > 3$.

Now, we have from the case 2 of the proposition 3.8 all the edges on the cycle C_n of $C_n \odot K_1$ has non-Zumkeller number on it. Thus, we get total n number of non m-Zumkeller numbers on the edges of the cycle C_n.

Again for the pendent edges $f^*(v_i u_j)$ of the crown graph we have,

$$f^*(v_1 u_1) = f(v_1) \cdot f(u_1) = 9 \times 3 = 27, \text{ an } m\text{-Zumkeller number.}$$

$$f^*(v_2 u_2) = f(v_2) \cdot f(u_2) = 5 \times 6 = 30, \text{ an } m\text{-Zumkeller number.}$$

$$f^*(v_i u_j) = f(v_i) \cdot f(u_j) = 2^{i+1}.2^j p_1, \text{ for } i, j \geq 3,\ i = j$$

$$= 2^{2i+1} p_1, \text{ an } m\text{-Zumkeller number.}$$

$$f^*(v_{i+1} u_{j+1}) = f(v_{i+1}) \cdot f(u_{j+1}) = 2^{i+1} p_2.p_1, \text{ for } i, j \geq 3,\ i = j$$

$$= 2^{i+1} p_1 p_2, \text{ an } m\text{-Zumkeller number.}$$

Thus, all the pendent edges of the crown graph has m-Zumkeller number on it. Therefore there are n number of m-Zumkeller numbers on the edges of the crown graph.

Hence, we can conclude that the crown graph $C_n \odot K_1$ admits an m-Zumkeller cordial labeling. □

Example 3.11. *The m-Zumkeller cordial labeling of the crown graph $C_8 \odot K_1$ for $p_1 = 5$ and $p_2 = 7$ is shown in figure 15.*

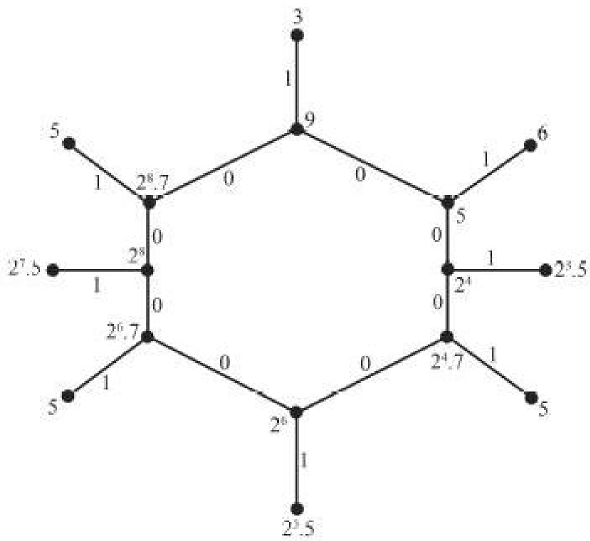

Figure 15: *m-Zumkeller cordial labeling of the crown graph $C_8 \odot K_1$.*

Definition 3.9. *A star graph S_n is a complete bipartite graph $K_{1,n}$.*

Proposition 3.11. *The star graph S_n admits an m-Zummkeller cordial labeling.*

Proof. Let, $V = \{v_0\} \cup \{v_i | 1 \leq i \leq n\}$ be the vertex set and $E = \{v_0v_i | 1 \leq i \leq n\}$ be the edge set of the star graph S_n.

Now define an injective function $f : V \to \mathbf{N}$ such that for $i = 1, 3, 5, ...$

$$f(v_0) = p_1,\ f(v_i) = 2^{i+1},\ f(v_{i+1}) = p_2$$

where p_1, p_2 are distinct odd primes and are less than 10.

Now,

$f^*(v_0v_i) = f(v_0) \cdot f(v_i) = p_1.2^{i+1} = 2^{i+1}p_1$, not an m-Zumkeller number.

$f^*(v_0v_{i+1}) = f(v_0) \cdot f(v_{i+1}) = p_1.p_2 = p_1p_2$, an m-Zumkeller number.

Therefore, $|e_{f^*}(0) - e_{f^*}(1)| = 1$ if n is odd.

and $|e_{f^*}(0) - e_{f^*}(1)| = 0$ if n is even.

Hence, we can conclude that the star graph S_n admits an m-Zumkeller cordial labeling. $\square$

Example 3.12. *The m-Zumkeller cordial labeling of the star graph S_7 for $p_1 = 5$ and $p_2 = 7$ is shown in figure 16.*

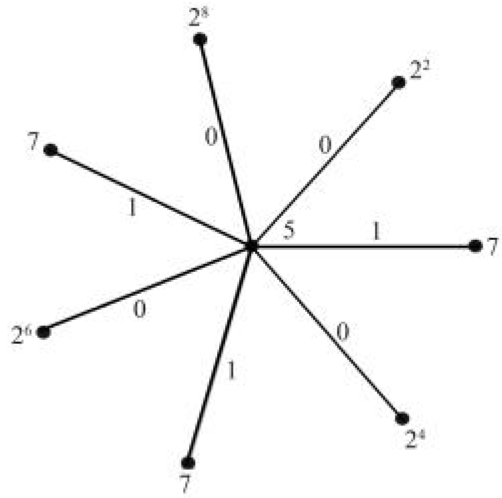

Figure 16: *m-Zumkeller cordial labeling of the star graph S_7.*

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