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Orbit equivalence of linear systems on manifolds and semigroup actions on homogeneous spaces*

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Abstract

In this paper we introduce the notion of orbit equivalence for semigroup actions and the concept of generalized linear control system on smooth manifold. We prove that, under certain conditions, the semigroup system of a generalized linear control system on a smooth manifold is orbit equivalent to the semigroup system of a linear control system on a homogeneous space.

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1. Introduction

Control theory on Lie groups has achieved significant advances in the last decades due to especially its relationship with the actions of semigroups on Lie groups (see e.g. Elliott [8], Jurdjevic [12], Rocio, San Martin and Santana [16], Rocio, Santana and Verdi [17] and Sachkov [18]).

Until the 1990s the theory of control systems on Lie groups was restricted, basically, to the control system of invariant vector fields. But in Ayala and Tirao [3] this study was expanded with the introduction of linear control systems on Lie groups and developed rapidly in recent years (see e.g. [2], [3] and [14]). In this context, in [11] Jouan considered a control system on a manifold given by complete linear vector fields that generate a finite dimensional Lie algebra and then showed a diffeomorphic relation between this system and a linear control system on a Lie group.

In our paper, the main objective is to introduce the concept of linear control system on manifolds, naming then generalized linear control system, and to establish conditions for the action of the semigroup of a generalized linear control system to be equivalent to the action of a semigroup on a homogeneous space. From these studies we hope to transfer the analysis of control issues on manifolds to Lie groups.

About the structure of this paper, in the second section we introduce the notion of orbit equivalence and topological conjugacy for semigroup actions and give some properties related with control sets. In the third section we fix the control theoretic notations and relates state equivalent control systems with diffeomorphic control systems. In the fourth section we present an improvement of Lie-Palais theorem. In the next section we prove that given a control system on M, the semigroup system on M is orbit equivalent to a semigroup action on a homogeneous space. In the sixth section we prove our main result which states that the semigroup system of the above system is orbit equivalent to a semigroup system of a linear control system on a G-homogeneous space. Finally, as application we show that invariance entropy is preserved by topological conjugation of control systems.

2. Orbit equivalence

In this section, we define the notions of orbit equivalence for semigroups actions and topological conjugacy for skew product which preserve some control properties. Then we begin recalling some concepts (for more details see e.g. San Martin [20] and San Martin and Tonelli [21]). Take a nonempty semigroup S acting on a topological space M. The semigroup S is said to be **accessible** if $\mathrm{int}Sx \neq \emptyset$ for every $x \in M$. A **control set** for the S-action on M is a subset $C \subset M$ such that $\mathrm{int}C \neq \emptyset$, $C \subset \mathrm{cl}(Sx)$ for all $x \in C$ and C is maximal with the first two properties. If $\mathrm{cl}C = \mathrm{cl}(Sx)$ for all $x \in C$, the control set C is named **invariant**. The **core** (or the **transitivity set**) of a control set C is the set $C^0 = \{x \in C : x \in (\mathrm{int}S)x\}$ and it holds that $C^0 = (\mathrm{int}S)x$ for all $x \in C^0$. We also recall the partial ordering between control sets given by $C_1 \prec C_2$ if there exists $x \in C_1$ such that $\mathrm{cl}(Sx) \cap C_2 \neq \emptyset$. Moreover, taking the topological space as flag manifolds, always exist control sets.

Now about equivalence of semigroups we have the following definition.

Definition 1. *Let M_1 and M_2 be topological spaces. Consider S and T semigroups acting on M_1 and M_2, respectively. The actions (M_1, S) and (M_2, T) are called* **orbit equivalent**, *if there exists an homeomorphism $f\colon M_1 \to M_2$ such that $f(Sx) = Tf(x)$ for all $x \in M_1$. The map f is called* **orbit equivalence map**.

Some authors call the pair (M, S) as transformation semigroup (see e.g. Ellis in [9] and Sousa in [22]). Locally, we have that the actions (M_1, S) and (M_2, T) are called orbit equivalent restricted to a subset $C \subset M_1$ if there exists an homeomorphism $f\colon M_1 \to M_2$ such that $f(Sx) = Tf(x)$ for all $x \in C$.

Now, supposing the existence of control sets, it is not difficult to show the following properties.

Proposition 2.

1. *Suppose that (M_1, S) and (M_2, T) are orbit equivalent. Hence if C_S is a control set for S then $f(C_S)$ is a control set for T in M_2. On the other hand, if C_T is a control set for T in M_2 then $f^{-1}(C_T)$ is a control set for S in M_1.*

2. *The orbit equivalence preserves the order of control sets.*

3. *Suppose that there exists a homeomorphism $f\colon M_1 \to M_2$ that send set of transitivity in set of transitivity, that is, if $C \subset M_1$ is the S invariant control set and C^0 its set of transitivity then $f(C)$ is the invariant control set for T with $f(C^0)$ its set of transitivity. Suppose also that S and T are accessible. With this hypotheses we have $(M_1, intS)$ and $(M_2, intT)$ are orbit equivalent restricted to C.*

4. *Suppose that* $(M_1, intS)$ *and* $(M_2, intT)$ *are orbit equivalent. Then* $f(C^0) = (f(C))^0$.

To finish this section, we establish a relation between conjugation and orbit equivalence. Suppose that the semigroups S and T contain the identities e_S and e_T. Let φ be a **cocycle** on X to T, that is, $\varphi: S \times X \to T$ continuous with

- $\varphi(st, x) = \varphi(s, tx)\,\varphi(t, x)$, for all $s, t \in S, x \in X$;
- $\varphi(e_S, x) = e_T$, for all $x \in X$.

A **skew-product transformation semigroup** on the product space $X \times Y$ is a map $\Phi: S \times X \times Y \to X \times Y$, with $\Phi(s, x, y) = (sx, \varphi(s, x)\,y)$. We write $s(x, y)$ instead of $\Phi(s, x, y)$.

We define the following subsemigroup of T, called **system semigroup**,

$$\mathcal{S} = \{\varphi(s_n, x_n)\,\varphi(s_{n-1}, x_{n-1}) \cdots \varphi(s_1, x_1) : s_j \in S, x_j \in X, n \in \mathbf{N}\}. \tag{2.1}$$

By considering the action σ restricted to the product $\mathcal{S}_\alpha \times Y$, we have the transformation semigroup $(\mathcal{S}, Y, \sigma)$ associated to the skew-product transformation semigroup $(S, X \times Y, \Phi)$.

To introduce the concepts of topological conjugacy and state equivalence we take the following two skew-product transformation semigroups

$$\Phi^i: S \times X^i \times Y^{ii} \times Y^i, \quad \Phi^i(s, x, y) = (sx, \varphi^i(s, x)\,y), 1 = 1, 2.$$

Definition 3. *Let* $\xi: Y^1 \to Y^2$ *and* $\iota: X^1 \to X^2$ *be maps such that* ξ *is continuous and satisfy:*

$$\xi(\varphi^1(s, x)y) = \varphi^2(s, \iota(x))\xi(y), \text{ for all } (s, x, y) \in S \times X^1 \times Y^1.$$

In this case, we say that the skew product Φ^1 *is* **topologically semiconjugate** *to* Φ^2*. If* ξ *is a homeomorphism and* ι *is invertible, then the skew products are called* **topologically conjugate**.

In particular, when Φ^1 and Φ^2 are topologically conjugate, $\iota = id_X$ and ξ is a diffeomorphism, we say that Φ^1 and Φ^2 are **state equivalent**. This terminology is inspired by the concept of state equivalence of control systems (for more details see Agrachev and Sachkov [1]).

Now we prove a result that relates the concepts of conjugation and orbit equivalence.

Proposition 4. *If* Φ^1 *and* Φ^2 *are topologically conjugate then the actions* $(Y^1, \mathcal{S}^1)$ *and* $(Y^2, \mathcal{S}^2)$ *are orbit equivalent, where* $\mathcal{S}^1$ *and* $\mathcal{S}^2$ *are the semigroup systems of* Φ^1 *and* Φ^2 *respectively.*

Proof. By hypothesis, there exists a homeomorphism $\xi: Y^1 \to Y^2$ and an invertible map $\iota: X^1 \to X^2$ such that $\xi(\varphi^1(s,x)y) = \varphi^2(s,\iota(x))\xi(y)$, for all $(s,x,y) \in S \times X^1 \times Y^1$.

Consider the following semigroups associated to Φ^i

$$\mathcal{S}^i = \{\varphi^i(s_n,x_n)\cdots\varphi^i(s_1,x_1); s_j \in S, x_j \in X^i, n \in \mathbf{N}\}, i = 1,2.$$

Take the homeomorphism h as ξ. Then given $a \in h(\mathcal{S}^1 y)$, we have $a = h(b)$, where $b \in \mathcal{S}^1 y$, i.e., $b = \varphi^1(s_n,x_n)\cdots\varphi^1(s_1,x_1)y = \varphi^1(s_n\cdots s_1,x)y$. Hence $a \in \mathcal{S}^2 h(y)$, since, $a = \xi(\varphi^1(s_n\cdots s_1,x)y) = \varphi^2(s_n\cdots s_1,\iota(x))h(y)$.

For the opposite inclusion, consider $a \in \mathcal{S}^2 h(y)$, then $a = bh(y)$, with $b \in \mathcal{S}^2$, hence $b = \varphi^2(s_m,v_m)\cdots\varphi^2(s_1,v_1) = \varphi^2(s_m\cdots s_1,v)$. Then, using a similar idea as above we prove that $a \in h(\mathcal{S}^1 y)$. $\square$

3. Conjugacy and state equivalence of control systems

In this section we recall the concepts of state equivalents and diffeomorphic control systems (see [1] and [11]) and prove that if two systems are diffeomorphic, then they are state equivalent.

Take M a differentiable and connected d-dimensional manifold. Consider in M the following control system

$$(\Sigma) \ : \ \dot{x}(t) = X_0(x(t)) + \sum_{j=1}^{m} u_j X_j(x(t)),$$

where X_i, $1 \le i \le m$ are differentiable vector fields on M and $u: \mathbf{R} \to U$ is a piecewise constant control with $U \subset \mathbf{R}^n$ compact and convex. Denote by $\mathcal{U}$ the set of all controls u. It is well known that $\mathcal{U}$ is a metric space (see e.g. Colonius and Kliemann [5]). We assume that for each u and $x \in M$ this system has a unique solution $\phi(t,u,x), t \in \mathbf{R}$, with $\phi(0,u,x) = x$.

As defined in [5], take

$$\Phi: \mathbf{R} \times \mathcal{U} \times M \to \mathcal{U} \times M, \ \Phi(t,u,x) = (\Theta_t(u), \phi(t,u,x)),$$

the control flow of system Σ. We know that it is a special case of skew-product transformation semigroup (see [22]).

Now consider two control systems Σ_1 and Σ_2 as above, take their control flows Φ_1 and Φ_2 and their correspondent system semigroups S_{Σ_1} and S_{Σ_2}. Take the map $\varphi_{t_1}^{u_1}: M_1 \to M_2$ given by $\varphi_{t_1}^{u_1}(x) = \varphi(t_1,u_1,x)$ then we have that S_{Σ_1} is a semigroup of diffeormophisms of M_1 given by

$$S_{\Sigma_1} = \{\varphi_{t_r}^{u_r} \circ \cdots \circ \varphi_{t_1}^{u_1}; u_i \in \mathcal{U}, t_i \ge 0, r \in \mathbf{N}\}.$$

The natural action of S_{Σ_1} on M_1 is defined by $\varphi\cdot x=\varphi(x)$. In the same way we have the semigroup S_{Σ_2}. Then as a consequence of Proposition 4 we have the following corollary:

Corollary 1. *Suppose that Σ_1 and Σ_2 are topologically conjugate then the actions (M_1, S_{Σ_1}) and (M_2, S_{Σ_2}) are orbit equivalent.*

Another concept, used to classify control systems, is the notion of **state equivalence**. In this case, ξ is a diffeomorphism and $\mathcal{U}=\mathcal{V}$. This concept is used to classify control systems preserving differentiable properties. A sufficient condition to guarantee that (Σ_1) and (Σ_2) be state equivalent is the existence of a diffeomorphism from M_1 to M_2 that preserves the control systems. Precisely, suppose that $\mathcal{U}=\mathcal{V}$ and that $\xi\colon M_1\to M_2$ be a diffeomorphism. For each $u\in\mathcal{U}$ consider the vector fields Z_u in M_1 and W_u in M_2 given by

$$Z_u(x)=X_0(x)+\sum_{j=1}^{m}u_jX_j(x)$$

and

$$W_u(\xi(x))=Y_0(\xi(x))+\sum_{j=1}^{m}u_jY_j(\xi(x)),$$

where $x\in M_1$. Hence,

Proposition 2. *If $\xi\colon M_1\to M_2$ is a diffeomorphism such that $\xi_*(Z_u(x))_x = W_u(\xi(x))$, for all $u\in\mathcal{U}$ and $x\in M_1$ then the control systems (Σ_1) and (Σ_2) are state equivalent.*

Proof. Given $u\in\mathcal{U}$ and $x\in M_1$ denote by $\varphi(t,u,x)$ the unique solution of the system (Σ_1) such that $\varphi(0,u,x)=x$ and by $\psi(t,u,\xi(x))$ the unique solution of (Σ_2) such that $\psi(0,u,\xi(x))=\xi(x)$. Then $\frac{d}{dt}\varphi(t,u,x)=Z_u(\varphi(t,u,x))$, for all $t\in\mathbf{R}$ and hence

$$\begin{aligned}\frac{d}{dt}\xi(\varphi(t,u,x)) &= (\xi_*)_{\varphi(t,u,x)}\frac{d}{dt}\varphi(t,u,x)=\xi_*(Z_u(\varphi(t,u,x))_{\varphi(t,u,x)}\\ &= W_u(\xi(\varphi(t,u,x))),\end{aligned}$$

showing that $\xi(\varphi(t,u,x))$ is also the solution of the differential equation

$$\dot{y}(t)=Y_0(y(t))+\sum_{j=1}^{m}u_jY_j(y(t))$$

on M_2, with initial value $\xi(\varphi(0,u,x)) = \xi(x)$. Therefore

$$\xi(\varphi(t,u,x)) = \psi(t,u,\xi(x)), \text{ for all } (t,u,x) \in \mathbf{R} \times \mathcal{U} \times M_1.$$

$\square$

Knowing that two control systems (Σ_1) and (Σ_2) are **diffeomorphic** if there exists a diffeomorphism $\xi\colon M_1 \to M_2$ such that $\xi_*(X_i) = Y_i$ for $0 \leq i \leq m$ (see e.g. [11]) we have the corollary.

Corollary 3. *If the control systems* (Σ_1) *and* (Σ_2) *are diffeomorphic then they are state equivalent.*

4. Lie-Palais Theorem

The Lie-Palais Theorem is fundamental to obtain the main results in the following sections. This well-known result was originally proved by Sophus Lie in a local way and generalized by Palais (see [15]). We start this section with the following definition.

Definition 1. *Let g be a Lie algebra and take M a differentiable manifold. An* **infinitesimal action** *of g on M is a homomorphism $\theta\colon g \to \mathcal{L}(TM)$.*

A differentiable action $\phi\colon G \times M \to M$ induces an infinitesimal action $\theta\colon g \to \mathcal{L}(TM)$ given by $\theta(X)(x) = d\phi_x \mid_1 (X)$, where $x \in M$ and 1 denote the identity element of G. One kind of converse is the Lie-Palais Theorem (see e.g. [10], [15] and [19]).

Theorem 2. [**Lie-Palais**] *Let g be a real and finite dimensional Lie algebra. Take G the connected and simply connected Lie group with Lie algebra g. Consider $\theta\colon g \to \mathcal{L}(TM)$ an infinitesimal action of g and suppose that the vector fields $\theta(X)$ is complete for all $X \in g$. Then there exists a differentiable action $\phi\colon G \times M \to M$ such that θ is the associated infinitesimal action.*

Over the time, this result was improved and applied in several situations (see e.g. [11]). In [10] the authors in different forms generalized the Lie-Palais Theorem relaxing the hypothesis that all vector fields $\theta(X)$ are complete on M. Independently, we showed a similar result from a different approach which will be presented from now on.

Define on $G \times M$ a distribution

$$\Delta_\theta(g,x) = \{(X(g), \theta(X)(x)) \in T_{(g,x)}G \times M : X \in g\}.$$

It is not difficult to prove that $\dim\Delta_\theta(g,x) = \dim G$ for all (g,x) and that Δ_θ is differentiable and integrable. Now take $I_\theta(g,x)$ the maximal connected integral manifold of Δ_θ, containing (g,x), then it is possible to prove that the restriction of the projection $p\colon G \times M \to G$ to $I_\theta(g,x)$ is a local diffeomorphism. In the approach of [19] the completeness hypothesis is only to prove two lemmas: the restriction of $p\colon G \times M \to G$ to $I_\theta(g,x)$ is surjective and it is hence a covering map. Then, to relax the completeness hypothesis we will prove that these results still hold supposing that there exists a subset $\mathcal{D} \subset g$ that spans g and such that the vector fields $\theta(X)$ are complete for all $X \in \mathcal{D}$.

We divide the proof of the Lie-Palais theorem in two lemmas.

Lemma 3. *Using the notations and assumptions from above and denoting by $\mathcal{D}$ the set which spans g, then the projection $p\colon G \times M \to G$ restricted to $I_\theta(g,x)$ is a local diffeomorphism. Moreover, if the vector fields $\theta(X)$ are complete for each $X \in \mathcal{D}$, then this restriction is surjective.*

Proof. Since the differential dp restrict to $\Delta_\theta(g,x)$ satisfies

$$dp\left(X^d(g), \theta(X)(x)\right) = X^d(g),$$

then dp is surjective and hence p is a local diffeomorphism.

In order to prove the surjectivity of p note that the trajectories $\left(e^{tX}g, \psi_t(x)\right)$ of $\left(X^d, \theta(X)\right)$, where ψ_t is the flow of $\theta(X)$, are in I_θ. As $\theta(X)$ is complete for all $X \in \mathcal{D}$, it follows that $p(I_\theta(g,x))$ contains $e^{tX}g$ for all $X \in \mathcal{D}$. Hence $p(I_\theta(g,x))$ contains all the elements of the form $e^{X_1}\cdots e^{X_n}g$ with $X_i \in \mathcal{D}$. Since G is connected, for all $g, h \in G$, there exist $X_1, \ldots, X_m \in \mathcal{D}$ such that $h = e^{X_1}\cdots e^{X_m}g$, therefore $h \in p(I_\theta(g,x))$, that is, $p(I_\theta(g,x)) = G$. □

Lemma 4. *With the previous notations and supposing that the vector fields $\theta(X)$ are complete for all $X \in \mathcal{D}$, then $p\colon I_\theta(g,x) \to G$ is a covering map for all maximal connected integral manifold $I_\theta(g,x)$ of $\Delta_\theta(g,x)$.*

Proof. We have to show that for all $h \in G$ there exists a neighbourhood of h, $U_h \subset G$, such that $p^{-1}(U_h)$ is a union of disjoint open sets $A_y \subset I_\theta(g,x)$, $h \in p^{-1}\{y\}$, with each A_y mapped homeomorphically onto U_h by p.

Take connected open sets $V \subset g$ and $U \in G$, containing the origins and such that $\exp\colon V \to U$ is diffeomorphism. Assume that $V = -V$. Note that

$U_h := U \cdot h$ is a neighbourhood of h, for each $h \in G$. Given $X \in g$, denote by ψ_t^X the flow of the field $\theta(X)$. The completeness of $\theta(X)$, $X \in \mathcal{D}$, implies that $\psi_1^X(y)$ is well defined for all $y \in M$.

Now we build the open set A_y. Take $Y_1, \ldots, Y_k$, $t_1, \ldots, t_k$ and $\rho(s_1, \ldots, s_k) = e^{s_1 Y_1} \cdots e^{s_k Y_k}$ satisfying $e^{s_1 Y_1} \cdots e^{s_k Y_k} = 1$. Then, there exists an open set $W \subset \mathbf{R}^k$ with $(t_1, \ldots, t_k) \in W$ such that ρ is a submersion in W. Hence, there are an open $V \subset \mathbf{R}^n$ ($n = \dim g$) and an immersion $\phi: V \to W$ such that $\rho \circ \phi$ is a diffeomorphism. Now define

$$f_{h,y}(u_1, \ldots, u_n) = \left(\rho \circ \phi(u_1, \ldots, u_n) h, \psi_{s_1}^{Y_1} \circ \cdots \circ \psi_{s_k}^{Y_k}(y)\right),$$

where $(s_1, \ldots, s_k) = \phi(u_1, \ldots, u_n)$. This map is a local diffeomorphism. With this we define

$$A_y = f_{h,y}(V).$$

The set A_y is open and its projection $B_y = p(f_{h,y}(V)) = \rho(W) h$ is also an open subset of G and does not depend on y. If W is connected then A_y and B_y are connected.

To finish the proof we need to show three assertions: A_y is connected, $A_{y_1} \cap A_{y_2} = \emptyset$ if $y_1 \neq y_2$ and $p^{-1}(Uh) = \bigcup_{y \in p^{-1}\{h\}} A_y$. For the first, we just need to take V connected. For the second one, suppose that

$$f_{h,y_1}(u_1, \ldots, u_n) = f_{h,y_2}(\overline{u}_1, \ldots, \overline{u}_n).$$

Then the elements u_i are all the same. Hence s_i are all the same. Therefore

$$\psi_{s_1}^{Y_1} \circ \cdots \circ \psi_{s_k}^{Y_k}(y_1) = \psi_{s_1}^{Y_1} \circ \cdots \circ \psi_{s_k}^{Y_k}(y_2)$$

which implies in $y_1 = y_2$. So we conclude that $A_{y_1} \cap A_{y_2} = \emptyset$ if $y_1 \neq y_2$. For the last assertion, we take $U = \rho \circ \phi(V) = \rho(W)$. Consider $(l, z) \in p^{-1}(U)$, i.e., $l = p(z) = e^{s_1 Y_1} \cdots e^{s_k Y_k} h$ with $(s_1, \ldots, s_k) = \phi(u_1, \ldots, u_n)$ e $(u_1, \ldots, u_n) \in V$. Let $\eta_t^{Y_i}$ be the flow of the complete vector field $\left(Y_i^d, \theta(Y_i)\right)$, which is given by

$$\eta_t^{Y_i}(h, y) = \left(e^{tY_i} h, \psi_t^{Y_i}(y)\right).$$

Then, $\eta_{-s_k}^{Y_k} \circ \cdots \circ \eta_{-s_1}^{Y_1}(z) = (h, y)$ with $y = \psi_{-s_k}^{Y_k} \circ \cdots \circ \psi_{-s_1}^{Y_1}(z)$. Hence $(l, z) = f_{h,y}(u_1, \ldots, u_n)$, implying that $(l, z) \in A_y$. □

If G is simply connected then $p: I_\theta(g, x) \to G$ is one-to-one. But p is a local diffeomorphism, then every $p: I_\theta(g, x) \subset G \times M \to G$ is a diffeomorphism. Therefore, $I_\theta(g,x)$ is the graph of a differentiable map $G \to M$. Denote by $\phi_x\colon G \to M$ the differentiable map whose graph is an integral manifold $I_\theta(1,x)$. With the following proposition we can prove that $\phi(g,x) = \phi_x(g)$ is the global action of the simply connected Lie group G on M associated to the infinitesimal action θ.

Proposition 5. *Given $x \in M$ and $X_1, \ldots, X_n \in g$, it holds*

$$\phi_x\left(e^{X_1}\cdots e^{X_n}\right) = \psi_1^{X_1} \circ \cdots \circ \psi_1^{X_n}(x). \tag{4.1}$$

Proof. The trajectories of $\left(X^d, \theta(X)\right)$ remain in the maximal connected integral manifold. The trajectory of this vector beginning in (g,y) is given by $\left(e^{tX}, \psi_t^X(y)\right)$. Then $\left(e^{X_1}\cdots e^{X_n}, \psi_1^{X_1} \circ \cdots \circ \psi_1^{X_n}(x)\right)$ belong to $I_\theta(1,x)$. Hence it holds the equality (4.1). □

Now we can prove an improvement of Lie-Palais Theorem.

Theorem 6. *Let g a finite dimensional Lie algebra and $\theta\colon g \to \Gamma(TM)$ an infinitesimal action. Suppose that there exists a subset $\mathcal{D} \subset g$ that generates g and such that the vector fields $\theta(X)$ are complete for all $X \in \mathcal{D}$. Then there exists a differentiable action $\phi\colon G \times M \to M$, such that θ is the associated infinitesimal action.*

Proof. Define $\phi(g,x) = \phi_x(g)$, where the graph of $\phi_x\colon G \to M$ is the integral manifold $I_\theta(1,x)$. This define an action of G in M. In fact,

1. if $x \in M$, then $\phi(1,x) = x$, since $(1,x)$ is the unique element of $I_\theta(1,x)$ that is projected in 1;
2. for $g, h \in G$, it holds $\phi(g, \phi(h,x)) = \phi(gh,x)$. In fact, we can write $g = e^{X_1}\cdots e^{X_n}$ and $h = e^{Y_1}\cdots e^{Y_n}$ with X_i and Y_i in $\mathcal{D}$ then by equality (4.1), it follows that

$$\begin{aligned} \phi(g, \phi(h,x)) &= \phi\left(g, \psi_1^{Y_1} \circ \cdots \circ \psi_1^{Y_m}(x)\right) \\ &= \psi_1^{X_1} \circ \cdots \circ \psi_1^{X_n} \circ \psi_1^{Y_1} \circ \cdots \circ \psi_1^{Y_m}(x) \\ &= \phi(gh,x). \end{aligned}$$

The differentiability of the action ϕ follows from the differentiability dependence and from the equality (4.1). □

Example 7. *To finish this section we presente an example that shows that our result is in fact a generalization of the Lie-Palais theorem, that is, the fact that a generating set of a Lie algebra is formed by complete vector fields does not guarantee in general that every element of the Lie algebra is complete. To do this we present three claims. First note that given a vector field* $a\frac{d}{dx}$ *in* $\mathbf{R}$*, the it is complete if the set of singularities of* a *is not limited neither superiorly nor inferiorly. As the second claim take* a *a strictly positive map in* $\mathbf{R}$ *and* g *a primitive of* $\frac{1}{a}$*, now suppose that* $\lim_{x\to\pm\infty} g(x) = \pm\infty$*, then* $a\frac{d}{dx}$ *is a complete field (for strictly negative map we have a similar result). Finally as the third claim, take the function* a *in* $\mathbf{R}$ *that is positive in a interval* $I = (\alpha, +\infty)$*, consider* g *the primitive of* $\frac{1}{a}$ *in* I *and suppose that* $\lim_{x\to+\infty} g(x)$ *is finite. Then* $a\frac{d}{dx}$ *is not complete. We have similar result for* $I = (-\infty, \omega)$ *and for strictly negative map.*

With this facts we can show that $\cos x\frac{d}{dx}$ *and* $x^n \sin x\frac{d}{dx}$ $(n \geq 2)$ *are complete but the Lie bracket* $\left[\cos x\frac{d}{dx}, x^n \sin x\frac{d}{dx}\right]$ *is not a complete vector field.*

5. Orbit equivalence of semigroup system on homogeneous space

Consider the same notations as the previous sections and take on M the control system

$$(\Sigma) \ : \ \dot{x}(t) = X_0(x(t)) + \textstyle\sum_{i=1}^{m} u_i X_i(x(t)).$$

The purpose of this section is to prove that (M, S_Σ) is orbit equivalent to a semigroup action on a homogeneous space. Then consider the Lie algebra $\mathcal{L}(TM)$ of all vector fields on M and take its Lie algebra $\mathcal{L}(\Gamma)$, generated by the set of vector field $\Gamma = \{X_0, X_1, \ldots, X_m\}$. Supposing that $\mathcal{L}(\Gamma)$ has finite dimension we take the connected and simply connected Lie group G with Lie algebra $\mathcal{L}(\Gamma)$. A natural way to define an action ϕ of G on M is: Denote by Ψ_t^X the flow of $X \in \mathcal{L}(\Gamma)$. As every $g \in G$ can be written as $g = e^{t_1 X_{i_1}} \cdots e^{t_s X_{i_s}}$, for some $t_{i_1}, \ldots, t_{i_s} \in \mathbf{R}$ and $X_{i_1}, \ldots, X_{i_s} \in \mathcal{L}(\Gamma)$, define ϕ as $\phi(g, x) = \Psi_{t_{i_1}}^{X_{i_1}} \circ \cdots \circ \Psi_{t_{i_s}}^{X_{i_s}}(x)$. The problem is that there is not just one way to write $g \in G$ as product of exponentials. But using Lie-Palais theorem, we can guarantee that this definition does not depend on this fact.

Theorem 1. *Let* $\Gamma = \{X_0, X_1, \ldots, X_m\}$ *be a family of transitive, complete and differentiable vector fields on the connected manifold* M*. Suppose* *that the Lie algebra $\mathcal{L}(\Gamma)$ has finite dimension and take G its associated connected and simply connected Lie group. Then, M is diffeomorphic to a G-homogeneous space.*

Proof. Consider the action $\phi: G\times M \to M$ given in the previous proposition. Then, $\phi(g,x) = \Psi_{t_{i_1}}^{X_{i_1}} \circ \cdots \circ \Psi_{t_{i_s}}^{X_{i_s}}(x)$ and as Γ is transitive we have that this action is transitive. Hence, fixing $x_0 \in M$ and considering the isotropy subgroup $H_{x_0} = \{g \in G : \phi(g, x_0) = x_0\}$ we that M is diffeomorphic to the homogeneous space G/H_{x_0}. □

Now we describe this above diffeomorphism. If $x \in M$, as Γ is transitive, there exist $X_{i_1}, \ldots, X_{i_s} \in \Gamma$ and $t_{i_1}, \ldots, t_{i_s} \in \mathbf{R}$ such that

$$x = \Psi_{t_{i_1}}^{X_{i_1}} \circ \cdots \circ \Psi_{t_{i_s}}^{X_{i_s}}(x_0) = \phi_{e^{t_{i_1}X_{i_1}} \cdots e^{t_{i_s}X_{i_s}}}(x_0).$$

In this case, the above diffeomorphism, denoted by $\xi: M \longrightarrow G/H_{x_0}$, is defined by $\xi(x) = (e^{t_{i_1}X_{i_1}} \cdots e^{t_{i_s}X_{i_s}})H_{x_0}$ and its inverse is given in the following way. Given $g \in G$, there exist $X_{i_1}, \ldots, X_{i_s} \in \Gamma$ and $t_{i_1}, \ldots, t_{i_s} \in \mathbf{R}$ such that $g = e^{t_{i_1}X_{i_1} t_{i_s}X_{i_s}}$. Remember that this choices are not unique. In this case, define

$$\xi^{-1}(gH_{x_0}) = \Psi_{t_{i_1}}^{X_{i_1}} \circ \cdots \circ \Psi_{t_{i_s}}^{X_{i_s}}(x_0), \tag{5.1}$$

note that this definition does not depend on the exponential form of g.

To finish this section we prove a result that relates a control system on M with his induced system on G/H_{x_0}. But first we show an important lemma to the sequence of this paper. Consider the map f defined as $\xi^{-1} \circ \pi: G \longrightarrow M$, where $\pi: G \longrightarrow G/H_{x_0}$ is the canonical projection. With this, $\pi(g) = \xi(f(g)), \forall g \in G$, and as ξ^{-1} and π are surjective maps it follows that f is surjective.

Lemma 2. *If $X \in \mathcal{L}(\Gamma)$ then $\pi_*(X) = \xi_*(X)$.*

Proof. Take $X \in \mathcal{L}(\Gamma)$, $g \in G$ and $x \in M$ such that $f(g) = x$. Consider $e^{tX}g$ the trajectory of X in G with initial point $g \in G$. Consider $\Psi_t^X(x)$ the trajectory of X in M with initial point $x \in M$. Then,

$$\frac{d}{dt}|_{t=0}(\xi(\Psi_t^X(x))) = d\xi|_x(X_x) \text{ and } \frac{d}{dt}|_{t=0}(\pi(e^{tX}g)) = d\pi|_g(X_g). \tag{5.2}$$

Note that there exist $X_{i_1}, \ldots, X_{i_s} \in \Gamma$ and $t_{i_1}, \ldots, t_{i_s} \in \mathbf{R}$ such that $g = e^{t_{i_1}X_{i_1}} \cdots e^{t_{i_s}X_{i_s}}$. Also there is $g_1 \in G$ such that $x = \phi(g_1, x_0)$. Analogously, there are $X_{j_1}, \ldots, X_{j_k} \in \Gamma$ and $t_{j_1}, \ldots, t_{j_k} \in \mathbf{R}$ such that $g_1 = e^{t_{j_1}X_{j_1}} \cdots e^{t_{j_k}X_{j_k}}$. Then $gg_1 = e^{t_{i_1}X_{i_1}} \cdots e^{t_{i_s}X_{i_s}} e^{t_{j_1}X_{j_1}} \cdots e^{t_{j_k}X_{j_k}}$. Hence

$$\xi(\phi(g,x)) \;=\; \xi(\phi(g,\phi(g_1,x_0))) = gg_1H_{x_0}.$$

As $\pi(gg_1) = gg_1H_{x_0}$, then $\pi(gg_1) = \xi(\phi(g,x))$. In particular, given $X \in \mathcal{L}(\Gamma)$ and $t \in \mathbf{R}$, $\pi(e^{tX}g) = \xi(\phi(e^{tX}, x))$. We have $\pi(e^{tX}g) = \xi(\Psi_t^X(x))$.

Hence, from (5.2) we have $\pi_*(X) = \xi_*(X)$. □

Returning to the control system (Σ) on M and taking the vector fields $\widetilde{X}_i = \pi_*(X_i), 0 \le i \le m$ on G/H_{x_0}, we define the following control system on G/H_{x_0}:

$$(\widetilde{\Sigma}) \;:\; \dot{\widetilde{x}}(t) = \widetilde{X}_0(\widetilde{x}(t)) + \textstyle\sum_{i=1}^m u_i \widetilde{X}_i(\widetilde{x}(t)).$$

Note that by Lemma 2, $\xi_*(X_i) = \widetilde{X}_i$ for $0 \le i \le m$, and knowing that $\xi\colon M \longrightarrow G/H_{x_0}$ is a diffeomorphism, we have that the control systems (Σ) and $(\widetilde{\Sigma})$ are diffeomorphic. Consequently, by Proposition 3 it follows that (Σ) and $(\widetilde{\Sigma})$ are state equivalent. Denoting by S_Σ and $S_{\widetilde{\Sigma}}$ the associated semigroups, using the Proposition 1 and recalling that state equivalent systems are topologically conjugate, we conclude the following theorem:

Theorem 3. *Suppose that $\Gamma = \{X_0, X_1, \ldots, X_m\}$ is transitive and complete on M. Suppose also that the Lie subalgebra of $\mathcal{L}(\Gamma)$ has finite dimension. Then, the action (M, S_Σ) is orbit equivalent to a semigroup action on a homogeneous space.*

Proof. As we see above, the action (M, S_Σ) is orbit equivalent to the action $(G/H_{x_0}, S_{\widetilde{\Sigma}})$. □

6. Generalized linear system on manifolds

Our goals in this section are to introduce the concept of linear control systems on smooth manifolds and using the results of the previous sections show that, under certain conditions, a linear control system on a manifold is orbit equivalent to a linear control system on a homogeneous space.

Recall that the concept of linear control system depends on the structure of the Lie group. Then to define this concept on general manifolds we must work around the lack of the Lie group. Now we define the generalized linear control system. Let M be a connected manifold with finite dimension and denote by $\mathcal{L}(TM)$ the Lie algebra of the differentiable vector fields on M.

Definition 1. *A* **generalized linear control system** *on M is a control system*

$$(\Lambda)\ :\ \dot{x} = \mathcal{F}(x) + \sum_{j=1}^{m} u_j Y_j(x) \tag{6.1}$$

where

1. *the set of vector fields $\Gamma = \{Y_1, \ldots, Y_m\}$ generates the finite dimensional Lie subalgebra $\mathcal{L}(\Gamma)$ of $\mathcal{L}(TM)$ and every vector field $Y_i \in \Gamma$ is complete;*
2. *$\mathcal{F} \in \mathcal{L}(TM)$, $[\mathcal{F}, X] \in \mathcal{L}(\Gamma), \forall X \in \mathcal{L}(\Gamma)$ and there exists $x_0 \in M$ such that $\mathcal{F}_{x_0} = 0$;*
3. *$u = (u_1, \ldots, u_m) \in \mathbf{R}^m$.*

It is clear that a linear control system on a Lie group is a generalized linear control system, but not all generalized linear control system is a linear control system. In fact, in case of generalized linear control system, the vector fields Y_i are not necessarily invariants.

Now we have our main result:

Theorem 2. *Consider a connected and simply connected smooth manifold M. Let (Λ) be a generalized linear control system (6.1) on M. If $\Gamma = \{Y_1, \ldots, Y_m\}$ is transitive on M, then the action (M, S_Λ) is orbit equivalent to a semigroup action associated to a linear control system on a homogeneous space.*

Proof. By Theorem 3 we need define a diffeomorphism ξ that carries Λ in a linear control system $\widetilde{\Lambda}$ on a homogeneous space. Now we define this homogeneous space, by Theorem 1 we take G the connected and simply connected Lie group with Lie algebra $\mathcal{L}(\Gamma)$. Note that G acts transitively on M. From this action, take $H \subset G$, the isotropy subgroup in $x_0 \in M$, then we have the diffeomorphism $\xi \colon M \to G/H$ given by $\xi(g \cdot x_0) = gH$, where $\cdot$ denotes the action of G on M. Hence, we need to show that when we apply ξ_* in (Λ) we get a linear control system on $\frac{G}{H}$, i.e., $\xi_*(\mathcal{F})$ is a linear vector field and $\xi_*(Y_j)$ is right invariant vector field for $i = \{1, \ldots, m\}$.

As ξ is a diffeomorphism, then $\xi_*(Y_j)$ and Y_j are ξ-related. Then, as Y_j is invariant we have that $\pi_*(Y_j)$ is invariant on G/H. Moreover, by Lemma

2 we have that $\pi_*(Y_j)=\xi_*(Y_j)$, for all $X\in g$, therefore $\xi_*(Y_j)$ is invariant on G/H.

We need to show that $\xi_*(\mathcal{F})$ is a linear vector field, i.e., $\xi_*(\mathcal{F})$ is π-related with a linear vector field on G. First, we find this linear vector field on G. By Lemma 2, if $X\in g$ then

$$[\xi_*(\mathcal{F}),\pi_*(X)]=\pi_*[\mathcal{F},X]. \tag{6.2}$$

Let $D\colon g\longrightarrow g$ be a derivation defined by $D(X)=[\mathcal{F},X]$. As G is connected and simply connected, there exists a linear vector field $\mathcal{X}$ on G such that $D(X)=[\mathcal{X},X]$, $\forall X\in g$.

Then we prove that $\xi_*(\mathcal{F})$ is $\pi-$related with $\mathcal{X}$. To do this, we prove that $\pi_*(\mathcal{X})$ is $\pi-$related with $\mathcal{X}$ and then we show that $\pi_*(\mathcal{X})=\xi_*(\mathcal{F})$.

Hence we first show that H is invariant by the flow ϕ_t of $\mathcal{X}$. Note that the vector field $\xi_*(\mathcal{F})$ in the point $H\in G/H$, $\xi_*(\mathcal{F})_H$, is equal to

$$d\xi\mid_{x_0}(\mathcal{F}_{x_0})=d\xi\mid_{x_0}(0)=0, \tag{6.3}$$

since $\xi(x_0)=\xi(1\cdot x_0)=1H=H$.

Note also that, $\pi_*(Y)_H=0$ for all Y in the Lie algebra h of H. In fact, as $Y\in h$ then $\exp(tY)\in H$, for all $t\in\mathbf{R}$. So, $\exp(tY)\cdot x_0=x_0$, for all $t\in\mathbf{R}$. Hence, $\pi_*(Y)_H=\frac{d}{dt}\mid_{t=0}(\exp(tY)\cdot H)=\frac{d}{dt}\mid_{t=0}(H)=0$. Therefore, as $\pi_*(Y)_H=0$ and $\xi_*(\mathcal{F})_H=0$, we have that

$$[\xi_*(\mathcal{F}),\pi_*(Y)]_H=0,\forall Y\in h.$$

Note that $\pi_*[\mathcal{X},Y]_H=0$. Hence, its flow given by $gH\mapsto(\exp t[\mathcal{X},Y])gH$ satisfies $(\exp t[\mathcal{X},Y])\cdot H=H$, for all $t\in\mathbf{R}$. Therefore, $\exp t[\mathcal{X},Y]\in H$, then $D(Y)=[\mathcal{X},Y]\in h,\forall Y\in h$.

This implies that

$$\phi_t(\exp Y)=\exp(e^{tD}Y)=\exp(I+tD+\frac{t^2D^2}{2!}+\cdots)Y\in H.$$

Then, $\phi_t(\exp Y)\in H$, $\forall t\in\mathbf{R}$ e $\forall Y\in h$. As M is connected, simply connected and diffeomorphic to $\frac{G}{H}$, it follows that $\frac{G}{H}$ is simply connected. Then H is connected. Hence, every element of H is product of exponentials of elements of h and as ϕ_t is an isomorphism then H is invariant by the flow ϕ_t. Consequently, $\pi_*(\mathcal{X})$ is a vector field on G/H π-related with $\mathcal{X}$.

To conclude the proof, we show that $\pi_*(\mathcal{X})=\xi_*(\mathcal{F})$. In fact, if $X\in g$, then $[\xi_*(\mathcal{F}),\pi_*(X)]=\pi_*[\mathcal{X},X]$. Note that $[\pi_*(\mathcal{X}),\pi_*(X)]$ and $[\mathcal{X},X]$ are π-related, hence $\pi_*[\mathcal{X},X]=[\pi_*(\mathcal{X}),\pi_*(X)]$, therefore $[\pi_*(\mathcal{X})-\xi_*(\mathcal{F}),\pi_*(X)]=0,\forall X\in g$.

Then the flow of $\pi_*(\mathcal{X}) - \xi_*(\mathcal{F})$ on G/H, denoted by α_t, commute with the flow of $\pi_*(X)$, given by $gH \mapsto (\exp tX)gH$.

As $\mathcal{X}$ is linear, then $\pi_*(\mathcal{X})_H = 0$. Moreover, from (6.3) we have $\xi_*(\mathcal{F})_H = 0$, then $\pi_*(\mathcal{X})_H = \xi_*(\mathcal{F})_H = 0$. Hence, $(\pi_*(\mathcal{X}) - \xi_*(\mathcal{F}))_H = 0$, so $\alpha_t(H) = H$, $\forall t \in \mathbf{R}$.

Consider, $g \in G$, as G is connected, there exist $Y_{i_1}, \dots, Y_{i_r} \in g$ e $t_{i_1}, \dots, t_{i_r} \in \mathbf{R}$ such that $g = \exp(t_{i_1} Y_{i_1}) \cdots \exp(t_{i_r} Y_{i_r})$. Then

$$\alpha_t(gH) = gH, \forall t \in \mathbf{R}.$$

Therefore, $(\pi_*(\mathcal{X}) - \xi_*(\mathcal{F}))_{gH} = 0$, i.e., $\pi_*(\mathcal{X}) = \xi_*(\mathcal{F})$. □

The following example presents a generalized linear control system which is not a linear control system on a Lie group.

Example 3. *Consider on* $\mathbf{S}^2 = \{p = (x, y, z) \in \mathbf{R}^3;\ x^2 + y^2 + z^2 = 1\}$ *the system*

$$\Lambda \ : \ \dot{p}(t) = \mathcal{F}(p(t)) + u_1(t) Y_2(p(t)) + u_2(t) Y_3(p(t)),$$

$u_1(t), u_2(t) \in \mathbf{R}$, *where* $\mathcal{F}(x, y, z) = Y_1(x, y, z)$ *and*

$$Y_1 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ Y_2 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},\ Y_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

In this case, $\Gamma = \{Y_2, Y_3\}$, $\mathcal{L}(\Gamma) = so(3, \mathbf{R})$ *and* Y_2, Y_3 *are clearly complete. The vector field* $\mathcal{F}$ *is smooth and it vanishes at* $(0, 0, 1)$ *and* $(0, 0, -1)$. *Moreover, for all* $(x, y, z) \in \mathbf{S}^2$, *we have*

- $[\mathcal{F}, Y_1] = 0 \in \mathcal{L}(\Gamma)$;
- $[\mathcal{F}, Y_2] = Y_3 \in \mathcal{L}(\Gamma)$;
- $[\mathcal{F}, Y_3] = -Y_2 \in \mathcal{L}(\Gamma)$.

Hence (Λ) *is a generalized linear control system on the connected and simply connected manifold* $\mathbf{S}^2$. *The discussion that precedes Theorem 3 shows that the system* Λ *is state equivalent to the linear control system*

$$\widetilde{\Lambda} : \dot{\widetilde{q}}(t) = \widetilde{\mathcal{X}}(\widetilde{q}(t)) + u_1 \widetilde{Y}_1(\widetilde{q}(t)) + u_2 \widetilde{Y}_2(\widetilde{q}(t)).$$

on $Sp(1)/H_{e_3}$, where $H_{e_3} = \{a + d\mathbf{k} \in \mathbf{H};\ a^2 + d^2 = 1\}$ is the isotropy subgroup at $e_3 = (0,0,1)$ of the transitive action of the connected and simply connected group $Sp(1) = \{q \in \mathbf{H};\ |q| = 1\}$ on $\mathbf{S}^2$, $\widetilde{\mathcal{X}} = \xi_*(\mathcal{X})$, $\widetilde{Y}_i = \xi_*(Y_i)$, $i = 1,2$, with $\mathcal{X} = Y_1$ and $\xi\colon \mathbf{S}^2 \to Sp(1)/H_{e_3}$ is the diffeomorphism $\xi(g \cdot e_3) = gH_{e_3}$.

In particular, Λ and $\widetilde{\Lambda}$ are topologically conjugate. By Theorem 2, the action $(\mathbf{S}^2, S_\Lambda)$ is orbit equivalent to $(Sp(1)/H_{e_3}, S_{\widetilde{\Lambda}})$, where S_Λ and $S_{\widetilde{\Lambda}}$ denote the semigroups associated to the systems Λ and $\widetilde{\Lambda}$, respectively.

Now we prove a generalization of this last theorem, where it is not necessary the simply connected hypothesis. First recall that given a universal covering $f\colon \tilde{M} \to M$, where $\tilde{M}$ is a differential manifold such that f is differentiable, we can lift the vector fields $Z \in TM$ to $T\tilde{M}$. In fact, given $\tilde{x} \in \tilde{M}$ there exist open neighbourhoods $\tilde{U}$ of $\tilde{x}$ in $\tilde{M}$ and U of x in M such that $f|_{\tilde{U}}\colon \tilde{U} \to U$ is diffeomorphism. Then define

$$\tilde{Z}_{\tilde{x}} = d(f|_{\tilde{U}})^{-1}|_x(Z_x).$$

Now consider the generalized linear control system

$$(\Lambda)\ :\ \dot{x} = \mathcal{F}(x) + \sum_{j=1}^{m} u_j Y_j(x),$$

on the connected differentiable manifold M, then we have the following theorem

Theorem 4. *Suppose that the family of vector fields $\Gamma = \{Y_1, \ldots, Y_m\}$ is transitive on M. Then the action (M, S_Λ) is orbit equivalent to a semigroup action associated to a linear control system on a homogeneous space.*

Proof. Let $f\colon \tilde{M} \to M$ be the above differentiable covering. Then from Λ we define the following system on $\tilde{M}$:

$$(\tilde{\Lambda})\ :\ \dot{x} = \tilde{\mathcal{F}}(x) + \sum_{j=1}^{m} u_j \tilde{Y}_j(x),$$

where $\tilde{\mathcal{F}}$ and $\tilde{Y}_j$ are defined above.

Consider $\tilde{\Gamma} = \{\tilde{Y}_1, \ldots, \tilde{Y}_m\}$. By definition of $\tilde{Y}_j$ we have that the family $\tilde{\Gamma}$ is complete and $\mathcal{L}(\Gamma)$ is isomorphic to $\mathcal{L}(\tilde{\Gamma})$.

Note that $\tilde{\Gamma}$ is transitive. In fact, every f-image of orbit is an orbit in M, moreover, the rank of f is constant in every orbit. As Γ is transitive in

M, the Γ-orbit has the same dimension as M, and therefore, as $\tilde{\Gamma}$. Then, the $\tilde{\Gamma}$-orbits in $\tilde{M}$ are submanifolds of the same dimension of $\tilde{M}$. As $\tilde{M}$ is connected and is the union of the $\tilde{\Gamma}$-orbits, it follows that exists just one $\tilde{\Gamma}$-orbit. Therefore, $\tilde{\Gamma}$ is transitive.

Moreover, we have that

$$[\tilde{\mathcal{F}}, \tilde{Y}_i] \in \mathcal{L}(\tilde{\Gamma}),$$

and as $\mathcal{F}_{x_0} = 0$ it follows that $\tilde{\mathcal{F}}_{\tilde{x_0}} = 0$ for all $\tilde{x_0} \in f^{-1}(x_0)$.

Consider the connected and simply connected Lie group G with Lie algebra $\mathcal{L}(\Gamma)$ (and $\mathcal{L}(\tilde{\Gamma})$).

We have the actions

$$G \times M \to M \quad \text{and} \quad G \times \tilde{M} \to \tilde{M}.$$

Take $\tilde{x_0} \in f^{-1}(x_0)$ then we have the isotropy subgroups

$$H = \{g \in G; gx_0 = x_0\} \quad \text{and} \quad \tilde{H} = \{g \in G; g\tilde{x}_0 = \tilde{x}_0\}.$$

Hence we have the diffeomorphisms $\xi: M \to G/H$ given by $\xi(g \cdot x_0) = gH$ and $\tilde{\xi}: \tilde{M} \to G/\tilde{H}$ with $\xi(g \cdot \tilde{x}_0) = g\tilde{H}$, here $\cdot$ denote the action of G on M or $\tilde{M}$. As $\tilde{M}$ is simply connected then $\tilde{H}$ is connected. As we see in the demonstration of the previous result, it follows that $\tilde{\Lambda}$ is diffeomorphic to linear control system on $G/\tilde{H}$. Now we describe this system on $G/\tilde{H}$.

Consider $D: \mathcal{L}(\tilde{\Gamma}) \to \mathcal{L}(\tilde{\Gamma})$ given by $D(Y) = [\tilde{\mathcal{F}}, Y]$, note that D is derivation. Then there exists a linear vector field $\mathcal{X}$ on $\tilde{G} = G$ such that $D(Y) = [\mathcal{X}, Y]$ for every $Y \in \mathcal{L}(\tilde{\Gamma})$. Let $\tilde{\pi}: G \to G/\tilde{H}$ be the canonical projection. By previous result, we have that $\tilde{\Lambda}$ is diffeomorphic to the following linear control system in $G/\tilde{H}$:

$$(\Lambda_{\tilde{\pi}}) \;:\; \dot{x} = \tilde{\pi}_*(\mathcal{X}) + \sum_{j=1}^{m} u_j \tilde{\pi}_*(\tilde{Y}_j),$$

where $\tilde{\pi}_*(\mathcal{X}) = \tilde{\xi}_*(\tilde{\mathcal{F}})$ and $\tilde{\pi}_*(\tilde{Y}_j) = \tilde{\xi}_*(\tilde{Y}_j)$.

Note that $\tilde{\pi}(\mathcal{X})$ exists, i.e., $\tilde{H}$ is invariant by the flow of $\mathcal{X}$.

It is not difficult to see that $l: G/\tilde{H} \to G/H$ defined by $l(g\tilde{H}) = gH$ is a differentiable covering.

Recall that we need to show that $\xi_*(\mathcal{F})$ is linear vector field on G/H and that $\xi_*(Y_j)$ are right invariant vector field for $i = \{1, \ldots, m\}$. As Y_j is invariant we have that $\pi_*(Y_j)$ is invariant on G/H. By Lemma 2, we have that $\pi_*(Y_j) = \xi_*(Y_j)$, then $\xi_*(Y_j)$ are invariants.

The vector field $\xi_*(\mathcal{F})$ is linear if $\xi_*(\mathcal{F})$ is π-related with a linear vector field on G. Then, we first show that $\pi_*(\mathcal{X})$ is linear on G/H, i.e., $\mathcal{X}$ is π-related with $\pi_*(\mathcal{X})$ in G/H. After this, we prove that $\xi_*(\mathcal{F}) = \pi_*(\mathcal{X})$.

First we note that $\xi_*(\mathcal{F})$ is null in $H/\tilde{H}$. As $\tilde{\xi_*}(\tilde{\mathcal{F}}) = \tilde{\pi_*}(\mathcal{X})$ then $\tilde{\pi_*}(\mathcal{X})$ is null in $H/\tilde{H}$. Then we can prove that H is invariant by the flow of $\mathcal{X}$. So ($\mathcal{X}$ is π-related with the vector field $\pi_*(\mathcal{X})$ on G/H.

Now we must prove that $\xi_*(\mathcal{F}) = \pi_*(\mathcal{X})$. As $\tilde{\mathcal{F}}_{\tilde{x}} = d(f|_{\tilde{U}})^{-1}|_x(\mathcal{F}_x)$ and $\tilde{\xi}$ and ξ are diffeomorphisms it follows that

$$\tilde{\xi_*}(\tilde{\mathcal{F}})|_{g\tilde{H}} = d(l|_{\tilde{V}})^{-1}(\xi_*(\mathcal{F})\ |_{gH})$$

and

$$\tilde{\pi_*}(\tilde{\mathcal{X}})|_{g\tilde{H}} = d(l|_{\tilde{W}})^{-1}(\pi_*(\mathcal{X})\ |_{gH})$$

then $\xi_*(\mathcal{F}) = \pi_*(\mathcal{X})$. □

7. Invariance Entropy

In this section we present the concept of invariance entropy and outer invariance entropy introduced by F. Colonius and C. Kawan in [4] for continuous-time control systems and prove that this concept is invariant for topological conjugace control systems. We start with the definition of admissible pair which will be used along this section.

Definition 1. *A pair (K, Q) of nonempty subsets of M is called* **admissible** *for the control system $\Sigma = (\mathbf{R}, M, U, \mathcal{U}, \varphi)$ if it satisfies the following properties:*

i) K is compact;

ii) For each $x \in K$, there exists $\omega \in \mathcal{U}$ such that $\varphi(t, x, \omega) \in Q$ for all $t \geq 0$.

Given $\tau > 0$ and an admissible pair (K, Q), we say that a set $\mathcal{S} \subset \mathcal{U}$ is a (τ, K, Q)**-spanning set** if

$$\forall\ x \in K,\ \exists\ \omega \in \mathcal{S};\ \varphi([0, \tau], x, \omega) \subset Q.$$

Denote by $r_{\text{inv}}(\tau, K, Q)$ the minimal number of elements such a set can have (if there is no finite set we say that $r_{\text{inv}}(\tau, K, Q) = \infty$).

The existence of (τ, K, Q)-spanning sets is guaranteed by property (ii); indeed, $\mathcal{U}$ is a (τ, K, Q)-spanning set for every $\tau > 0$.

Definition 2. *Given an admissible pair (K, Q), we define the* **invariance entropy of** (K, Q) *by*

$$h_{inv}(K, Q) = h_{inv}(K, Q; \Sigma) := \limsup_{\tau \to \infty} \frac{1}{\tau} \log r_{inv}(\tau, K, Q).$$

Here, we consider $\log = \log_e = \ln$. *If $K = Q$, again we omit the argument K and write $h_{inv}(Q)$. Moreover, we let* $\log \infty := \infty$.

Hence, invariance entropy is a nonnegative (possibly infinite) quantity which is assigned to an admissible pair (K, Q). In fact, the invariance entropy of (K, Q) measures the exponential growth rate of the minimal number of different control functions sufficient to stay in Q when starting in K, as time tends to infinity.

Another notion of entropy (whose definition requires a metric) associated with an admissible pair is given in sequence.

Definition 3. *Given an admissible pair (K, Q) such that Q is closed in M, and a metric ϱ on M, we define the* **outer invariance entropy** *of (K, Q) by*

$$h_{inv,out}(K, Q) := h_{inv,out}(K, Q; \varrho; \Sigma) := \lim_{\varepsilon \searrow 0} h_{inv}(K, N_\varepsilon(Q)) = \sup_{\varepsilon > 0} h_{inv}(K, N_\varepsilon(Q)),$$

where $N_\varepsilon(Q) = \{y \in M; \exists\, x \in Q \text{ with } d(x, y) < \varepsilon\}$ denotes the ε-neighborhood of Q.

These two quantities are related as follows

$$0 \leq h_{\mathrm{inv,out}}(K, Q) \leq h_{\mathrm{inv}}(K, Q) \leq \infty.$$

Although in general these quantities do not coincide, this fact is verified (under some assumption which we expose in sequence) in the case of linear control systems on $\mathbf{R}^n$ (see [13, Corollary 5.3]):

Theorem 4. *Consider a linear control system given by the family of differential equation*

$$(\Sigma_{lin}) \;:\; \dot{x}(t) = Ax(t) + B\omega(t), \quad \omega \in \mathcal{U},$$

where the matrix pair (A, B) is controllable and such that A has no eigenvalues on the imaginary axis (that is, A is hyperbolic). Further assume that the control range U is a compact and convex set with $0 \in intU$. Let

$C \subset \mathbf{R}^d$ be the unique control set for Σ_{lin} with nonempty interior. Then for every compact set $K \subset C$ it holds that

$$h_{inv}(K,Q) \leq \sum_{\lambda \in \sigma(A)} \max\{0, n_\lambda Re(\lambda)\},$$

where $\sigma(A)$ denotes the spectrum of A and n_λ is the multiplicity of $\lambda \in \sigma(A)$. If, additionally, K has positive Lebesgue measure and $Q := \overline{C}$ it holds that

$$h_{inv}(K,Q) = h_{inv,out}(K,Q) = \sum_{\lambda \in \sigma(A)} \max\{0, n_\lambda Re(\lambda)\}.$$

In the case of linear control systems on lie groups, there is an upper bound for the outer invariance entropy in terms of the real parts of the eigenvalues of the derivation associated to the linear vector field, as shown in [7]:

Theorem 5. *Let (K,Q) be an admissible pair of the linear control system on a Lie group G. Assume that Q is compact. Then, the outer invariance entropy satisfies*

$$h_{inv,out}(K,Q) \leq \sum_{\lambda_D > 0} \lambda_D,$$

where λ_D are the real parts of the eigenvalues of the derivation D.

The next proposition shows that the invariance entropy is preserved by topological conjugacy. The ideas of the proof are based on [13, Proposition 2.13].

Proposition 6. *Let*

$$(\Sigma_j) \ : \ \dot{x}(t) = X_0^j(x(t)) + \sum_{i=1}^{m_j} u_i X_i^j(x(t)), \ u_i \in \mathcal{U}_j, \ j = 1,2,$$

be two control systems on differentiable manifolds M_1 and M_2, respectively. Assume that Σ_1 is topologically conjugate to Σ_2. Denote by $\xi\colon M_1 \to M_2$ the homeomorphism and $\iota\colon \mathcal{U}_1 \to \mathcal{U}_2$ the invertible map such that $\xi(\varphi_1(t,u,x)) = \varphi_2(t,\iota(u),\xi(x))$, for all $x \in M_1$ and $u \in \mathcal{U}_1$, where φ_j is the solutions of Σ_j, $j = 1,2$. Then for all admissible pair (K,Q) for Σ_1, with Q compact, we have that $(\xi(K),\xi(Q))$ is an admissible pair (K,Q) for Σ_2 and

$$h_{inv,out}(K,Q;\Sigma_1) = h_{inv,out}(\xi(K),\xi(Q);\Sigma_2).$$

Proof. In order to show that $(\xi(K), \xi(Q))$ is an admissible pair, note that since ξ is continuous, the sets $\xi(K)$ and $\xi(Q)$ are compact. Let $y \in \xi(K)$, then $y = \xi(x)$ for some $x \in K$. Since (K, Q) is an admissible pair, there is $u \in \mathcal{U}$ such that $\varphi_1(t, u, x) \subset Q$, for all $t \geq 0$, and we obtain

$$\varphi_2(t, \iota(u), y) \quad = \quad \varphi_2(t, \iota(u), \xi(x)) = \xi(\varphi_1(t, u, x)) \in \xi(Q),$$

for all $t \geq 0$. Therefore $(\xi(K), \xi(Q))$ is an admissible pair for Σ_2.

Denote by ϱ_j the metric on M_j, $j = 1, 2$. Since ξ is continuous on the compact Q, ξ is uniformly continuous on Q. Hence, given $\varepsilon > 0$, there is $\delta > 0$ such that

$$\varrho_1(x, y) < \delta \Rightarrow \varrho_2(\xi(x), \xi(y)) < \varepsilon.$$

Let $\mathcal{S} \subset \mathcal{U}$ be a $(\tau, K, N_\delta(Q))$-spanning set. The set $\iota(\mathcal{S})$ is a $(\tau, \xi(K), N_\varepsilon(\xi(Q)))$-spanning set. In fact, given $y \in \xi(K)$, consider $x \in K$ with $y = \xi(x)$ and take $u \in \mathcal{S}$ such that $\varphi_1(t, u, x) \in N_\delta(Q)$, for all $t \in [0, \tau]$. But this implies that for each $t \in [0, \tau]$, there exists $x_t \in Q$ with $\varrho_1(\varphi_1(t, u, x), x_t) < \delta$. Hence

$$\begin{aligned} \varrho_2(\varphi_2(t, \iota(u), y), \xi(x_t)) &= \varrho_2(\varphi_2(t, \iota(u), \xi(x), \xi(x_t)) \\ &= \varrho_2(\xi(\varphi_1(t, u, x)), \xi(x_t)) < \varepsilon, \end{aligned}$$

for all $t \in [0, \tau]$. Therefore $\xi(\varphi_1(t, u, x)) \in N_\varepsilon(\xi(Q))$. Hence

$$r_{\text{inv}}(\xi(K), N_\varepsilon(\xi(Q))) \leq r_{\text{inv}}(K, N_\delta(Q)).$$

So, $h_{\text{inv,out}}(\xi(K), \xi(Q); \Sigma_2) \leq h_{\text{inv,out}}(K, Q; \Sigma_1)$. Since ξ is a homeomorphism and ι is invertible and satisfies

$$\varphi_2(t, \iota(u), \xi(x)) = \xi(\varphi_1(t, u, x)),$$

for all $t \geq 0$, $x \in M_1$ and $u \in \mathcal{U}_1$, then

$$\xi^{-1}(\varphi_2(t, v, y)) = \varphi_1(t, \iota^{-1}(v), \xi^{-1}(y)),$$

for all $t \geq 0$, $y \in M_2$ and $v \in \mathcal{U}_2$. Hence we can also obtain that

$$\begin{aligned} h_{\text{inv,out}}(K, Q; \Sigma_1) &= h_{\text{inv,out}}(\xi^{-1}(\xi(K)), \xi^{-1}(\xi(Q)); \Sigma_1) \\ &\leq h_{\text{inv,out}}(\xi(K), \xi(Q); \Sigma_2) \end{aligned}$$

Therefore $h_{\text{inv}}(K, Q; \Sigma_1) = h_{\text{inv}}(\xi(K), \xi(Q); \Sigma_2)$. □

The following example shows an application of Proposition 6.

Example 7. *In Example 3 we showed that the system Λ and $\widetilde{\Lambda}$ are topologically conjugate. The maps that conjugate these systems are the diffeomorphism $\xi\colon \mathbf{S}^2 \to Sp(1)/H_{e_3}$, $\xi(g \cdot e_3) = gH_{e_3}$, and the identity map $\iota = id_{\mathcal{U}}\colon \mathcal{U} \to \mathcal{U}$. Since $\widetilde{\Lambda}$ is a linear system on $Sp(1)/H_{e_3}$, then it is the projection of a linear system*

$$(\Sigma) \ : \ \dot{g}(t) = \mathcal{X}(g(t)) + \sum_{j=1}^{m} u_j(t) Y_j(g(t)), \quad u_1, \dots, u_m \in \mathcal{U},$$

on $Sp(1)$.

Denote by $\pi\colon Sp(1) \to Sp(1)/H_{e_3}$ the canonical projection. It is not difficult to see that the maps π and ι a semi-conjugacy from Σ to $\widetilde{\Lambda}$ (see [13, Definition 2.4]).

Consider an admissible pair (K, Q) of Λ, so $(\xi(K), \xi(Q))$ is an admissible pair of $\widetilde{\Lambda}$. Define the subsets $K' := \pi^{-1}(\xi(K))$ and $Q' := \pi^{-1}(\xi(Q))$ of $Sp(1)$. Then (K', Q') is an admissible pair of Σ. In fact, since $Sp(1)$ is compact, π is continuous and $Sp(1)/H_{e_3}$ is Hausdorff, then K' is compact. Moreover, given $x \in K'$, then $\pi(x) \in \xi(K)$. Hence there exists $u \in \mathcal{U}$ such that

$$\widetilde{\varphi}(\mathbf{R}_+, \pi(x), u) \subset \xi(Q),$$

where $\widetilde{\varphi}$ denotes the solution of $\widetilde{\Lambda}$. But

$$\pi(\phi(\mathbf{R}_+, x, u)) = \widetilde{\varphi}(\mathbf{R}_+, \pi(x), u) \subset \xi(Q),$$

where ϕ denotes the solution of Σ. Therefore,

$$\phi(\mathbf{R}_+, x, u) \subset \pi^{-1}(\xi(Q)) = Q',$$

which shows the claim.

Hence the pair (K, Q) induces an admissible pair (K', Q') of Σ and

$$h_{inv,out}(K, Q; \Lambda) = h_{inv,out}(\xi(K), \xi(Q); \widetilde{\Lambda}) \leq h_{inv,out}(K', Q'; \Sigma).$$

But $h_{inv,out}(K', Q'; \Sigma) = 0$, because Σ is a linear control system on the compact Lie group $Sp(1)$ (see [6, Corollary 4.1.19]), hence

$$h_{inv,out}(K, Q, \Lambda) = 0.$$

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