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# A generalization of O'Neil's theorems for projections of measures and dimensions

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**Abstract**

*In this paper, more general versions of O'Neil's projection theorems and other related theorems. In particular, we study the relationship between the $\varphi$-multifractal dimensions and its orthogonal projections in Euclidean space.*

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## 1. Introduction

The dimensional properties of projections of sets and measures have been the focus of a lot of researchers for the last decades [2, 3, 18, 19, 21, 22, 23, 24, 28, 34]. One of the most fundamental results in this area is Marstrand's theorem. In his paper [27], he proved, using geometric arguments, that for a Borel set $E$ in the plane,

$$\dim(\pi_V(E)) = \min\Big(\dim E, 1\Big)$$

for almost every line V (here $\pi_V$ denotes orthogonal projection onto V). In [24], Kaufman gave an alternative proof of the same result using potential-theoretic methods. He also provided a dimension estimate for the set of exceptional directions. Mattila [28], generalized the latter concept to higher dimensions, his proof combined the methods of Marstrand and Kaufman. More specifically, he proved that for a Borel measure $\mu$ on $\mathbf{R}^n$,

$$\dim \pi_V(\mu) = \min\Big\{\dim \mu\,,\ m\Big\},$$

for almost all $m$-dimensional subspace $V$. Let us mention that the authors in [18, 21] have extended these results to packing dimensions of both sets and measures, such that, for $\mu$ a Borel measure on $\mathbf{R}^n$, if Dim $\mu \leq m$, then Dim $\pi_V(\mu) =$ Dim $\mu$, for almost all $m$-dimensional subspace $V$ of $\mathbf{R}^n$. Other works were carried out in this sense for classes of similar measures in euclidean and symbolic spaces [19, 23, 34].

As a continuity to this research, many authors have been studying the relationship between multifractal features of a measure $\mu$ on $\mathbf{R}^n$ and those of the projection of the measure onto $m$-dimensional subspaces. In [19, 20, 22] the authors studied the behavior of the $L^q$-spectrum of a measure $\mu$ on $\mathbf{R}^n$ under orthogonal projections onto lower dimensional linear subspaces. O'Neil and Selmi [34, 37] compared the generalized Hausdorff and packing dimensions of a set $E$ of $\mathbf{R}^n$ with respect to a measure $\mu$ with those of their projections onto $m$-dimensional subspaces. In [3], the authors studied the multifractal analysis of the orthogonal projections onto $m$-dimensional linear subspaces of singular measures on $\mathbf{R}^n$ satisfying the multifractal formalism. Later on, Douzi and Selmi [13], considered the relative multifractal formalism developed by Cole [9], as they studied the relationship between the relative multifractal spectra of orthogonal projections of a measure $\mu$ in Euclidean space and those of $\mu$. Recently, as a generalization of these results, Douzi and Selmi proved in [14, 15] a relationship between the mutual multifractal spectra of a couple of measures $(\mu, \nu)$ and its orthogonal projections in Euclidean spaces.

In the present paper, we pursue this direction of work, and that is by considering the $\varphi$-multifractal formalism developed in [29]. The aim of this work is twofold. Firstly, we investigate the behavior of the $\varphi$-multifractal Hausdorff and packing dimensions under projection. Secondly, we take interest in the relationship between the $\varphi$-multifractal spectrum and its projection onto a lower dimensional linear subspace.

## 2. Preliminaries and main results

We denote the family of Borel probability measures on $\mathbf{R}^d$ by $\mathcal{P}(\mathbf{R}^d)$ and for $\mu \in \mathcal{P}(\mathbf{R}^d)$ $supp(\mu)$ is the topological support of $\mu$. In this paper, we will work with the $\varphi$-multifractal formalism introduced in [29]. Let us recall the $\varphi$-multifractal Hausdorff and packing measures.

Let $\varphi : \mathbf{R}_+ \longrightarrow \mathbf{R}$ be such that $\varphi$ is non-decreasing and $\varphi(r) < 0$ for r small enough. For $\mu \in \mathcal{P}(\mathbf{R}^n)$, $q, t \in \mathbf{R}$, $E \subseteq \mathbf{R}^n$ and $\delta > 0$, we define

$$\overline{\mathcal{P}}^{q,t}_{\mu,\varphi,\delta}(E) = \sup\left\{\sum_i \mu\Big(B(x_i, r_i)\Big)^q e^{t\varphi(r_i)}\right\}, \quad E \neq \emptyset,$$

where the supremum is taken over all centered $\delta$-packings of $E$. Moreover, we can set $\overline{\mathcal{P}}^{q,t}_{\mu,\varphi,\delta}(\emptyset) = 0$.

The $\varphi$-multifractal packing pre-measure is then given by

$$\overline{\mathcal{P}}^{q,t}_{\mu,\varphi}(E) = \inf_{\delta>0} \overline{\mathcal{P}}^{q,t}_{\mu,\varphi,\delta}(E).$$

In a similar way, we define

$$\overline{\mathcal{H}}^{q,t}_{\mu,\varphi,\delta}(E) = \inf\left\{\sum_i \mu\Big(B(x_i, r_i)\Big)^q e^{t\varphi(r_i)}\right\}, \quad E \neq \emptyset,$$

where the infinimum is taken over all centered $\delta$-coverings of E. Moreover, we can set $\overline{\mathcal{H}}^{q,t}_{\mu,\varphi,\delta}(\emptyset) = 0$.

The $\varphi$-multifractal Hausdorff pre-measure is defined by

$$\overline{\mathcal{H}}^{q,t}_{\mu,\varphi}(E) = \sup_{\delta>0} \overline{\mathcal{H}}^{q,t}_{\mu,\varphi,\delta}(E).$$

We also have the following conventions $0^q = \infty$ for $q \le 0$ and $0^q = 0$ for $q > 0$.

$\overline{\mathcal{H}}^{q,t}_{\mu,\varphi}$ is $\sigma$-subadditive but not increasing whereas $\overline{\mathcal{P}}^{q,t}_{\mu,\varphi}$ is increasing but not $\sigma$-subadditive. For this particular reason, the authors introduced the following modifications on the $\varphi$-multifractal Hausdorff and packing measures $\mathcal{H}^{q,t}_{\mu,\varphi}$ and $\mathcal{P}^{q,t}_{\mu,\varphi}$,

$$\mathcal{H}^{q,t}_{\mu,\varphi}(E) = \sup_{F\subseteq E} \overline{\mathcal{H}}^{q,t}_{\mu,\varphi}(F) \quad \text{and} \quad \mathcal{P}^{q,t}_{\mu,\varphi}(E) = \inf_{E\subseteq\bigcup_i E_i} \sum_i \overline{\mathcal{P}}^{q,t}_{\mu,\varphi}(E_i).$$

The functions $\mathcal{H}^{q,t}_{\mu,\varphi}$ and $\mathcal{P}^{q,t}_{\mu,\varphi}$ are metric outer measures and hence, measures on the Borel family of subsets of $\mathbf{R}^n$. It is clear that $\mathcal{P}^{q,t}_{\mu,\varphi} \le \overline{\mathcal{P}}^{q,t}_{\mu,\varphi}$, there exists an integer $\xi \in \mathbf{N}$ such that $\mathcal{H}^{q,t}_{\mu,\varphi} \le \xi\mathcal{P}^{q,t}_{\mu,\varphi}$.

The measures $\mathcal{H}^{q,t}_{\mu,\varphi}$ and $\mathcal{P}^{q,t}_{\mu,\varphi}$ and the pre-measure $\overline{\mathcal{P}}^{q,t}_{\mu,\varphi}$ assign, in the usual way, a multifractal dimension to each subset $E$ of $\mathbf{R}^n$. More precisely, we have the following result.

**Proposition 1.** *[29]*

1. *There exists a unique number* $\dim^q_{\mu,\varphi}(E) \in [-\infty, +\infty]$ *such that*

$$\mathcal{H}^{q,t}_{\mu,\varphi}(E) = \begin{cases} \infty & \textit{if} \quad t < \dim^q_{\mu,\varphi}(E), \\ \\ 0 & \textit{if} \quad \dim^q_{\mu,\varphi}(E) < t. \end{cases}$$

2. *There exists a unique number* $Dim^q_{\mu,\varphi}(E) \in [-\infty, +\infty]$ *such that*

$$\mathcal{P}^{q,t}_{\mu,\varphi}(E) = \begin{cases} \infty & \textit{if} \quad t < Dim^q_{\mu,\varphi}(E), \\ \\ 0 & \textit{if} \quad Dim^q_{\mu,\varphi}(E) < t. \end{cases}$$

3. *There exists a unique number* $\Delta^q_{\mu,\varphi}(E) \in [-\infty, +\infty]$ *such that*

$$\overline{\mathcal{P}}^{q,t}_{\mu,\varphi}(E) = \begin{cases} \infty & \textit{if} \quad t < \Delta^q_{\mu,\varphi}(E), \\ \\ 0 & \textit{if} \quad \Delta^q_{\mu,\varphi}(E) < t. \end{cases}$$

**Remark 1.** *In the case* $\varphi = \log$, *we have*

$$\dim^0_{\mu,\varphi}(E) = \dim(E), \quad Dim^0_{\mu,\varphi}(E) = Dim(E) \text{ and } \Delta^0_{\mu,\varphi}(E) = \Delta(E),$$

*where* $\dim$, *Dim and* $\Delta$ *are respectively the Hausdorff dimension, the packing dimension and the Bouligand-Minkowski dimension.*

Next, for $q \in \mathbf{R}$, we define the separator functions $b_{\mu,\varphi}$, $B_{\mu,\varphi}$ and $\Lambda_{\mu,\psi}$ by

$$b_{\mu,\varphi}(q) = \dim(\mu), \varphi^q(\operatorname{supp}(\mu)),$$

$$B_{\mu,\varphi}(q) = \operatorname{Dim}^q_{\mu,\varphi}\operatorname{supp}(\mu)$$

$$\text{and } \Lambda_{\mu,\varphi}(q) = \Delta^q_{\mu,\varphi}(\operatorname{supp}(\mu)).$$

It is well known that

$$b_{\mu,\varphi} \leq B_{\mu,\varphi} \leq \Lambda_{\mu,\varphi}. \tag{2.1}$$

**Proposition 2.** *[29] One has*

1. $b_{\mu,\varphi}$ *and* $B_{\mu,\varphi}$ *are non decreasing with respect to the inclusion property in* $\mathbf{R}^n$.
2. $b_{\mu,\varphi}$ *and* $B_{\mu,\varphi}$ *are* $\sigma$*-stable.*
3. $0 \leq b_{\mu,\varphi}(q) \leq B_{\mu,\varphi}(q) \leq \Lambda_{\mu,\varphi}(q)$, *whenever* $q < 1$.
4. $b_{\mu,\varphi}(1) = B_{\mu,\varphi}(1) = \Lambda_{\mu,\varphi}(1) = 0$.
5. $b_{\mu,\varphi}(q) \leq B_{\mu,\varphi}(q) \leq \Lambda_{\mu,\varphi}(q) \leq 0$ *whenever* $q > 1$.

Now, we define both upper and lower multifractal bouligand-minkowski $\varphi$-dimensions in $\mathbf{R}^n$. Let $\mu \in \mathcal{P}(\mathbf{R}^n)$ and $q \in \mathbf{R}$. For $E \subseteq \mathbf{R}^n$ and $r > 0$, we write

$$S^q_{\mu,r}(E) = \sup\left\{\sum_i \mu\Big(B(x_i, r)\Big)^q\right\},$$

where $\Big\{B(x_i,r)\Big\}_i$ is a centered packing of $E\cap \mathrm{supp}(\mu)$. The upper and lower Bouligand-Minkowski $\varphi$-dimensions of $E$ denoted respectively by $\overline{\mathcal{R}}^q_{\mu,\varphi}$ and $\underline{\mathcal{R}}^q_{\mu,\varphi}$ are defined s follows:

$$\overline{\mathcal{R}}^q_{\mu,\varphi}(E)=\limsup_{r\to 0}\frac{\log S^q_{\mu,r}(E)}{-\varphi(r)} \quad \text{and} \quad \underline{\mathcal{R}}^q_{\mu,\varphi}(E)=\liminf_{r\to 0}\frac{\log S^q_{\mu,r}(E)}{-\varphi(r)}.$$

When the two limits coincide, we denote the common value by $\mathcal{R}^q_{\mu,\varphi}$.

Another natural way to define the Bouligand-Minkowski $\varphi$-dimensions is given by

$$T^q_{\mu,r}(E)=\inf\left\{\sum_i \mu\Big(B(x_i,r)\Big)^q\right\},$$

where $\Big\{B(x_i,r)\Big\}_i$ is a centered covering of $E\cap \mathrm{supp}\mu$. Now, we write

$$\overline{\mathcal{L}}^q_{\mu,\varphi}(E)=\limsup_{r\to 0}\frac{\log T^q_{\mu,r}(E)}{-\varphi(r)} \quad \text{and} \quad \underline{\mathcal{L}}^q_{\mu,\varphi}(E)=\liminf_{r\to 0}\frac{\log T^q_{\mu,r}(E)}{-\varphi(r)}.$$

If $\overline{\mathcal{L}}^q_{\mu,\varphi}(E)=\underline{\mathcal{L}}^q_{\mu,\varphi}(E)$, their common value at $q$ is denoted by $\mathcal{L}^q_{\mu,\varphi}(E)$.

**Remark 2.** *In the special case $q=0$ and $\varphi=\log$, the Bouligand-Minkowski $\varphi$-dimensions represent the upper and lower box-dimension, i.e.,*

$$\begin{aligned}\overline{\mathcal{L}}^0_{\mu,\log}(E)=\overline{\mathcal{R}}^0_{\mu,\log}(E)=\overline{dim}_B(E)\\ \underline{\mathcal{L}}^0_{\mu,\log}(E)=\underline{\mathcal{R}}^0_{\mu,\log}(E)=\underline{dim}_B(E)\end{aligned}$$

Next, for $q\in\mathbf{R}$, we denote by

$$\overline{\mathcal{R}}^\varphi_\mu(q)=\overline{\mathcal{B}}^q_{\mu,\varphi}(\mathrm{supp}(\mu)),$$

$$\underline{\mathcal{B}}^\varphi_\mu(q)=\underline{\mathcal{B}}^q_{\mu,\varphi}(\mathrm{supp}(\mu))$$

and

$$\overline{\mathcal{L}}^\varphi_\mu(q)=\overline{\mathcal{L}}^q_{\mu,\varphi}(\mathrm{supp}(\mu)), \quad \underline{\mathcal{L}}^\varphi_\mu(q)=\underline{\mathcal{L}}^q_{\mu,\varphi}(\mathrm{supp}(\mu)).$$

Now, we define a subclass of measures. For $\mu\in\mathcal{P}(\mathbf{R}^n)$ and $a>1$, we write

$$\mathcal{T}_a(\mu)=\limsup_{r\searrow 0}\left(\sup_{x\in\mathrm{supp}\mu}\frac{\mu\Big(B(x,ar)\Big)}{\mu\Big(B(x,r)\Big)}\right).$$

We will now say that the measure $\mu$ satisfies the doubling condition if there exists $a > 1$ such that $\mathcal{T}_a(\mu) < \infty$. It is easily seen that the exact value of the parameter $a$ is unimportant: $\mathcal{T}_a(\mu) < \infty$, for some $a > 1$ if and only if $\mathcal{T}_a(\mu) < \infty$, for all $a > 1$. Also, we will write $\mathcal{P}_d(\mathbf{R}^n)$ for the family of Borel probability measures on $\mathbf{R}^n$ which satisfy the doubling condition.

**Proposition 3.** *[29] For $q \in \mathbf{R}$ and $E \subseteq \mathbf{R}^n$ we have*

1. $\overline{\mathcal{L}}^q_{\mu,\varphi}(E) \leq \overline{\mathcal{R}}^q_{\mu,\varphi}(E)$ *and* $\underline{\mathcal{L}}^q_{\mu,\varphi}(E) \leq \underline{\mathcal{R}}^q_{\mu,\varphi}(E)$, *for* $\mu \in \mathcal{P}(\mathbf{R}^n)$.
2. $\overline{\mathcal{L}}^q_{\mu,\varphi}(E) = \overline{\mathcal{R}}^q_{\mu,\varphi}(E)$ *and* $\underline{\mathcal{L}}^q_{\mu,\varphi}(E) = \underline{\mathcal{R}}^q_{\mu,\varphi}(E)$, *for* $\mu \in \mathcal{P}_d(\mathbf{R}^n)$.

**Theorem 1.** *[29] Let $q \in \mathbf{R}$ and $\mu \in \mathcal{P}(\mathbf{R}^n)$. Then for $E \subseteq \mathbf{R}^n$, we have $\Delta^q_{\mu,\varphi}(E) = \overline{\mathcal{R}}^q_{\mu,\varphi}(E)$.*

**Theorem 2.** *[29] Let $q \in \mathbf{R}$ and $\mu \in \mathcal{P}_d(\mathbf{R}^n)$. Then for $E \subseteq \mathbf{R}^n$, we have $\dim^q_{\mu,\varphi}(E) \leq \underline{\mathcal{L}}^q_{\mu,\varphi}(E)$.*

In the next, let $m$ be an integer with $0 < m < n$ and $G_{n,m}$ stand for the Grassmannian manifold of all $m$-dimensional linear subspaces of $\mathbf{R}^n$ and we denote $\gamma_{n,m}$ the invariant Haar measure on $G_{n,m}$ such that $\gamma_{n,m}(G_{n,m}) = 1$. For $V \in G_{n,m}$, we define the projection map, $\pi_V : \mathbf{R}^n \longrightarrow V$ as the usual orthogonal projection onto $V$. Now, for a Borel probability measure $\mu$ on $\mathbf{R}^n$, supported on the compact set supp$\mu$ and for $V \in G_{n,m}$ we define $\mu_V$, the projection of $\mu$ onto $V$ by

$$\mu_V(A) = \mu(\pi_V^{-1}(A)), \quad \forall A \subseteq V.$$

Since $\mu$ has a compact support, $\mathrm{supp}(\mu_V) = \pi_V(\mathrm{supp}(\mu))$ for all $V \in G_{n,m}$ then for any continuous function $f : V \longrightarrow \mathbf{R}$

$$\int_V f d\mu_V = \int f(\pi_V(x)) d\mu(x)$$

whenever these integrals exist.

**Proposition 4.** *Let $\mu$ be a compactly supported Borel probability measure on $\mathbf{R}^n$ and $E \subseteq supp(\mu)$. Then, for $q \leq 1$ and all $V \in G_{n,m}$, we have*

$$\Delta^q_{\mu_V,\varphi}(\pi_V(E)) \leq \Delta^q_{\mu,\varphi}(E).$$

**Corollary 1.** *Let $\mu$ be a compactly supported Borel probability measure on $\mathbf{R}^n$. Then, for $q \leq 1$ and all $V \in G_{n,m}$, we have*

$$\Lambda_{\mu_V,\varphi}(q) \leq \Lambda_{\mu,\varphi}(q).$$

**Proof.** The proof is a straightforward consequence of Proposition 4. □

**Proposition 5.** *Let $\mu$ be a compactly supported Borel probability measure on $\mathbf{R}^n$. Then, for $q \leq 1$ and all $V \in G_{n,m}$, we have*

$$B_{\mu_V,\varphi}(q) \leq B_{\mu,\varphi}(q).$$

**Proposition 6.** *Let $\mu$ be a compactly supported Borel probability measure on $\mathbf{R}^n$ and $E \subseteq supp(\mu)$. Then, for all $V \in G_{n,m}$, we have*

1. *$b^q_{\mu_V,\varphi}(E) \leq b^q_{\mu,\varphi}(E)$, for $q \leq 0$.*
2. *$b_{\mu_V,\varphi}(q) \geq b_{\mu,\varphi}(q)$, for $q \geq 1$.*

**Remark 3.** *If $\varphi = \log$, we obtain the results of O'Neil (see [34]).*

In the sequel, we focus on the behavior of the Bouligand-Minkowski $\varphi$-dimensions under orthogonal projections.

**Theorem 3.** *Let $\mu$ be a compactly supported Borel probability measure on $\mathbf{R}^n$ and $E \subseteq supp(\mu)$. Then for $q \leq 1$ and for all $m$-dimensional subspaces $V$, we have*

$$\underline{\mathcal{R}}^q_{\mu_V,\varphi}(\pi_V(E)) \leq \underline{\mathcal{R}}^q_{\mu,\varphi}(E)$$

*and*

$$\overline{\mathcal{R}}^q_{\mu_V,\varphi}(\pi_V(E)) \leq \overline{\mathcal{R}}^q_{\mu_V,\varphi}(E). \tag{2.2}$$

**Theorem 4.** *Let $\mu$ be a compactly supported Borel probability measure on $\mathbf{R}^n$ and $E \subseteq supp(\mu)$. Then for $q \leq 1$ and for all $m$-dimensional subspaces $V$, we have*

$$\underline{\mathcal{L}}^q_{\mu_V,\varphi}(\pi_V(E)) \leq \underline{\mathcal{L}}^q_{\mu,\varphi}(E) \quad \text{and} \quad \overline{\mathcal{L}}^q_{\mu_V,\varphi}(\pi_V(E)) \leq \overline{\mathcal{L}}^q_{\mu,\varphi}(E).$$

**Theorem 5.** *Let $\mu$ be a compactly supported Borel probability measure on $\mathbf{R}^n$ and $E \subseteq supp(\mu)$. Then for $q \geq 1$ and for all $m$-dimensional subspaces $V$,*

$$\underline{\mathcal{L}}^q_{\mu_V,\varphi}(\pi_V(E)) \geq \underline{\mathcal{L}}^q_{\mu,\varphi}(E) \quad \text{and} \quad \underline{\mathcal{L}}^q_{\mu_V,\varphi}(\pi_V(E)) \geq \underline{\mathcal{L}}^q_{\mu,\varphi}(E).$$

## 3. Proofs of the main results

**Proof of Proposition 4**. We have too cases :

• **Case** 1 : $q < 0$. It is easy to see that if $\Big(B(x_i, r_i)\Big)_i$ is a centered $\delta$-packing of $\pi_V(E)$, then $\Big(B(y_i, r_i)\Big)_i$ is a centered $\delta$-packing of $E$ (where $y_i \in E$ such that $x_i = \pi_V(y_i)$). Then, one has

$$\mu(B(y_i, r_i)) \le \mu(\pi_V^{-1}(B(x_i, r_i))),$$

which implies that

$$\mu_V(B(x_i, r_i))^q \le \mu(B(y_i, r_i))^q.$$

Hence, we get the desired result.

• **Case** 2 : $0 \le q \le 1$. Let $t \in \mathbf{R}$ such that $\Delta_{\mu,\varphi}(q) < t$ and consider $V \in G_{n,m}$. Fix $\delta > 0$ and let $\Big(B(x_i, r_i)\Big)_i$ be a centered $\delta$-packing of $\pi_V(E)$. There exists an integer $k_m$ depending only on m such that the balls $B(x_i, 2r_i)$ can be divided into $K \le k_m$ families of disjoint balls $\mathcal{B}_1, \dots, \mathcal{B}_K$. Let $1 \le \ell \le K$. For each $B(x_i, r_i) \in \mathcal{B}_\ell$, denote $E_i = E \cap \pi_V^{-1}\Big(B(x_i, r_i)\Big)$. We have $E_i \subseteq \bigcup_{y \in E_i} B(y, r_i)$, so Besicovitch's covering theorem provides a positive integer $\xi_n$ as well as $K_i \le \xi_n$ families of pairwise disjoint balls $\mathcal{B}_{i,k} = \Big\{B\Big(y_j^{(i,k)}, r_{ijk}\Big) : r_{ijk} = \dfrac{r_i}{2}\Big\}$, $1 \le k \le K_i$, extracted from $\Big(B(y, r_i)\Big)_{y \in E_i}$ such that

$$E_i \subseteq \bigcup_{k=1}^{K_i} \bigcup_j B\Big(y_j^{(i,k)}, r_{ijk}\Big).$$

Therefore, we get

$$\begin{aligned}
\sum_i \mu_V(B(x_i, r_i))^q e^{t\varphi(r_i)} &\le \sum_i \mu\Big(\bigcup_{k=1}^{K_i} \bigcup_j B(y_j^{(i,k)}, r_{ijk})\Big)^q e^{t\varphi(r_i)} \\
&\le \sum_i \sum_j \sum_{k=1}^{K_i} \mu(B(y_j^{(i,k)}, r_{ijk}))^q e^{t\varphi(r_i)} \\
&\le \sum_i \sum_j \sum_{k=1}^{K_i} \mu(B(y_j^{(i,k)}, 2r_{ijk}))^q e^{t\varphi(2r_{ijk})}.
\end{aligned}$$

In both cases and by construction, since the balls $B(x_i, 2r_i) \in \mathcal{B}_\ell$ are pairwise disjoint, if $B(y,r) \in \mathcal{B}_{i,k}$ and $B(y',r') \in \mathcal{B}_{i',k'}$ with $i \neq i'$, then $B(y,r) \cap B(y',r') = \emptyset$. So, we can collect the balls $B(y,r)$ invoked in the above sum into at most $\xi_n$ centered packing of $E$. This holds for all $1 \leq \ell \leq K$, so

$$\sum_i \mu_V(B(x_i,r_i))^q e^{t\varphi(r_i)} \leq k_m \xi_n \sup\Big\{\sum_j \mu(B(y_j,2r_j))^q e^{t\varphi(2r_j)}\Big\},$$

where the supremum is taken over all centered packing of $E$ by closed balls of radius $r$. It results that

$$\overline{\mathcal{P}}^{q,t}_{\mu_V,\varphi,\delta}(\pi_V(E)) \leq k_m \xi_n \overline{\mathcal{P}}^{q,t}_{\mu,\varphi,2\delta}(E).$$

When $\delta \downarrow 0$ in the above expression, we obtain

$$\overline{\mathcal{P}}^{q,t}_{\mu_V,\varphi}(\pi_V(E)) \leq k_m \xi_n \overline{\mathcal{P}}^{q,t}_{\mu,\varphi}(E). \tag{3.1}$$

**Proof of Proposition 5**. Let $t \in \mathbf{R}$ such that $B_{\mu,\varphi}(q) < t$. Consider $F \subseteq \mathbf{R}^n$ and $V \in G_{n,m}$. By the inequality (3.1), one has

$$\overline{\mathcal{P}}^{q,t}_{\mu_V,\varphi}(\pi_V(F)) \leq k_m \xi_n \overline{\mathcal{P}}^{q,t}_{\mu,\varphi}(F).$$

As $B_{\mu,\varphi}(q) < t$, then $\mathcal{P}^{q,t}_{\mu,\varphi}(\mathrm{supp}(\mu)) < \infty$, and there exists $(E_i)_i$ a covering of $\mathrm{supp}(\mu)$ such that

$$\sum_i \overline{\mathcal{P}}^{q,t}_{\mu,\varphi}(E_i) < 1.$$

We clearly have $\pi_V(\mathrm{supp})(\mu) \subseteq \bigcup_i \pi_V(E_i)$, and then one gets

$$\begin{aligned}
\mathcal{P}^{q,t}_{\mu_V,\varphi}(\mathrm{supp}(\mu_V)) &\leq \sum_i \overline{\mathcal{P}}^{q,t}_{\mu_V,\varphi}(\pi_V(E_i)) \\
&\leq k_m \xi_n \sum_i \overline{\mathcal{P}}^{q,t}_{\mu,\varphi}(E_i) \\
&\leq k_m \xi_n < \infty.
\end{aligned}$$

It results that

$$B_{\mu_V,\varphi}(q) \leq t, \quad \forall t > B_{\mu,\varphi}(q).$$

Therefore, we can deduce that

$$B_{\mu_V,\varphi}(q) \leq B_{\mu,\varphi}(q).$$

**Proof of Proposition 6**.

1. The proof is similar to that of case one in Proposition 4.

2. Fix $V \in G_{n,m}$ and $\delta > 0$ and suppose that $\Big(B(x_i, r_i)\Big)_i$ is a $\delta$-cover of $\pi_V(E)$. For each i, we may use the Besicovitch covering theorem to find a constant $\xi$, depending only on n and a family of balls $\Big\{B(x_{ij}, r_{ij})\Big\}$ with $r_{ij} = \frac{r_i}{2}$, which is a $\delta$-cover of $\pi_V^{-1}\Big(B(x_i, r_i)\Big) \cap E$ such that $\bigcup_j B(x_{ij}, r_{ij}) \subseteq \pi_V^{-1}\Big(B(x_i, 2r_i) \cap V\Big)$, which leads to

$$\begin{aligned}\sum_i \mu_V(B(x_i, 2r_i))^q e^{t\varphi(2r_i)} &\geq \xi^{-q} \sum_i \Big(\sum_j \mu(B(x_{ij}, r_{ij}))\Big)^q e^{t\varphi(2r_i)} \\ &= \xi^{-q} \sum_i \sum_j \mu(B(x_{ij}, r_{ij}))^q e^{t\varphi(r_{ij})}.\end{aligned}$$

Then, as $\Big(B(x_i, r_i)\Big)_i$ represent a $\delta$-cover of $\pi_V(E)$, we deduce that

$$\overline{\mathcal{H}}^{q,t}_{\mu,\varphi,\delta}(E) \leq \xi^q \overline{\mathcal{H}}^{q,t}_{\mu_V,\varphi,2\delta}(\pi_V(E)).$$

When $\delta$ tends to 0, we get

$$\overline{\mathcal{H}}^{q,t}_{\mu,\varphi}(E) \leq \xi^q \overline{\mathcal{H}}^{q,t}_{\mu_V,\varphi}(\pi_V(E)).$$

Since $E \subseteq \operatorname{supp}(\mu)$, $\pi_V(E) \subseteq \operatorname{supp}(\mu_V)$ and we find

$$\overline{\mathcal{H}}^{q,t}_{\mu,\varphi}(E) \quad \leq \quad \xi^q \overline{\mathcal{H}}^{q,t}_{\mu_V,\varphi}(\pi_V(E)) \leq \xi^q \mathcal{H}^{q,t}_{\mu_V,\varphi}(\operatorname{supp}(\mu_V)).$$

Finaly, the arbitrary on $E$, means that

$$\mathcal{H}^{q,t}_{\mu,\varphi}(\operatorname{supp}(\mu)) \leq \xi^q H^{q,t}_{\mu_V,\varphi} \operatorname{supp}((\mu_V)),$$

and we deduce the desired result.

**Proof of Theorem 3**. Fix $V \in G_{n,m}$ and let $\Big(B(x_i,r)\Big)_i$ be a centered packing of $\pi_V(E)$. There exists an integer $k_m$ depending only on $m$ such that we can divide up the balls $B(x_i,2r)$ into $K \leq k_m$ families of disjoint balls $\mathcal{B}_1,\dots,\mathcal{B}_K$. Let $1 \leq \ell \leq K$. For each $B(x_i,r) \in \mathcal{B}_\ell$, denote $E_i = E \cap \pi_V^{-1}\Big(B(x_i,r)\Big)$. We have $E_i \subseteq \bigcup_{y\in E_i} B(y,r)$, so Besicovitch's covering theorem provides a positive integer $\xi_n$ as well as $k_i \leq \xi_n$ families of pairwise disjoint balls $\mathcal{B}_{i,k} = \Big\{B\Big(y_j^{(i,k)},r\Big)\Big\}$, $1 \leq k \leq k_i$, extracted from $\Big\{B(y,r)\Big\}_{y\in E_i}$ such that

$$E_i \subseteq \bigcup_{k=1}^{k_i}\bigcup_j B\Big(y_j^{(i,k)},r\Big).$$

• __Case1 :__ If $q<0$, we get

$$\begin{aligned}\sum_i \mu_V\Big(B(x_i,r)\Big)^q &\leq \sum_i \mu\left(B\Big(y_j^{(i,k)},r\Big)\right)^q\\ &\leq \sum_{i,j}\sum_{k=1}^{k_i}\mu\Big(B\Big(y_j^{(i,k)},r\Big)\Big)^q.\end{aligned}$$

• __Case2__ If $0 \leq q \leq 1$, then

$$\begin{aligned}\sum_i \mu_V\Big(B(x_i,r)\Big)^q &\leq \sum_i \mu\Big(\bigcup_{k=1}^{k_i}\bigcup_j B\Big(y_j^{(i,k)},r\Big)\Big)^q\\ &\leq \sum_{i,j}\sum_{k=1}^{k_i}\mu\Big(B\Big(y_j^{(i,k)},r\Big)\Big)^q.\end{aligned}$$

We remark that the balls $B(x_i,2r) \in \mathcal{B}_\ell$ are pairwise disjoint therefore, in both cases and by construction, if $B(y,r) \in B_{i,k}$ and $B(y',r) \in B_{i',k'}$ with $i \neq i'$ then $B(y,r) \cap B(y',r) = \emptyset$. Then, we can collect the balls $B(y,r)$ involved in the above sum into at most $\xi_n$ centered packings of $E$. This holds for all $1 \leq \ell \leq K$ and thus

$$\sum_i \mu_V\Big(B(x_i,r)\Big)^q \leq k_m\xi_n \sup\Big\{\sum_j \mu\Big(B(y_j,r)\Big)^q\Big\},$$

where the supremum is taken over all centered packing of $E$ by closed balls of radius $r$. Which implies that

$$S^q_{\mu_V,r}(\pi_V(E)) \leq k_m\xi_n S^q_{\mu,r}(E).$$

**Proof of Theorem 4**. Consider $V \in G_{n,m}$. Let $\Big\{B(x_i,r)\Big\}_i$ be a centered covering of $\pi_V(E)$. Denote by $E_i = E\bigcap \pi_V^{-1}\Big(B(x_i,r)\Big)$.

We have $E_i \subseteq \bigcup_{y\in E_i\cap\pi_V^{-1}(\{x_i\})} B\Big(y,\dfrac{r}{n}\Big)$. From Besicovitch covering theorem, there exists an integer $\xi_n$, depending only on $n$ as well as $k_i \leq \xi_n$ families of pairwise disjoint balls $\Big\{B\Big(y_j^{(i,k)},\dfrac{r}{n}\Big)\Big\}$, $1 \leq k \leq k_i$ such that

$$E\cap\pi_V^{-1}\Big(B(x_i,r)\Big) \subseteq \bigcup_{k=1}^{k_i}\bigcup_j B\Big(y_j^{(i,k)},r\Big).$$

• <u>**Case1**</u> : If $q<0$, then

$$\begin{aligned}\sum_i \mu_V\Big(B(x_i,r)\Big)^q &\leq \sum_i \mu\left(B\Big(y_j^{(i,k)},r\Big)\right)^q\\ &\leq \sum_{i,j}\sum_{k=1}^{k_i}\mu\left(B\Big(y_j^{(i,k)},r\Big)\right)^q.\end{aligned}$$

• <u>**Case2**</u> : If $0\leq q\leq 1$, one gets

$$\begin{aligned}\sum_i \mu_V\Big(B(x_i,r)\Big)^q &\leq \sum_i \mu\Big(\bigcup_{k=1}^{k_i}\bigcup_j B\Big(y_j^{(i,k)},r\Big)\Big)^q\\ &\leq \sum_{i,j}\sum_{k=1}^{k_i}\mu\left(B\Big(y_j^{(i,k)},r\Big)\right)^q.\end{aligned}$$

Therefore

$$T^q_{\mu_V,r}(\pi_V(E)) \leq T^q_{\mu,r}(E).$$

**Proof of Theorem 5**. Fix $V \in G_{n,m}$ and let $\Big\{B(y_i,r)\Big\}_i$ be a centered covering of $\pi_V(E)$. For each $i$, let $E_i = \pi_V^{-1}(B(y_i,r))\bigcap E$.

Since $E_i \subseteq \bigcup_{z \in E_i \cap \pi_V^{-1}(\{y_i\})} B\Big(z, r\Big)$, by Besicovitch covering theorem, we find a constant $\xi$, depending only on $n$, and a family of balls $\Big\{B(x_{ij}, r)\Big\}_{j \in \mathbf{N}}$ which is a centered packing of $E_i$ such that

$$\bigcup_j B(x_{ij}, r) \subseteq \pi_V^{-1}\Big(B(y_i, 2r) \cap V\Big).$$

Then, one gets

$$\begin{aligned} \sum_i \mu_V\Big(B(y_i, 2r)\Big)^q &\geq \xi^{-q} \sum_i \mu\Big(\bigcup_j B(x_{ij}, r)\Big)^q \\ &= \xi^{-q} \sum_i \Big(\sum_j \mu\left(B(x_{ij}, r)\right)\Big)^q \\ &\geq \xi^{-q} \sum_{i,j} \mu(B(x_{ij}, r))^q. \end{aligned}$$

Nest, since $\Big\{B(y_i, r)\Big\}_i$ may be any a centered covering of $\pi_V(E)$, we obtain

$$T^q_{\mu,r}(E) \leq \xi^q T^q_{\mu_V, 2r}(\pi_V(E)).$$

## Appendix

**Besicovitch covering theorem**. There exists a constant $\xi = \xi(n)$, depending only on $n$ such that: if $\mathcal{C}$ is a collection of nondegenerate closed balls in $\mathbf{R}^n$ with

$$\sup\Big\{\text{diam } B; \;\; B \in \mathcal{C}\Big\} < +\infty$$

and if $C$ is the set of centres of balls in $\mathcal{C}$, then there exist $\mathcal{C}_1, \mathcal{C}_2, ..., \mathcal{C}_\xi \subset \mathcal{C}$ such that each $\mathcal{C}_i$ is a countable collection of disjoint balls in $\mathcal{C}$ and

$$C \subseteq \bigcup_{i=1}^{\xi} \bigcup_{B \in \mathcal{C}_i} B.$$

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