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Equitable chromatic number of weak modular product of Some graphs

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Abstract

An equitable coloring of a graph G is a proper coloring of the vertices of G such that the number of vertices in any two color classes differ by at most one. The equitable chromatic number $\chi_=(G)$ of a graph G is the minimum number of colors needed for an equitable coloring of G. In this paper, we obtain the equitable chromatic number of weak modular product of two graphs G and H, denoted by $G \diamond H$. First, we consider the graph $G \diamond H$, where G is the path graph, and H be any simple graph like the path, the cycle graph, the complete graph. Secondly, we consider G and H as the complete graph and cycle graph respectively. Finally, we consider G as the star graph and H be the complete graph and star graph.

Keywords: *Equitable coloring, weak modular product, path graph, cycle graph, complete graph and star graph.*

1. Introduction

All graphs considered in this paper are finite, undirected graphs with neither loops nor multiple edges. Let $V(G)$ and $E(G)$ be the vertex set and edge set of a graph G. A *proper k-coloring* of a graph G is a labeling $f : V(G) \rightarrow \{1, 2, \ldots, k\}$ such that the adjacent vertices have different labels. The labels are called *colors*; the vertices of one color form a *color class*. The *chromatic number* of a graph G, written $\chi(G)$, is the least k such that G has a proper k-coloring. A graph G is said to be *equitably k-colorable* [1, 2, 3, 6, 7, 8, 10, 11, 12, 13, 14, 15, 16, 18, 19] if its vertices can be partitioned into k-classes $V_1, V_2, \ldots, V_k$ such that each V_i is an independent set and the condition $\Big||V_i| - |V_j|\Big| \leq 1$ holds for every i, j. The smallest integer k for which G is equitably k-colorable is known as the *equitable chromatic number* of G and is denoted by $\chi_=(G)$. For any graph G, we have $\chi_=(G) \geq \chi(G)$. Every time when we have to divide a system with binary conflicting relations into equal or almost equal conflict free subsystems we can model this situation by means of equitable graph coloring. The notion of equitable coloring was introduced by Meyer in 1973 [16]. The application given by Tucker [17] where vertices represented garbage collection routes and two such vertices were joined when the corresponding routes should not be run on the same day, motivated Meyer to introduce equitable coloring. Application of equitable coloring is found scheduling and timetabling problem.

In 1964, Erdös [4] conjectured that any graph G with maximum degree $\Delta(G) \leq k$ has an equitable $(k + 1)$-coloring. This conjecture was later proved by Hajnal and Szemerédi in 1970 [5]. In 1973, Meyer [16] gave the conjecture which states that "For any connected graph G, other than the complete graph and the odd cycle, $\chi_=(G) \leq \Delta(G)$". Later in 1994, Chen et al. [2] proposed the following conjecture which states that, "If G is a connected graph with maximum degree Δ other than $K_{\Delta+1}$, $K_{\Delta,\Delta}$ and odd cycle, then G is equitably Δ-colorable". The Equitable Δ-coloring conjecture has been proved for outerplanar graphs [10, 19], planar graphs with $\Delta \geq 13$ [18] and for some other families of graphs [12, 13, 14]. In 1998, Chen et al. [3] has written a manuscript, concerning the equitable coloring of graph products. The results were then extended by Furmańczyk [6] in 2006. W.H.Lin and G.J.Chang [15] in 2012 establish the exact value of equitable chromatic numbers of Cartesian products of an odd cycle or an odd path with a bipartite graph, an even cycle or an even path with a complete bipartite graph and two stars. They also gave the upper bounds on the equitable chromatic number of Cartesian product of two complete bipartite graphs. In 2013, Furmańczyk et al. [8] give the exact value of the equitable chromatic number of corona product of graphs G and H where G is an equitable 3- or 4-colorable graph and H is an r-partite graph, a path, a cycle, or a complete graph. These results were then extended for corona multiproducts of graphs [7].

2. Preliminaries

A trail is called a path if all its vertices are distinct. A closed trail whose origin and internal vertices are distinct is called a cycle. A star graph is a complete bipartite graph in which $n-1$ vertices have degree 1 and a single vertex have degree $(n-1)$. It is denoted by $K_{1,n}$. We consider the vertex set of $K_{1,n}$ be the order of n. $V(K_{1,n}) = \{u_0\} \cup \{u_i : 1 \leq i \leq n\}$. A graph G is complete if every pair of distinct vertices of G are adjacent in G. A complete graph on n vertices is denoted by K_n.

Graph products were first defined by Sabidussi 1960 and Vizing 1963. The *weak modular product* [9] of two graphs G and H denoted by $G \diamond H$ is the graph with vertex set $V(G) \times V(H)$ in which two vertices (a, b) and $(c, d) \in V(G \diamond H)$ are adjacent if;

$$ac \in E(G) \text{ and } bd \in E(H), \text{ or}$$
$$ac \notin E(G) \text{ and } bd \notin E(H).$$

In this paper, we obtain the equitable chromatic number of weak modular product of two graphs G and H. First, we consider the graph G be the path graph or complete graph and H be the path graph or the complete graph. Second, we consider the graph G be the path graph and H be the cycle graph. Third, we consider the graph G and H be the cycle graphs. Fourth, we consider the graph G be the complete graph and H be the cycle graph respectively. Finally, we consider the graph G be the star graph and H be the complete graph and star graph.

3. Main Results

First we consider G and H be the graphs of order m and n. Let graph G be isomorphic to the path graph, the complete graph and the cycle graph of m and H be isomorphic to the path graph, complete graph and cycle graph of n. Let $V(G) = \{u_i : 1 \leq i \leq m\}$ and $V(H) = \{v_j : 1 \leq j \leq n\}$.

By the definition of the weak modular product, we denote the vertices of $G \diamond H$ as follows:

$$V(G \diamond H) = \bigcup_{i=1}^{m} \{s_{i,j} : 1 \leq j \leq n\},$$

where $s_{i,j}$ are the vertices (u_i, v_j) $(1 \leq i \leq m,\ 1 \leq j \leq n)$.

Theorem 1. *Let G and H be the path graphs or complete graphs of order $m > 1$ and $n > 1$, then the equitable chromatic number of weak modular product of G and H is m for $m \leq n$.*

Proof: Define the mapping $f : V(G \diamond H) \rightarrow N$, as follows:

$$f(s_{i,j}) = i,\ 1 \leq i \leq m, 1 \leq j \leq n.$$

Since, each color $(1, 2, \ldots, m)$ appears exactly m times. Hence, the difference does not exceed one.

Now we assume that $\chi_{=}(G \diamond H) \leq m$. Since there exists cliques of order m in $V(G \diamond H)$, we have $\chi(G \diamond H) \geq m$ and also since $\chi_{=}(G \diamond H) \geq \chi(G \diamond H) \geq m$, hence $\chi_{=}(G \diamond H) \geq m$. Therefore, $\chi_{=}(G \diamond H) = m$. □

Theorem 2. *Let $G \cong P_m$ be of order $m > 1$ and $H \cong C_n$ be of order $n > 2,\ n \geq m$, then*

$$\chi_{=}(G \diamond H) = \begin{cases} 2, & \text{when } m \text{ is even},\ n = 3 \\ m, & \text{otherwise.} \end{cases}$$

Proof:

Case (i): When m is even and $n = 3$

Define the mapping $f : V(G \diamond H) \rightarrow \{1, 2\}$ as follows:

For $1 \leq j \leq 3$,

$$\begin{aligned} f(s_{2i-1,j}) &= 1,\ 1 \leq i \leq \left\lceil \frac{m}{2} \right\rceil; \\ f(s_{2i,j}) &= 2,\ 1 \leq i \leq \left\lfloor \frac{m}{2} \right\rfloor \end{aligned}$$

Obviously, $\chi_{=}(G \diamond H) = 2$.

Case (ii): When m is not even and $n = 3$ or m is not even and $n \neq 3$ or m is even and $n \neq 3$.
Define the mapping $f : V(G \diamond H) \rightarrow N$, as follows:

$$f(s_{i,j}) = i,\ 1 \leq i \leq m, 1 \leq j \leq n.$$

Since, each color $(1, 2, \ldots, m)$ appears exactly m times. Hence, the difference does not exceed one.

Now we assume that $\chi_{=}(G \diamond H) \leq m$. Since there exists cliques of order m in $V(G \diamond H)$, we have $\chi(G \diamond H) \geq m$ and also since $\chi_{=}(G \diamond H) \geq \chi(G \diamond H) \geq m$, hence $\chi_{=}(G \diamond H) \geq m$. Therefore, $\chi_{=}(G \diamond H) = m$.

□

Theorem 3. *Let $G \cong C_m$ be of order $m > 2$ and $H \cong C_n$ be of order $n > 2$, $m \geq n$, then*

$$\chi_{=}(G \diamond H) = \begin{cases} 2, & \textit{when } m \textit{ is even},\ n = 3 \\ n, & \textit{otherwise}. \end{cases}$$

Proof:

Case (i): When $m = 3$ and n is even
Define the mapping $f : V(G \diamond H) \rightarrow \{1, 2\}$ as follows:
For $1 \leq i \leq 3$,

$$\begin{aligned} f(s_{i,2j-1}) &= 1,\ 1 \leq j \leq \frac{n}{2}; \\ f(s_{i,2j}) &= 2,\ 1 \leq j \leq \frac{n}{2}. \end{aligned}$$

Obviously, $\chi_{=}(G \diamond H) = 2$.

Case (ii): When $m \geq n$

Define the mapping $f : V(G \diamond H) \rightarrow \{1, 2, \ldots, n\},\ n \geq 3$, as follows:

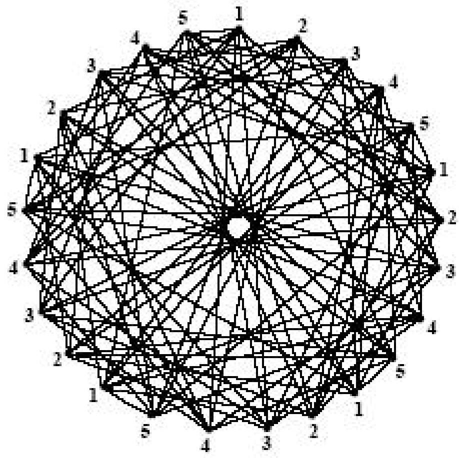

Figure 1: Equitable Coloring of $C_5 \diamond C_5$ is 5.

$f(s_{i,j}) = j,\ 1 \leq i \leq m, 1 \leq j \leq n.$

Thus $\chi_=(G \diamond H) \leq n$. Since $\chi(G \diamond H) \geq n$ and also since $\chi_=(G \diamond H) \geq \chi(G \diamond H) \geq n$, we have $\chi_=(G \diamond H) \geq n$. Therefore, $\chi_=(G \diamond H) = n$. Hence $\chi_=(G \diamond H) = n$.

□

Theorem 4. *Let G be the path graph of order $m > 1$ and H be the complete graph of order $n > 1$, then*

$$\chi_=(G \diamond H) = \begin{cases} 2, & \textit{when } m \textit{ is even, } n > 2 \textit{ or } m > 1,\ n = 2 \\ 3, & \textit{otherwise.} \end{cases}$$

Proof: The graph $G \diamond H$ is a bipartite graph $(X \cup Y, E)$ such that every second row belongs to Y.

(i.e.,) $|X| = \left\lceil \frac{m}{2} \right\rceil n$ and $|Y| = \left\lfloor \frac{m}{2} \right\rfloor n$.

Case (i): Define the mapping $f : V(G \diamond H) \rightarrow \{1, 2\}$ as follows:

Subcase (i): When m is even and $n > 2$

For $1 \leq i \leq m$, $1 \leq j \leq n$,

$$f(s_{i,j}) = \begin{cases} 1, & i \equiv 12, \\ 2, & i \equiv 02. \end{cases}$$

Obviously, $\chi_{=}(G \diamond H) = 2$.

Subcase (ii): When $m > 1$ and $n = 2$

For $1 \leq i \leq m$,

$$f(s_{i,j}) = \begin{cases} 1, & \text{for } j = 1, \\ 2, & \text{for } j = 2. \end{cases}$$

Obviously, $\chi_{=}(G \diamond H) = 2$.

Since, each color 1 and 2 appears $\left\lceil \frac{m}{2} \right\rceil n$ and $\left\lfloor \frac{m}{2} \right\rfloor n$ times. Hence, the difference does not exceed one.

Case (ii): When m is odd and $n \geq 3$.

Define the mapping $f : V(G \diamond H) \to \{1, 2, 3\}$ as follows:

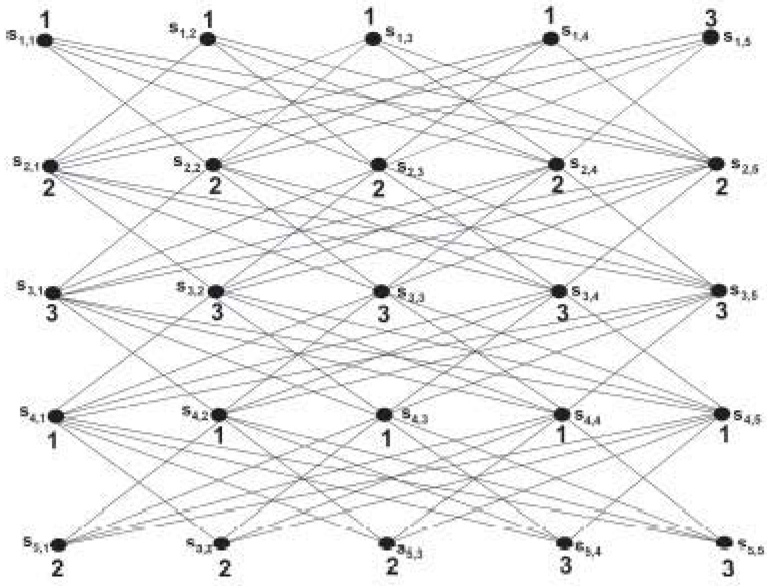

Figure 2: Equitable Coloring of $P_5 \diamond K_5$ is 3.

Subcase (i): When $m = 3k$, $k \geq 1$, k is odd

For $1 \leq i \leq m, 1 \leq j \leq n$,

$$f(s_{i,j}) = \begin{cases} 1, & \text{for } i \equiv 13, \\ 2, & \text{for } i \equiv 23, \\ 3, & \text{for } i \equiv 03, \end{cases}$$

Subcase (ii): We first color the i-th row with i mod 3, then change the colors of some vertices in the first and last row in the following ways.

1. If $m = 6k + 1,\ k \geq 1$, vertices in the first and last row are colored with 1. We change the color of $\left\lceil \frac{(n-2)}{3} \right\rceil$ vertices in the first row to 3 and $\left\lceil \frac{(n-1)}{3} \right\rceil$ vertices in the last row to 2.
2. If $m = 6k - 1,\ k \geq 1$, then we change the colors of $n - \left\lceil \frac{2n}{3} \right\rceil$ vertices from the first row and $n - \left\lceil \frac{(2n-1)}{3} \right\rceil$ vertices from the last row to 3.

So, the obtained colorings are equitable. Moreover, since m is odd, $|X| - |Y| = n,\ n \geq 3$ and therefore we cannot use less than three colors.

$\square$

Theorem 5. *Let G be isomorphic to the star graph of order $m+1$ and $H \cong K_n$ or $K_{1,n}$ be of order of $n > 1$, then the equitable chromatic number of weak modular product of G and H is $m+1$ for $n \geq m+1$.*

Proof: Let $V(G) = \{u_1\} \cup \{u_i : 2 \leq i \leq m+1\}$.

Case (i): Let $H \cong K_n$. Let $V(H) = \{v_j : 1 \leq j \leq n\}$. By the definition of the weak modular product, the vertices of $G \diamond H$ are denoted as follows: $V(G \diamond H) = \bigcup_{i=1}^{m+1} \{s_{i,j} : 1 \leq j \leq n\}$, where $s_{i,j}$ are the vertices (u_i, v_j) $(1 \leq i \leq m+1, 1 \leq j \leq n)$.

Define the mapping $f : V(G \diamond H) \rightarrow \{1, 2, \ldots, m+1\},\ m \geq 1$ as follows:

$$f(s_{i,j}) = i,\ 1 \leq i \leq m+1, 1 \leq j \leq n.$$

Since, each color $(1, 2, \ldots, m+1)$ appears exactly $m+1$ times. Hence, the difference does not exceed one.

Now we assume that $\chi_{=}(G \diamond H) \leq m+1$. First we assign the color $1, 2, \ldots m$ to the vertices $s_{i,j}$ in such a way that, they have received the same colors in any one pair of vertices . Which is a contradiction. So we need one more color. Hence $\chi_{=}(G \diamond H) \geq m+1$. Therefore, $\chi_{=}(G \diamond H) = m+1$.

Case (ii): Let $H \cong K_{1,n}$. Let $V(H) = \{v_1\} \cup \{v_j : 2 \leq j \leq n+1\}$. By the definition of the weak modular product, the vertices of $G \diamond H$ are denoted as follows: $V(G \diamond H) = \bigcup\limits_{i=1}^{m+1} \{s_{i,j} : 1 \leq j \leq n+1\}$, where $s_{i,j}$ are the vertices (u_i, v_j) $(1 \leq i \leq m+1, 1 \leq j \leq n+1)$.

Define the mapping $f : V(G \diamond H) \to \{1, 2, \ldots, m+1\}$ as follows:

$$f(s_{i,j}) = i, \ 1 \leq i \leq m+1, 1 \leq j \leq n+1.$$

Since, each color $(1, 2, \ldots, m+1)$ appears exactly $m+1$ times. Hence, the difference does not exceed one.

Now we assume that $\chi_{=}(G \diamond H) \leq m+1$. Since there exists cliques of order $m+1$ in $V(G \diamond H)$, we have $\chi(G \diamond H) \geq m+1$ and also since $\chi_{=}(G \diamond H) \geq \chi(G \diamond H) \geq m+1$. Hence $\chi_{=}(G \diamond H) \geq m+1$. Therefore, $\chi_{=}(G \diamond H) = m+1$.

□

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