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On graded $G2$-absorbing and graded strongly $G2$-absorbing second submodules

Shatha Alghueiri
Jordan University of Science and Technology, Jordan
and
Khaldoun Al-Zoubi
Jordan University of Science and Technology, Jordan
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Abstract

In this paper, we introduce the concepts of graded $G2$-absorbing and graded strongly $G2$-absorbing second submodules of graded modules over graded commutative rings. We give a number of results concerning these classes of graded submodules and their homogeneous components.

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1. Introduction and Preliminaries

Throughout this paper all rings are commutative with identity and all modules are unitary.

The concept of graded 2-absorbing ideals was introduced and studied by Al-Zoubi, Abu-Dawwas and Ceken in [3]. Al-Zoubi and Abu-Dawwas in [4] extended graded 2-absorbing ideals to graded 2-absorbing submodules. The concept of graded primary ideals was introduced and studied by Refai and Al-Zoubi in [19]. Al-Zoubi and Sharafat in [5] introduced the concept of graded 2-absorbing primary ideals which is a generalization of graded primary ideals. In [7], the concept of graded second submodules was introduced and studied in [2, 8, 13]. Also, graded secondary modules have been introduced by Atani and Farzalipour in [12]. The concepts of graded 2-absorbing second submodules and graded strongly 2-absorbing second submodules were introduced and studied in [1, 18].

Recently, Ansari-Toroghy, Farshadifar and Maleki-Roudposhti, in [9] introduced the concept of generalized 2-absorbing second and strongly generalized 2-absorbing second submodules.

The purpose of this paper is to introduce and study the concepts of graded generalized 2-absorbing second and graded generalized 2-absorbing second submodules and investigate some properties of these new classes of graded submodules and their homogeneous components.

First, we recall some basic properties of graded rings and modules which will be used in the sequel. We refer to [14, 15, 16, 17] for these basic properties and more information on graded rings and modules.

Let G be an abelian multiplicative group with identity e and R be a commutative ring with identity 1_R. Then R is a G-graded ring if there exist additive subgroups R_g of R such that $R = \oplus_{g\in G} R_g$ and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. The non-zero elements of R_g are said to be homogeneous of degree g where the R_g's are additive subgroups of R indexed by the elements $g \in G$. Moreover, $h(R) = \cup_{g\in G} R_g$. If $x \in R$, then x can be written uniquely as $\sum_{g\in G} x_g$, where x_g is the component of x in R_g. Moreover, R_e is a subring of R with $1_R \in R_e$. Let R be a G-graded ring and I be an ideal of R. Then I is called a graded ideal of R if $I = \sum_{g\in G}(I \cap R_g)$. Thus, if $x \in I$, then $x = \sum_{g\in G} x_g$ with $x_g \in I$, (see [17]).

Let R be a G-graded ring and M an R-module. We say that M is a G-graded R-module (or graded R-module) if there exists a family of additive subgroups $\{M_g\}_{g\in G}$ of M such that $M = \oplus_{g\in G} M_g$ (as abelian groups) and $R_g M_h \subseteq M_{gh}$ for all $g, h \in G$. Also, we write $h(M) = \cup_{g\in G} M_g$ and the non-zero elements of $h(M)$ are said to be homogeneous. Moreover, M_g is an R_e-module for all $g \in G$. Let M be a graded R-module and N a submodule of M. Then N is called a graded submodule of M if $N = \sum_{g \in G} N_g$ where $N_g = N \cap M_g$ for $g \in G$. In this case, N_g is called the g-component of N (see [17]).

Let R be a G-graded ring, M a graded R-module and N a graded submodule of M. Then $(N :_R M)$ is defined as $(N :_R M) = \{r \in R : rM \subseteq N\}$. It is shown in [10, Lemma 2.1] that if N is a graded submodule of M, then $(N :_R M)$ is a graded ideal of R. The annihilator of M is defined as $(0 :_R M)$ and is denoted by $Ann_R(M)$. A proper graded submodule C of M is said to be a completely graded irreducible if $C = \cap_{\alpha \in \Delta} C_\alpha$, where $\{C_\alpha\}_{\alpha \in \Delta}$ is a family of graded submodules of M, implies that $C = C_\beta$ for some $\beta \in \Delta$, (see [1]). A non-zero graded submodule N of M is said to be a graded second if for each $r \in h(R)$, the endomorphism of N given by multiplication by r is either surjective or zero, (see [7]). Let K be a graded ideal of R. The *graded radical* of K, denoted by $Gr(K)$, is the set of all $r = \sum_{g \in G} r_g \in R$ such that for each $g \in G$ there exists $n_g > 0$ with $r_g^{n_g} \in K$. Note that, if r is a homogeneous element, then $r \in Gr(K)$ if and only if $r^n \in K$ for some $n \in \mathbf{N}$, (see [19]). A proper graded ideal K of R is said to be a graded primary ideal if whenever $r_g, s_h \in h(R)$ with $r_g s_h \in K$, then either $r_g \in K$ or $s_h \in Gr(K)$, (see [19]).

2. Results

Definition 2.1. *Let R be a G-graded ring, M a graded R-module, $N = \oplus_{g \in G} N_g$ a graded submodule of M and $g \in G$.*

1. *We say that N_g is a generalized g-2-absorbing second or g-$G2$-absorbing second submodule of the R_e-module M_g if $N_g \neq \{0\}_g$ and whenever $r_e, s_e \in R_e$ and K_g is a completely irreducible submodule of the R_e-module M_g with $r_e s_e N_g \subseteq K_g$, then either $r_e s_e \in Ann_{R_e}(N_g)$ or $r_e \in Gr((K_g :_{R_e} N_g))$ or $s_e \in Gr((K_g :_{R_e} N_g))$.*
2. *We say that N is a graded generalized 2-absorbing second or graded $G2$-absorbing second submodule of M if $N \neq \{0\}$ and whenever $r_g, s_h \in h(R)$ and K is a completely graded irreducible submodule of M with $r_g s_h N \subseteq K$, then either $r_g s_h \in Ann_R(N)$ or $r_g \in Gr((K :_R N))$ or $s_h \in Gr((K :_R N))$.*

A graded R-module M is said to be a graded cocyclic if the sum of all graded minimal submodules of M is a large and graded simple submodule of M, (see [18]). Recall from [5] that a proper graded ideal I of a G-graded ring R is said to be a graded 2-absorbing primary ideal if whenever $r_g, s_h, t_\lambda \in h(R)$ with $r_g s_h t_\lambda \in I$, then either $r_g s_h \in I$ or $r_g t_\lambda \in Gr(I)$ or $s_h t_\lambda \in Gr(I)$.

Theorem 2.2. *Let R be a G-graded ring, M a graded R-module and $N = \oplus_{g\in G} N_g$ a graded submodule of M. If N is a graded $G2$-absorbing second submodule of M, then the following statements hold.*

1. *If K is a completely graded irreducible submodule of M with NK, then $(K :_R N)$ is a graded 2-absorbing primary ideal of R.*
2. *If M is a graded cocyclic module, then $Ann_R(N)$ is a graded 2-absorbing primary ideal of R.*
3. *If $Ann_R(N)$ is a graded primary ideal of R and K is a completely graded irreducible submodule of M with NK, then $(K :_R N)$ is a graded primary ideal of R.*

Proof. (i) Let K be a completely graded irreducible submodule of M with NK. Since NK, $(K :_R N)$ is a proper graded ideal of R. Now, let $r_g, s_h, t_\lambda \in h(R)$ such that $r_g s_h t_\lambda \in (K :_R N)$. Thus, $r_g s_h N \subseteq (K :_M t_\lambda)$. But $(K :_M t_\lambda)$ is a completely graded irreducible submodule of M by [18, Lemma 2.11], so we get either $r_g^{m_1} N \subseteq (K :_M t_\lambda)$ for some $m_1 \in \mathbf{Z}^+$ or $s_h^{m_2} N \subseteq (K :_M t_\lambda)$ for some $m_2 \in \mathbf{Z}^+$ or $r_g s_h N = \{0\}$ as N is a graded $G2$-absorbing second submodule of M. Hence, either $r_g t_\lambda \in Gr((K :_R N))$ or $s_h t_\lambda \in Gr((K :_R N))$ or $r_g s_h \in (K :_R N)$. Therefore, $(K :_R N)$ is a graded 2-absorbing primary ideal of R.

(ii) Since $M = M/\{0\}$ is a graded cocyclic module, $\{0\}$ is a completely graded irreducible submodule of M by [18, Lemma 2.10]. Then we get the result by part (i).

(iii) Let K be a completely graded irreducible submodule of M with NK. Since NK, $(K :_R N)$ is a proper graded ideal of R. Now, let $r_g, s_h \in h(R)$ with $r_g s_h \in (K :_R N)$. Hence, $r_g s_h N \subseteq K$ and then we get either $r_g \in Gr((K :_R N))$ or $s_h \in Gr((K :_R N))$ or $r_g s_h \in Ann_R(N)$. If either $r_g \in Gr((K :_R N))$ or $s_h \in Gr((K :_R N))$, then we get the result. Now, if $r_g s_h \in Ann_R(N)$, then $r_g \in Ann_R(N) \subseteq (K :_R N)$ or $s_h \in Gr(Ann_R(N)) \subseteq Gr((K :_R N))$. Therefore, $(K :_R N)$ is a graded primary ideal of R. $\square$

Lemma 2.3. *Let R be a G-graded ring, M a graded R-module, $I = \oplus_{g\in G} I_g$ a graded ideal of R and N a graded $G2$-absorbing second submodule of M. If $r_g \in h(R)$, $h \in G$ and K is a completely graded irreducible* *submodule of M with $r_gI_hN \subseteq K$, then either $r_g \in Gr((K :_R N))$ or $I_h \subseteq Gr((K :_R N))$ or $r_gI_h \subseteq Ann_R(N)$.*

Proof. Suppose that $r_g \in h(R)$, $h \in G$ and K is a completely graded irreducible submodule of M with $r_gI_hN \subseteq K$, $r_g \notin Gr((K :_R N))$ and $r_gI_h \nsubseteq Ann_R(N)$. Then there exists $i_{h_1} \in I_h$ such that $r_gi_{h_1}N \neq \{0\}$. Hence, since N is a graded $G2$-absorbing second submodule of M and $r_gi_{h_1}N \subseteq K$, $i_{h_1} \in Gr((K :_R N))$. Now, let $i_{h_2} \in I_h$. Then $r_g(i_{h_1}+i_{h_2})N \subseteq K$, so either $i_{h_1} + i_{h_2} \in Gr((K :_R N))$ or $r_g(i_{h_1} + i_{h_2}) \in Ann_R(N)$. If $i_{h_1} + i_{h_2} \in Gr((K :_R N))$, then $i_{h_2} \in Gr((K :_R N))$ since $i_{h_1} \in Gr((K :_R N))$. If $r_g(i_{h_1}+i_{h_2}) \in Ann_R(N)$, then $r_gi_{h_2} \notin Ann_R(N)$. But $r_gi_{h_2}N \subseteq K$, so $i_{h_2} \in Gr((K :_R N))$. Therefore, $I_h \subseteq Gr((K :_R N))$. $\square$

Theorem 2.4. *Let R be a G-graded ring, M a graded R-module, $I = \oplus_{g\in G}I_g$, $J = \oplus_{g\in G}J_g$ graded ideals of R and N a non-zero graded submodule of M. Then the following statement are equivalent:*

1. *N is a graded $G2$-absorbing second submodule of M.*
2. *If K is a completely graded irreducible submodule of M and $g, h \in G$ with $I_gJ_hN \subseteq K$, then either $I_g \subseteq Gr((K :_R N))$ or $J_h \subseteq Gr((K :_R N))$ or $I_gJ_h \subseteq Ann_R(N)$.*

Proof. $(i) \Rightarrow (ii)$ Let K be a completely graded irreducible submodule of M and $g, h \in G$ with $I_gJ_hN \subseteq K$, $I_g \nsubseteq Gr((K :_R N))$ and $J_h \nsubseteq Gr((K :_R N))$. We show that $I_gJ_h \subseteq Ann_R(N)$, so let $i_g \in I_g$ and $j_h \in J_h$. Since $I_g \nsubseteq Gr((K :_R N))$, there exists $i'_g \in I_g$ such that $i'_g \notin Gr((K :_R N))$. But $i'_gJ_hN \subseteq K$, so by Lemma 2.3, we get $i'_gJ_h \subseteq Ann_R(N)$ and so $(I_g\backslash Gr((K :_R N)))J_h \subseteq Ann_R(N)$. Similarly, since $J_h \nsubseteq Gr((K :_R N))$, there exists $j'_h \in J_h$ such that $j'_h \notin Gr((K :_R N))$. But $I_gj'_hN \subseteq K$, so by Lemma 2.3, we get $I_gj'_h \subseteq Ann_R(N)$ and so $I_g(J_h\backslash Gr((K :_R N))) \subseteq Ann_R(N)$. Hence, we have $i'_gj'_h \in Ann_R(N)$, $i'_gj_h \in Ann_R(N)$ and $i_gj'_h \in Ann_R(N)$. Now, as $i_g + i'_g \in I_g$ and $j_h + j'_h \in J_h$, we have $(i_g + i'_g)(j_h + j'_h)N \subseteq K$. Thus, we get either $i_g + i'_g \in Gr((K :_R N))$ or $j_h + j'_h \in Gr((K :_R N))$ or $(i_g + i'_g)(j_h + j'_h) \in Ann_R(N)$. If $i_g + i'_g \in Gr((K :_R N))$, then $i_g \notin Gr((K :_R N))$. Then $i_g \in I_g\backslash Gr((K :_R N))$ which yields that $i_gj_h \in Ann_R(N)$. Similarly, if $j_h + j'_h \in Gr((K :_R N))$, then $i_gj_h \in Ann_R(N)$. Now, if $(i_g+i'_g)(j_h+j'_h) = i_gj_h+i_gj'_h+i'_gj_h+i'_gj'_h \in Ann_R(N)$, then $i_gj_h \in Ann_R(N)$. Therefore, $I_gJ_h \subseteq Ann_R(N)$.

$(ii) \Rightarrow (i)$ Assume that (ii) holds. Let $r, s, \in h(R)$ and K be a completely graded irreducible submodule of M with $rsN \subseteq K$. Let I and J be ideals of R generated by r and s, respectively. That is $I = rR$ and $J = sR$. So, $I = \oplus_{g\in G} rR_g$ and $J = \oplus_{g\in G} sR_g$. Moreover, for each $g \in G$, $I_g = rR_g$ and $J_g = sR_g$. Now, by our assumption $I_e J_e N \subseteq K$. So, we obtain either $I_e \subseteq Gr((K :_R N))$ or $J_e \subseteq Gr((K :_R N))$ or $I_e J_e \subseteq Ann_R(N)$. Hence, either $r \in Gr((K :_R N))$ or $s \in Gr((K :_R N))$ or $rs \in Ann_R(N)$. Therefore, N is a graded $G2$-absorbing second submodule of M. □

Definition 2.5. *Let R be a G-graded ring, M a graded R-module, $N = \oplus_{g\in G} N_g$ a graded submodule of M and $g \in G$.*

1. *We say that N_g is a strongly g-$G2$-absorbing second submodule of the R_e-module M_g, if $N_g \neq \{0\}_g$; and whenever $r_e, s_e \in R_e$ and K_g is a submodule of M_g with $r_e s_e N_g \subseteq K_g$, then either $r_e \in Gr((K_g :_{R_e} N_g))$ or $s_e \in Gr((K_g :_{R_e} N_g))$ or $r_e s_e \in Ann_{R_e}(N_g)$.*
2. *We say that N is a graded strongly $G2$-absorbing second submodule of M, if $N \neq \{0\}$; and whenever $r_g, s_h \in h(R)$ and K is a graded submodule of M with $r_g s_h N \subseteq K$, then either $r_g \in Gr((K :_R N))$ or $s_h \in Gr((K :_R N))$ or $r_g s_h \in Ann_R(N)$.*

Theorem 2.6. *Let R be a G-graded ring, M a graded R-module and N a non-zero graded submodule of M. Let $I = \oplus_{g\in G} I_g$, $J = \oplus_{g\in G} J_g$ be graded ideals of R and K a graded submodule of M. Then the following statements are equivalent:*

1. *N is a graded strongly $G2$-absorbing second submodule of M.*
2. *If $g, h \in G$ with $I_g J_h N \subseteq K$, then either $I_g \subseteq Gr((K :_R N))$ or $J_h \subseteq Gr((K :_R N))$ or $I_g J_h \subseteq Ann_R(N)$.*

Proof. The proof is similar to that in Theorem 2.4. □

A graded R-module M is said to be gr-Artinian if satisfies the descending chain condition for graded submodules, (see [17]).

Theorem 2.7. *Let R be a G-graded ring, M a gr-Artinian R-module and N a non-zero graded submodule of M. Then the following statements are equivalent.*

1. *N is a graded strongly $G2$-absorbing second submodule of M.*

2. *If $r_g, s_h \in h(R)$ and L_1, L_2 are two completely graded irreducible submodules of M with $r_g s_h N \subseteq L_1 \cap L_2$, then either $r_g \in Gr((L_1 \cap L_2 :_R N))$ or $s_h \in Gr((L_1 \cap L_2 :_R N))$ or $r_g s_h \in Ann_R(N)$.*

Proof. $(ii) \Rightarrow (i)$ Let $r_g, s_h \in h(R)$ and K be a graded submodule of M such that $r_g s_h N \subseteq K$ and $r_g s_h \notin Ann_R(N)$. Since M is a gr-Artinian and by [1, Theorem 2.1], there exist completely graded irreducible submodules $L_1, ..., L_n$ where $n \in \mathbf{Z}^+$ such that $K = \cap_{i=1}^n L_i$. So, for each $i = 1, ..., n$, $r_g s_h N \subseteq L_i = L_i \cap L_i$ which yields that $r_g \in Gr((L_i :_R N))$ or $s_h \in Gr((L_i :_R N))$. Hence, if $r_g \in Gr((L_i :_R N))$ for each $i = 1, ..., n$, then $r_g \in \cap_{i=1}^n Gr((L_i :_R N)) = Gr(\cap_{i=1}^n (L_i :_R N)) = Gr((\cap_{i=1}^n L_i :_R N)) = Gr((K :_R N))$. Similarly, if $s_h \in Gr((L_i :_R N))$ for each $i = 1, ..., n$, then $s_h \in Gr((K :_R N))$. Now, suppose that there exist $j_1, j_2 \in \{1, ..., n\}$ such that $r_g \notin Gr((L_{j_1} :_R N))$ and $s_h \notin Gr((L_{j_2} :_R N))$. Then we get $r_g \in Gr((L_{j_2} :_R N))$ and $s_h \in Gr((L_{j_1} :_R N))$. But $r_g s_h N \subseteq L_{j_1} \cap L_{j_2}$, so either $r_g \in Gr((L_{j_1} \cap L_{j_2} :_R N)) \subseteq Gr((L_{j_1} :_R N))$ or $s_h \in Gr((L_{j_1} \cap L_{j_2} :_R N)) \subseteq Gr((L_{j_2} :_R N))$, a contradiction. Therefore, N is a graded strongly $G2$-absorbing second submodule of M.

$(i) \Rightarrow (ii)$ Trivial. □

Let R be a G-graded ring and M a graded R-module. A non-zero graded submodule N of M is said to be a graded secondary if $N \neq 0$; and for each $r \in h(R)$, the endomorphism of N given by multiplication by r is either surjective or nilpotent. It is immediate that $Gr(Ann_R(N)) = P$ is a graded prime ideal of R, and N is said to be graded P-secondary, (see [12]).

Theorem 2.8. *Let R be a G-graded ring and M a graded R-module. If N is a graded secondary submodule of M or N is a finite sum of graded P-secondary submodules of M, then N is a graded strongly $G2$-absorbing second submodule of M.*

Proof. Suppose that N is a graded secondary submodule of M. Now, let $r_g, s_h \in h(R)$ and K be a graded submodule of M such that $r_g s_h N \subseteq K$. If $r_g^{m_1} N = \{0\}$ for some $m_1 \in \mathbf{Z}^+$ or $s_h^{m_2} N = \{0\}$ for some $m_2 \in \mathbf{Z}^+$, then $r_g^{m_1} N \subseteq K$ or $s_h^{m_2} N \subseteq K$. So, assume that $r_g N = s_h N = N$. If $r_g s_h N = N$, then $r_g N = N = r_g s_h N \subseteq K$ and $s_h N = N = r_g s_h N \subseteq K$. Also, if $(r_g s_h)^n N = \{0\}$ for some $n \in \mathbf{Z}^+$, then $r_g^n N = (r_g s_h)^n N = \{0\} \subseteq K$ and $s_h^n N = (r_g s_h)^n N = \{0\} \subseteq K$. Now, suppose N is a finite sum of graded P-secondary submodules of M, then N is a graded P-secondary by [11, Lemma 3.2] and then we get the result. □

Theorem 2.9. *Let R be a G-graded ring, M a graded R-module and N and L be two graded submodules of M with LN. If N is a graded strongly $G2$-absorbing second submodule of M, then N/L is a graded strongly $G2$-absorbing second submodule of M/L.*

Proof. Let $r_g, s_h \in h(R)$ and K/L be a graded submodule of M/L such that $r_g s_h(N/L) \subseteq K/L$. Hence, $r_g s_h N \subseteq K$ and then either $r_g^{n_1} N \subseteq K$ for some $n_1 \in \mathbf{Z}^+$ or $s_h^{n_2} N \subseteq K$ for some $n_2 \in \mathbf{Z}^+$ or $r_g s_h N = 0$ as N is a graded strongly $G2$-absorbing second submodule of M. Thus, either $r_g^{n_1}(N/L) \subseteq K/L$ for some $n_1 \in \mathbf{Z}^+$ or $s_h^{n_2}(N/L) \subseteq K/L$ for some $n_2 \in \mathbf{Z}^+$ or $r_g s_h(N/L) = L$. □

Let R be a G-graded ring and $I = \oplus_{g \in G} I_g$ a graded ideal of R. Then I_e is called an e-2-absorbing primary ideal of R_e if $I_e \neq R_e$ and whenever $r_e, s_e, t_e \in R_e$ with $r_e s_e t_e \in I_e$, then either $r_e s_e \in I_e$ or $r_e t_e \in Gr(I_e)$ or $s_e t_e \in Gr(I_e)$, (see [5]).

Theorem 2.10. *Let R be a G-graded ring, M a graded R-module, $N = \oplus_{g \in G} N_g$ a graded submodule of M and $g \in G$. Let N_g be a strongly g-$G2$-absorbing second submodule of the R_e-module M_g. Then the following statements hold.*

1. *$Ann_{R_e}(N_g)$ is an e-2-absorbing primary ideal of R_e.*
2. *If K_g is a submodule of M_g with $N_g \nsubseteq K_g$, then $(K_g :_{R_e} N_g)$ is an e-2-absorbing primary ideal of R_e.*

Proof. (i) Let $r_e, s_e, t_e \in R_e$ such that $r_e s_e t_e \in Ann_{R_e}(N_g)$. Now, since $r_e s_e N_g \subseteq r_e s_e N_g$ and N_g is a strongly g-$G2$-absorbing second submodule of M_g, either $r_e s_e N_g = \{0\}_g$ or $r_e^{m_1} N_g \subseteq r_e s_e N_g$ for some $m_1 \in \mathbf{Z}^+$ or $s_e^{m_2} N_g \subseteq r_e s_e N_g$ for some $m_2 \in \mathbf{Z}^+$. If $r_e s_e N_g = \{0\}_g$, then $r_e s_e \in Ann_{R_e}(N_g)$. Now, if $r_e^{m_1} N_g \subseteq r_e s_e N_g$, then $(r_e t_e)^{m_1} N_g \subseteq r_e^{m_1} t_e N_g \subseteq r_e s_e t_e N_g = \{0\}_g$. Hence, $r_e t_e \in Gr(Ann_{R_e}(N_g))$. Also, if $s_e^{m_2} N_g \subseteq r_e s_e N_g$, then $s_e t_e \in Gr(Ann_{R_e}(N_g))$. Therefore, $Ann_{R_e}(N_g)$ is an e-2-absorbing primary ideal of R_e.

(ii) Since $N_g \nsubseteq K_g$, $(K_g :_{R_e} N_g)$ is a proper ideal of R_e. Now, let $r_e, s_e, t_e \in R_e$ such that $r_e s_e t_e \in (K_g :_{R_e} N_g)$. So, $r_e s_e N_g \subseteq (K_g :_{M_g} t_e)$ and then either $r_e s_e N_g = \{0\}_g$ or $r_e^{m_1} t_e N_g \subseteq K_g$ for some $m_1 \in \mathbf{Z}^+$ or $s_e^{m_2} t_e N_g \subseteq K_g$ for some $m_2 \in \mathbf{Z}^+$ as N_g is a strongly g-$G2$-absorbing second submodule of M_g. Hence, either $r_e s_e \in (K_g :_{R_e} M_g)$ or $r_e t_e \in Gr((K_g :_{R_e}$

N_g)) or $s_e t_e \in Gr((K_g :_{R_e} N_g))$. Therefore, $(K_g :_{R_e} N_g)$ is an e-2-absorbing primary ideal of R_e. $\square$

Let R be a G-graded ring, M a graded R-module and $g \in G$. Then M is said to be a graded comultiplication module (gr-comultiplication module) if for every graded submodule N of M there exists a graded ideal I of R such that $N = (0 :_M I)$, where $(0 :_M I) = \{m \in M : mI = 0\}$. Equivalently, M is a gr-comultiplication module if and only if for each graded submodule N of M, $N = (0 :_M Ann_R(N))$, (see [2]). Also, the R_e-module M_g is said to be g-comultiplication module if for every submodule N_g of M_g there exists an ideal I_e of R_e such that $N_g = (0_g :_{M_g} I_e)$. Equivalently, M_g is a g-comultiplication module if and only if for each submodule N_g of M_g, $N_g = (0_g :_{M_g} Ann_{R_e}(N_g))$, (see [2]).

Let R be a G-graded ring, M a graded R-module, $N = \oplus_{g\in G} N_g$ a graded submodule of M and $g \in G$. Then N_g is said to be a strongly g-2-absorbing second submodule of the R_e-module M_g, if $N_g \neq \{0\}_g$; and whenever $r_e, s_e \in R_e$ and K_g is a submodule of M_g with $r_e s_e N_g \subseteq K_g$, then either $r_e N_g \subseteq K_g$ or $s_e N_g \subseteq K_g$ or $r_e s_e \in Ann_{R_e}(N_g)$, (see [1]).

Let R be a G-graded ring, M a graded R-module, $N = \oplus_{g\in G} N_g$ a graded submodule of M and $g \in G$. We say that N_g is a g-coidempotent submodule of the R_e-module M_g if $N_g = (0 :_{M_g} Ann_{R_e}(N_g)^2)$. Also, M_g is called a fully g-coidempotent if every submodule is a g-coidempotent. It is easy to see that every fully g-coidempotent module is a g-comultiplication.

Corollary 2.11. *Let R be a G-graded ring, M a graded R-module, $N = \oplus_{g\in G} N_g$ a graded submodule of M and $g \in G$. If M_g is a g-comultiplication R_e-module and N_g is a strongly g-$G2$-absorbing second submodule of M_g with $Gr(Ann_{R_e}(N_g)) = Ann_{R_e}(N_g)$, then N_g is a strongly g-2-absorbing second submodule of M_g.*

Proof. Since N_g is a strongly g-$G2$-absorbing second submodule of M_g, $Ann_{R_e}(N_g)$ is an e-2-absorbing primary ideal of R_e by Theorem 2.10. Now, let $r_e, s_e \in R_e$ and K_g be a submodule of M_g such that $r_e s_e N_g \subseteq K_g = (\{0\}_g :_{M_g} Ann_{R_e}(K_g))$. So, we get $r_e s_e Ann_{R_e}(K_g) N_g = \{0\}_g$ and then $r_e s_e Ann_{R_e}(K_g) = Ann_{R_e}(N_g)$ which yields that either $r_e Ann_{R_e}(K_g) \subseteq Gr(Ann_{R_e}(N_g)) = Ann_{R_e}(N_g)$ or $s_e Ann_{R_e}(K_g) \subseteq Gr(Ann_{R_e}(N_g)) = Ann_{R_e}(N_g)$ or $r_e s_e \in Ann_{R_e}(N_g)$ by [6, Lemma 3.1]. Hence, either $r_e N_g \subseteq K_g$ or $s_e N_g \subseteq K_g$ or $r_e s_e \in Ann_{R_e}(N_g)$. Therefore, N_g is a strongly g-2-absorbing second submodule of M_g. $\square$

A G-graded ring R is called gr-Noetherian if it satisfies the ascending chain condition on graded ideals of R. Equivalently, R is gr-Noetherian if and only if every graded ideal of R is finitely generated, (see [17]).

Theorem 2.12. *Let R be a G-graded gr-Noetherian ring, M a graded R-module, $N = \oplus_{g\in G} N_g$ a graded submodule of M and $g \in G$. Let M_g be a fully g-coidempotent R_e-module. If $Ann_{R_e}(N_g)$ is an e-2-absorbing primary ideal of R_e, then N_g is a strongly g-$G2$-absorbing second submodule of M_g.*

Proof. Let $r_e, s_e \in R_e$ and $K = \oplus_{h\in G} K_h$ be a graded submodule of M such that $r_e s_e N_g \subseteq K_g$. Thus, $r_e s_e Ann_{R_e}(K_g) N_g = \{0\}$ and then $r_e s_e Ann_{R_e}(K_g) \subseteq Ann_{R_e}(N_g)$. So, by [6, Lemma 3.1], we get either $r_e s_e \in Ann_{R_e}(N_g)$ or $r_e Ann_{R_e}(K_g) \subseteq Gr(Ann_{R_e}(N_g))$ or
$s_e Ann_{R_e}(K_g) \subseteq Gr(Ann_{R_e}(N_g))$ as $Ann_{R_e}(N_g)$ is an e-2-absorbing primary ideal of R_e. If $r_e s_e \in Ann_{R_e}(N_g)$, then we get the result. Now, since R is a gr-Noetherian, then so R_e. So, if $r_e Ann_{R_e}(K_g) \subseteq Gr(Ann_{R_e}(N_g))$, then $(r_e Ann_{R_e}(K_g))^t N_g = r_e^t Ann_{R_e}(K_g)^t N_g = \{0\}$ for some $t \in \mathbf{Z}^+$ which yields that $Ann_{R_e}(K_g)^t \subseteq Ann_{R_e}(r_e^t N_g)$. Hence, as M_g is a fully g-coidempotent we get $r_e^t N_g = (\{0\} :_{M_g} Ann_{R_e}(r_e^t N_g)) \subseteq (\{0\} :_{M_g} Ann_{R_e}(K_g)^t) = (\{0\} :_{M_g} Ann_{R_e}(K_g)^2) = K_g$. Also, if $s_e Ann_{R_e}(K_g) \subseteq Gr(Ann_{R_e}(N_g))$, then $s_e^l N_g \subseteq K_g$ for some $l \in \mathbf{Z}^+$. Therefore, N_g is a strongly g-$G2$-absorbing second submodule of M_g. $\square$

The following example shows that Theorem 2.12 is not true in general.

Example 2.13. *Let $G = \mathbf{Z}_2$ and $R = \mathbf{Z}$. Then R is a G-graded ring with $R_0 = \mathbf{Z}$ and $R_1 = 0$. Let $M = \mathbf{Z}$. Then M is a graded R-module with $M_0 = \mathbf{Z}$ and $M_1 = 0$ where M_0 is not a fully 0-coidempotent. Now, consider the graded submodule $N = 2\mathbf{Z}$ of M. Then N_0 is not a strongly 0-$G2$-absorbing second submodule of M_0 since $3 \cdot 5 \cdot 2\mathbf{Z} \subseteq 30\mathbf{Z}$ but neither $3 \cdot 5 \in Ann_{\mathbf{Z}}(2\mathbf{Z}) = 0$ nor $3^{t_1} \cdot 2\mathbf{Z} \not\subseteq 30\mathbf{Z}$ for all $t_1 \in \mathbf{Z}^+$ nor $5^{t_2} \cdot 2\mathbf{Z} \not\subseteq 30\mathbf{Z}$ for all $t_2 \in \mathbf{Z}^+$. However, easy computations show that $Ann_{\mathbf{Z}}(2\mathbf{Z}) = 0$ is a 0-2-absorbing primary ideal of R_0.*

Let R be a G-graded ring and M, M' graded R-modules. Let $f : M \to M'$ be an R-module homomorphism. Then f is said to be a graded homomorphism if $f(M_g) \subseteq M'_g$ for all $g \in G$, (see [17]).

Theorem 2.14. *Let R be a G-graded ring, M and M' be two graded R-modules and $f : M \to M'$ a graded monomorphism.*

1. *If N is a graded strongly $G2$-absorbing second submodule of M, then $f(N)$ is a graded strongly $G2$-absorbing second submodule of M'.*

2. *If N' is a graded strongly $G2$-absorbing second submodule of M' such that $N' \subseteq f(M)$, then $f^{-1}(N')$ is a graded strongly $G2$-absorbing second submodule of M.*

Proof. (i) It is easy to see that $f(N)$ is a graded non-zero submodule of M'. Now, let $r_g, s_h \in h(R)$ and K' a graded submodule of M' such that $r_g s_h f(N) \subseteq K'$. Hence, $r_g s_h N \subseteq f^{-1}(K')$ which yields that either $r_g^{t_1} N \subseteq f^{-1}(K')$ for some $t_1 \in \mathbf{Z}^+$ or $s_h^{t_2} N \subseteq f^{-1}(K')$ for some $t_2 \in \mathbf{Z}^+$ or $r_g s_h N = 0$ as N is a graded strongly $G2$-absorbing second submodule of M. So, either $r_g^{t_1} f(N) \subseteq f(f^{-1}(K')) \subseteq K'$ or $s_h^{t_2} f(N) \subseteq f(f^{-1}(K')) \subseteq K'$ or $r_g s_h f(N) = 0$. Therefore, $f(N)$ is a graded strongly $G2$-absorbing second submodule of M'. (ii) Clearly, $f^{-1}(N')$ is a graded non-zero submodule of M. Now, let $r_g, s_h \in h(R)$ and K a graded submodule of M such that $r_g s_h f^{-1}(N') \subseteq K$. Thus, $r_g s_h N' = r_g s_h f(f^{-1}(N')) \subseteq f(K)$ which yields either $r_g^{t_1} N' \subseteq f(K)$ for some $t_1 \in \mathbf{Z}^+$ or $s_h^{t_2} N' \subseteq f(K)$ for some $t_2 \in \mathbf{Z}^+$ or $r_g s_h N' = 0$ as N' is a graded strongly $G2$-absorbing second submodule of M'. Hence, $r_g^{t_1} f^{-1}(N') \subseteq K$ or $s_h^{t_2} f^{-1}(N') \subseteq K$ or $r_g s_h f^{-1}(N') = 0$. Therefore, $f^{-1}(N')$ is a graded strongly $G2$-absorbing second submodule of M. □

Corollary 2.15. *Let R be a G-graded ring, M a graded R-module and N and K be two graded submodules of M with $N \subseteq K$. Then N is a graded strongly $G2$-absorbing second submodule of K if and only if N is a graded strongly $G2$-absorbing second submodule of M.*

Proof. By using the graded natural monomorphism $f : K \to M$ and Theorem 2.14 we get the result. □

Lemma 2.16. *Let R_i be a G-graded ring, M_i a graded R_i-module, for $i = 1, 2$ and $g \in G$. Let $R = R_1 \times R_2$ and $M = M_1 \times M_2$. Then M_{i_g} is a fully g-coidempotent R_{i_e}-module, for $i = 1, 2$ if and only if M_g is a fully g-coidempotent R_e-module.*

Proof. Suppose that M_g is a fully g-coidempotent R_e-module. Now, let N_{1_g} be a submodule of an R_{1_e}-module M_{1_g}. Then $N_g = N_{1_g} \times \{0\}_{2_g}$ is a submodule of M_g. Hence, $N_g = (\{(0,0)\}_g :_{M_g} Ann_{R_e}(N_g)^2) = (\{0\}_{1_g} :_{M_{1_g}} Ann_{R_{1_e}}(N_{1_g})^2) \times \{0\}_{2_g}$. Thus, $N_{1_g} = (\{0\}_{1_g} :_{M_{1_g}} Ann_{R_{1_e}}(N_{1_g})^2)$. Therefore, M_{1_g} is a fully g-coidempotent R_{1_e}-module. Similarly, M_{2_g} is a fully g-coidempotent R_{2_e}-module. Conversely, M_{i_g} is a fully g-coidempotent R_{i_e}-module, for $i = 1, 2$. So, let N_g be a submodule of M_g. Then $N_g = N_{1_g} \times N_{2_g}$

for some submodules N_{1_g} of M_{1_g} and N_{2_g} of M_{2_g}. But $N_{i_g} = (\{0\}_{i_g} :_{M_{i_g}} Ann_{R_{i_e}}(N_{i_g})^2)$, for $i = 1, 2$, so $N_g = (\{0\}_{1_g} :_{M_{1_g}} Ann_{R_{1_e}}(N_{1_g})^2) \times (\{0\}_{2_g} :_{M_{2g}} Ann_{R_{2_e}}(N_{2_g})^2) = (\{0\}_g :_{M_g} Ann_{R_e}(N_g)^2)$. Therefore, M_g is a fully g-coidempotent R_e-module. □

Lemma 2.17. *Let R be a G-graded ring, M a graded R-module and $N = \oplus_{h \in G} N_h$ a graded submodule of M. Let $g \in G$ such that M_g is a g-comultiplication R_e-module. Then N_g is a g-secondary submodule of M_g if and only if $Ann_{R_e}(N_g)$ is an e-primary ideal of R_e.*

Proof. The necessity is clear. Conversely, let $r_e \in R_e$, so $r_e N_g = (\{0\}_g :_{M_g} I_e)$ for some ideal I_e of R_e as M_g is a g-comultiplication R_e-module. Thus, $r_e I_e \subseteq Ann_{R_e}(N_g)$ which yields that $I_e \subseteq Ann_{R_e}(N_g)$ or $r_e^k \in Ann_{R_e}(N_g)$ for some $k \in \mathbf{Z}^+$. Hence, $N_g \subseteq (\{0\}_g :_{M_g} Ann_{R_e}(N_g)) \subseteq (\{0\}_g :_{M_g} I_e) = r_e N_g \subseteq N_g$ or $r_e^k N_g = \{0\}_g$ for some $k \in \mathbf{Z}^+$. Therefore, N_g is a g-secondary submodule of M_g. □

Theorem 2.18. *Let R_i be a G-graded gr-Noetherian ring, M_i a graded R_i-module, $N_i = \oplus_{g \in G} N_{i_g}$ a graded submodule of M_i for $i = 1, 2$ and $g \in G$. Let $R = R_1 \times R_2$ and $M = M_1 \times M_2$ such that M_g is a fully g-coidempotent R_e-module. Then:*

1. *N_{1_g} is a strongly g-$G2$-absorbing second submodule of M_{1_g} if and only if $N_{1_g} \times \{0\}_{2_g}$ is a strongly g-$G2$-absorbing second submodule of M_g.*

2. *N_{2_g} is a strongly g-$G2$-absorbing second submodule of M_{2_g} if and only if $\{0\}_{1_g} \times N_{2_g}$ is a strongly g-$G2$-absorbing second submodule of M_g.*

3. *If N_{1_g} is a g-secondary submodule of M_{1_g} and N_{2_g} is a g-secondary submodule of M_{2_g}, then $N_{1_g} \times N_{2_g}$ is a strongly g-$G2$-absorbing second submodule of M_g.*

Proof. (i) Since M_g is a fully g-coidempotent R_e-module, M_{i_g} is a fully g-coidempotent R_{i_e}-module, for $i = 1, 2$ by Lemma 2.16. Now, suppose that N_{1_g} is a strongly g-$G2$-absorbing second submodule of M_{1_g}, then by Theorem 2.10, we get $Ann_{R_{1_e}}(N_{1_g})$ is an e-2-absorbing primary ideal of R_{1_e}. Since $Ann_{R_e}(N_{1_g} \times \{0\}_{2_g}) = Ann_{R_{1_e}}(N_{1_g}) \times R_{2_e}$, $Ann_{R_e}(N_{1_g} \times \{0\}_{2_g})$ is an e-2-absorbing primary ideal of R_e by [6, Theorem 3.6]. Hence, by Theorem

2.12, we get $N_{1_g} \times \{0\}_{2_g}$ is a strongly g-$G2$-absorbing second submodule of M_g. Conversely, suppose that $N_{1_g} \times \{0\}_{2_g}$ is a strongly g-$G2$-absorbing second submodule of M_g, then $Ann_{R_e}(N_{1_g} \times \{0\}_{2_g}) = Ann_{R_{1_e}}(N_{1_g}) \times R_{2_e}$ is an e-2-absorbing primary ideal of R_e by Theorem 2.10. Hence, $Ann_{R_{1_e}}(N_{1_g})$ is an e-2-absorbing primary ideal of R_{1_e} by [6, Theorem 3.6]. Thus, by Theorem 2.12, we get N_{1_g} is a strongly g-$G2$-absorbing second submodule of M_{1_g}.

(ii) The proof is similar to that in part (i).

(iii) Let N_{i_g} be a g-secondary submodule of M_{i_g}, then by Lemma 2.17, we get $Ann_{R_{i_e}}(N_{i_g})$ is an e-primary ideal of R_{i_e}, for $i = 1, 2$. Now, since $Ann_{R_e}(N_{1_g} \times N_{2_g}) = Ann_{R_{1_e}}(N_{1_g}) \times Ann_{R_{2_e}}(N_{2_g})$, $Ann_{R_e}(N_{1_g} \times N_{2_g})$ is an e-2-absorbing primary ideal of R_e by [6, Theorem 3.6]. Therefore, $N_{1_g} \times N_{2_g}$ is a g-$G2$-absorbing submodule of M_g by Theorem 2.12. □

Theorem 2.19. *Let R_i be a G-graded gr-Noetherian ring, M_i a graded R_i-module, $N_i = \oplus_{g \in G} N_{i_g}$ a graded submodule of M_i for $i = 1, 2$ and $g \in G$. Let $R = R_1 \times R_2$, $M = M_1 \times M_2$ such that M_g is a fully g-coidempotent R_e-module and $N = N_1 \times N_2$. Then the following statements are equivalent:*

1. *N_g is a strongly g-$G2$-absorbing second submodule of M_g.*
2. *Either $N_{1_g} = \{0\}_{1_g}$ and N_{2_g} is a strongly g-$G2$-absorbing second submodule of M_{2_g} or $N_{2_g} = \{0\}_{2_g}$ and N_{1_g} is a strongly g-$G2$-absorbing second submodule of M_{1_g} or N_{1_g} and N_{2_g} are g-secondary submodules of M_{1_g} and M_{2_g}, respectively.*

Proof. $(i) \Rightarrow (ii)$ Assume that $N_g = N_{1_g} \times N_{2_g}$ is a strongly g-$G2$-absorbing second submodule of M_g. So, by Theorem 2.10, we get $Ann_{R_e}(N_g) = Ann_{R_{1_e}}(N_{1_g}) \times Ann_{R_{2_e}}(N_{2_g})$ is an e-2-absorbing primary ideal of R_e. Thus, by [6, Theorem 3.6] we get either $Ann_{R_{1_e}}(N_{1_g}) = R_{1_e}$ and $Ann_{R_{2_e}}(N_{2_g})$ is an e-2-absorbing primary ideal of R_{2_e} or $Ann_{R_{2_e}}(N_{2_g}) = R_{2_e}$ and $Ann_{R_{1_e}}(N_{1_g})$ is an e-2-absorbing primary ideal of R_{1_e} or $Ann_{R_{1_e}}(N_{1_g})$ and $Ann_{R_{2_e}}(N_{2_g})$ are e-primary ideals of R_{1_e} and R_{2_e}, respectively. Now, if $Ann_{R_{1_e}}(N_{1_g}) = R_{1_e}$ and $Ann_{R_2}(N_{2_g})$ is an e-2-absorbing primary ideal of R_{2_e}, then $N_{1_g} = \{0\}_{1_g}$ and N_{2_g} is a strongly g-$G2$-absorbing second submodule of M_{2_g} by, Theorem 2.12. Similarly, if $Ann_{R_{2_e}}(N_{2_g}) = R_{2_e}$ and $Ann_{R_{1_e}}(N_{1_g})$ is an e-2-absorbing primary ideal of R_{1_e}, then $N_{2_g} = \{0\}_{2_g}$ and N_{1_g} is a strongly g-$G2$-absorbing second submodule of M_{1_g}. If $Ann_{R_{i_e}}(N_{i_g})$ is an e-primary ideal of R_{i_e}, since M_{i_g} is a g-comultiplication R_{i_e}-module by Lemma 2.17, N_{i_g} is a g-secondary submodule of M_{i_g}, for $i = 1, 2$.

$(ii) \Rightarrow (i)$ Clearly, by Theorem 2.18. $\square$

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Shatha Alghueiri
Department of Mathematics and Statistics,
Jordan University of Science and Technology,
P. O. Box 3030,
Irbid 22110,
Jordan
e-mail: ghweiri64@gmail.com

and

Khaldoun Al-Zoubi
Department of Mathematics and Statistics,
Jordan University of Science and Technology,
P. O. Box 3030,
Irbid 22110,
Jordan
e-mail: kfzoubi@just.edu.jo
Corresponding author