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Mappings preserving sum of products $a \diamond b + b^* a$ (resp., $a^* \diamond b + ab^*$) on $*$-algebras

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Abstract

Let $\mathcal{A}$ and $\mathcal{B}$ be two prime complex $*$-algebras. We proved that every bijective mapping $\Phi : \mathcal{A} \to \mathcal{B}$ satisfying $\Phi(a \diamond b + b^* a) = \Phi(a) \diamond \Phi(b) + \Phi(b)^* \Phi(a)$ (resp., $\Phi(a^* \diamond b + ab^*) = \Phi(a)^* \diamond \Phi(b) + \Phi(a)\Phi(b)^*$), where $a \diamond b = ab + ba^*$, for all elements $a, b \in \mathcal{A}$, is a $*$-ring isomorphism.

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1. Introduction

Throughout this paper all algebras are assumed to be complex. Let $\mathcal{A}$ and $\mathcal{B}$ be algebras. For $a, b \in \mathcal{A}$ (resp., $a, b \in \mathcal{B}$), denote by $a \circ b = ab + ba$ the *Jordan product* of $\mathcal{A}$ (resp., *Jordan product* of $\mathcal{B}$) and denote by $[a, b] = ab - ba$ the *Lie product* of $\mathcal{A}$ (resp., *Lie product* of $\mathcal{B}$). We say that a mapping $\Phi : \mathcal{A} \to \mathcal{B}$ *preserves product* if $\Phi(ab) = \Phi(a)\Phi(b)$, for all elements $a, b \in \mathcal{A}$, *preserves Jordan product* if $\Phi(a \circ b) = \Phi(a) \circ \Phi(b)$, for all elements $a, b \in \mathcal{A}$ and *preserves Lie product* if $\Phi([a, b]) = [\Phi(a), \Phi(b)]$, for all elements $a, b \in \mathcal{A}$. We say that a mapping $\Phi : \mathcal{A} \to \mathcal{B}$ is *additive* if $\Phi(a + b) = \Phi(a) + \Phi(b)$, for all elements $a, b \in \mathcal{A}$ and that is a *ring isomorphism* if Φ is an additive bijection that preserves product.

An algebra $\mathcal{A}$ is called *prime* if for $a, b \in \mathcal{A}$, $a\mathcal{A}b = \{0\}$ (or simply $a\mathcal{A}b = 0$) implies that either $a = 0$ or $b = 0$.

Let $\mathcal{A}$ be an algebra. An *involution in* $\mathcal{A}$ is a mapping $a \to a^*$ of $\mathcal{A}$ onto itself satisfying the following involution axioms:

(i) $(a^*)^* = a$, for all element $a \in \mathcal{A}$,

(ii) $(a + b)^* = a^* + b^*$, for all elements $a, b \in \mathcal{A}$,

(iii) $(\lambda a)^* = \overline{\lambda} a^*$, for all elements $\lambda \in \mathbf{C}$ and $a \in \mathcal{A}$,

(iv) $(ab)^* = b^* a^*$, for all elements $a, b \in \mathcal{A}$.

An algebra with an involution is called a $*$-*algebra* (or *involution algebra*). A *projection* is any idempotent element $p \in \mathcal{A}$ satisfying the condition $p^* = p$. A projection which is neither the zero nor the identity element is said to be *nontrivial*.

We say that a $*$-algebra $\mathcal{A}$ is *prime* if the associated algebra is prime.

Let $\mathcal{A}$ and $\mathcal{B}$ be $*$-algebras. We say that a mapping $\Phi : \mathcal{A} \to \mathcal{B}$ *preserves involution* if $\Phi(a^*) = \Phi(a)^*$, for all element $a \in \mathcal{A}$, and that Φ is a $*$-*ring isomorphism* if Φ is a ring isomorphism that preserves involution. For $a, b \in \mathcal{A}$ (resp., $a, b \in \mathcal{B}$), consider the following new products on $\mathcal{A}$ (resp., $\mathcal{B}$): $[a, b]_* = ab - ba^*$, $a \diamond b = ab + ba^*$, $a \bullet b = a^*b + b^*a$ and $a^* \circ b = a^*b + ba^*$. These products play an important role in some research topics and their studies have recently attracted the attention of some authors (for example, see [1], [4], [5] and [6] and for other products see [2], [3], [7] and [8]). In particular, the authors in [1], [4] and [6] studied bijective mappings preserving the new products mentioned above. They showed that such mappings on factor von Neumann algebras are $*$-ring isomorphisms.

Let $\mathcal{A}$ and $\mathcal{B}$ be two $*$-algebras. We say that a mapping $\Phi : \mathcal{A} \to \mathcal{B}$ *preserves sum of products* $a \diamond b + b^*a$ (resp., $a^* \diamond b + ab^*$) if $\Phi(a \diamond b + b^*a) = \Phi(a) \diamond \Phi(b) + \Phi(b)^*\Phi(a)(resp., \Phi(a^* \diamond b + ab^*) = \Phi(a)^* \diamond \Phi(b) + \Phi(a)\Phi(b)^*)$, for all elements $a, b \in \mathcal{A}$.

The following lemma follows directly from the above definition and hence its proof is omitted.

Lemma 1.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be two $*$-algebras and $\Phi : \mathcal{A} \to \mathcal{B}$ a mapping. Then, the following statements are equivalents:*

(i) *Φ preserves sum of products $a \diamond b + b^*a$,*

(ii) *Φ preserves sum of products $a^* \diamond b + ab^*$.*

Inspired by the research described in [1], [4], [6], the aim of this paper is to prove that a bijective mapping preserving sum of products $a \diamond b + b^*a$ (resp., $a^* \diamond b + ab^*$) on prime $*$-algebras is a $*$-ring isomorphism.

Our main result reads as follows.
Main Theorem. *Let $\mathcal{A}$ and $\mathcal{B}$ be two prime $*$-algebras with $1_{\mathcal{A}}$ and $1_{\mathcal{B}}$ the identities of them, respectively, and such that $\mathcal{A}$ has a nontrivial projection. Then every bijective mapping $\Phi : \mathcal{A} \to \mathcal{B}$ satisfying $\Phi(a \diamond b + b^*a) = \Phi(a) \diamond \Phi(b) + \Phi(b)^*\Phi(a)$ (resp., $\Phi(a^* \diamond b + ab^*) = \Phi(a)^* \diamond \Phi(b) + \Phi(a)\Phi(b)^*$), for all elements $a, b \in \mathcal{A}$, is a $*$-ring isomorphism.*

2. The proof of main theorem

Due to Lemma 1.1, the proof of the Main Theorem is made by proving several lemmas, considering only the mapping that preserves sums of products $a^* \diamond b + ab^*$. We begin with the following lemma.

Lemma 2.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be two $*$-algebras and $\Phi : \mathcal{A} \to \mathcal{B}$ a mapping that preserves sum of products $a^* \diamond b + ab^*$. If $\Phi(c) = \Phi(a) + \Phi(b)$, for some elements $a, b, c \in \mathcal{A}$, then hold the following identities:*

(i) *$\Phi(t^* \diamond c + tc^*) = \Phi(t^* \diamond a + ta^*) + \Phi(t^* \diamond b + tb^*)$, for all element $t \in \mathcal{A}$,*

(ii) *$\Phi(c^* \diamond t + ct^*) = \Phi(a^* \diamond t + at^*) + \Phi(b^* \diamond t + bt^*)$, for all element $t \in \mathcal{A}$.*

Proof. For an arbitrary element $t \in \mathcal{A}$, we have

$$\begin{aligned}\Phi(t^* \diamond c + tc^*) &= \Phi(t)^* \diamond \Phi(c) + \Phi(t)\Phi(c)^* \\ &= \Phi(t)^* \diamond (\Phi(a) + \Phi(b)) + \Phi(t)(\Phi(a) + \Phi(b))^* \\ &= \Phi(t)^* \diamond \Phi(a) + \Phi(t)\Phi(a)^* + \Phi(t)^* \diamond \Phi(b) + \Phi(t)\Phi(b)^* \\ &= \Phi(t^* \diamond a + ta^*) + \Phi(t^* \diamond b + tb^*).\end{aligned}$$

Similarly, we obtain (ii). □

Lemma 2.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be two $*$-algebras and $\Phi : \mathcal{A} \to \mathcal{B}$ a surjective mapping that preserves sum of products $a^* \diamond b + ab^*$. Then $\Phi(0) = 0$.*

Proof. From the surjectivity of Φ there exists an element $b \in \mathcal{A}$ such that $\Phi(b) = 0$. It follows that

$$\begin{aligned}&\Phi(0) = \Phi(0^* \diamond b + 0b^*) = \Phi(0)^* \diamond \Phi(b) + \Phi(0)\Phi(b)^* \\ &= \Phi(0)^* \diamond 0 + \Phi(0)0^* = 0.\end{aligned}$$

□

Lemma 2.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be two prime $*$-algebras such that $\mathcal{A}$ has the identity $1_{\mathcal{A}}$ and a nontrivial projection. Then every bijective mapping $\Phi : \mathcal{A} \to \mathcal{B}$ that preserves sum of products $a^* \diamond b + ab^*$ is additive.*

We will establish the proof of Lemma 2.3 in a series of Properties, based on the techniques used in [1], [2], [3], [4] and [6]. We begin, though, with a well-known result that will be used throughout this paper.

Let p_1 be an arbitrary nontrivial projection of $\mathcal{A}$ and write $p_2 = 1_{\mathcal{A}} - p_1$. Then $\mathcal{A}$ has a Peirce decomposition $\mathcal{A} = \mathcal{A}_{11} \oplus \mathcal{A}_{12} \oplus \mathcal{A}_{21} \oplus \mathcal{A}_{22}$, where $\mathcal{A}_{ij} = p_i \mathcal{A} p_j$ $(i, j = 1, 2)$, satisfying the following multiplicative relations: $\mathcal{A}_{ij}\mathcal{A}_{kl} \subseteq \delta_{jk}\mathcal{A}_{il}$, where δ_{jk} is the *Kronecker delta function.*

Property 2.1. *For arbitrary elements $a_{11} \in \mathcal{A}_{11}$, $b_{12} \in \mathcal{A}_{12}$, $c_{21} \in \mathcal{A}_{21}$ and $d_{22} \in \mathcal{A}_{22}$ hold:*

(i) $\Phi(a_{11} + b_{12}) = \Phi(a_{11}) + \Phi(b_{12})$,

(ii) $\Phi(a_{11} + c_{21}) = \Phi(a_{11}) + \Phi(c_{21})$,

(iii) $\Phi(b_{12} + d_{22}) = \Phi(b_{12}) + \Phi(d_{22})$,

(iv) $\Phi(c_{21} + d_{22}) = \Phi(c_{21}) + \Phi(d_{22})$.

Proof. From the surjectivity of Φ there exists $f = f_{11}+f_{12}+f_{21}+f_{22} \in \mathcal{A}$ such that $\Phi(f) = \Phi(a_{11}) + \Phi(b_{12})$. By Lemma 2.1(i), we have

$$\Phi(p_2^* \diamond f + p_2 f^*) = \Phi(p_2^* \diamond a_{11} + p_2 a_{11}^*) + \Phi(p_2^* \diamond b_{12} + p_2 b_{12}^*)$$
$$= \Phi(b_{12} + b_{12}^*).$$

This implies that $p_2^* \diamond f + p_2 f^* = b_{12} + b_{12}^*$ resulting that $f_{12} = b_{12}$, $f_{21} = 0$ and $f_{22} = 0$. Thus, $\Phi(f_{11} + b_{12}) = \Phi(a_{11}) + \Phi(b_{12})$. It follows that, for an arbitrary element $d_{21} \in \mathcal{A}_{21}$, we have

$$\Phi(d_{21}^* \diamond (f_{11} + b_{12}) + d_{21}(f_{11} + b_{12})^*)$$
$$= \Phi(d_{21}^* \diamond a_{11} + d_{21} a_{11}^*) + \Phi(d_{21}^* \diamond b_{12} + d_{21} b_{12}^*)$$

which implies that $\Phi(d_{21} f_{11}^* + b_{12} d_{21}) = \Phi(d_{21} a_{11}^*) + \Phi(b_{12} d_{21})$. Hence,

$$\Phi(p_2^* \diamond (d_{21} f_{11}^* + b_{12} d_{21}) + p_2 (d_{21} f_{11}^* + b_{12} d_{21})^*)$$
$$= \Phi(p_2^* \diamond (d_{21} a_{11}^*) + p_2 (d_{21} a_{11}^*)^*) + \Phi(p_2^* \diamond (b_{12} d_{21}) + p_2 (b_{12} d_{21})^*)$$

which yields $\Phi(d_{21} f_{11}^*) = \Phi(d_{21} a_{11}^*)$. This shows that $d_{21} f_{11}^* = d_{21} a_{11}^*$. Therefore $f_{11} = a_{11}$.

Similarly, we prove the cases (ii), (iii) and (iv). □

Property 2.2. *For arbitrary elements $a_{11} \in \mathcal{A}_{11}$, $b_{12} \in \mathcal{A}_{12}$, $c_{21} \in \mathcal{A}_{21}$ and $d_{22} \in \mathcal{A}_{22}$ hold:*

(i) $\Phi(a_{11} + b_{12} + d_{22}) = \Phi(a_{11}) + \Phi(b_{12}) + \Phi(d_{22})$,

(ii) $\Phi(a_{11} + c_{21} + d_{22}) = \Phi(a_{11}) + \Phi(c_{21}) + \Phi(d_{22})$.

Proof. From the surjectivity of Φ there exists $f = f_{11}+f_{12}+f_{21}+f_{22} \in \mathcal{A}$ such that $\Phi(f) = \Phi(a_{11}) + \Phi(b_{12}) + \Phi(d_{22})$. By Lemma 2.1(i) and Property 2.1(i), we have

$$\Phi(p_1^* \diamond f + p_1 f^*) = \Phi(p_1^* \diamond a_{11} + p_1 a_{11}^*) + \Phi(p_1^* \diamond b_{12} + p_1 b_{12}^*)$$
$$+\Phi(p_1^* \diamond d_{22} + p_1 d_{22}^*) = \Phi(2a_{11} + a_{11}^*) + \Phi(b_{12}) = \Phi(2a_{11} + a_{11}^* + b_{12})$$

which implies that $p_1^* \diamond f + p_1 f^* = 2a_{11} + a_{11}^* + b_{12}$. As consequence we obtain $f_{11} = a_{11}$, $f_{12} = b_{12}$ and $f_{21} = 0$. Now, by Lemma 2.1(ii) we have

$$\Phi(f^* \diamond p_2 + f p_2^*) = \Phi(a_{11}^* \diamond p_2 + a_{11} p_2^*) + \Phi(b_{12}^* \diamond p_2 + b_{12} p_2^*)$$
$$+\Phi(d_{22}^* \diamond p_2 + d_{22} p_2^*) = \Phi(b_{12}) + \Phi(2d_{22} + d_{22}^*) = \Phi(b_{12} + 2d_{22} + d_{22}^*)$$

that allows us to obtain $f_{22} = d_{22}$.

Similarly, we prove the case (ii). □

Property 2.3. *For arbitrary elements $a_{12}, b_{12} \in \mathcal{A}_{12}$ and $b_{21}, c_{21} \in \mathcal{A}_{21}$ hold:*

(i) $\Phi(a_{12}+b_{12})=\Phi(a_{12})+\Phi(b_{12})$,

(ii) $\Phi(b_{21}+c_{21})=\Phi(b_{21})+\Phi(c_{21})$.

Proof. First, we note that the following identity is valid

$$(p_1+a_{12})^*\diamond(p_2+b_{12})+(p_1+a_{12})(p_2+b_{12})^*$$
$$=a_{12}+b_{12}+a_{12}b_{12}^*+a_{12}^*b_{12}.$$

Hence, by Property 2.2(i) we have

$$\Phi(a_{12}+b_{12})+\Phi(a_{12}b_{12}^*)+\Phi(a_{12}^*b_{12})$$
$$=\Phi(a_{12}+b_{12}+a_{12}b_{12}^*+a_{12}^*b_{12})$$
$$=\Phi((p_1+a_{12})^*\diamond(p_2+b_{12})+(p_1+a_{12})(p_2+b_{12})^*)$$
$$=\Phi(p_1+a_{12})^*\diamond\Phi(p_2+b_{12})+\Phi(p_1+a_{12})\Phi(p_2+b_{12})^*$$
$$=(\Phi(p_1)^*+\Phi(a_{12})^*)\diamond(\Phi(p_2)+\Phi(b_{12}))$$
$$+(\Phi(p_1)+\Phi(a_{12}))(\Phi(p_2)^*+\Phi(b_{12})^*)$$
$$=\Phi(p_1)^*\diamond\Phi(p_2)+\Phi(p_1)\Phi(p_2)^*$$
$$+\Phi(p_1)^*\diamond\Phi(b_{12})+\Phi(p_1)\Phi(b_{12})^*$$
$$+\Phi(a_{12})^*\diamond\Phi(p_2)+\Phi(a_{12})\Phi(p_2)^*$$
$$+\Phi(a_{12})^*\diamond\Phi(b_{12})+\Phi(a_{12})\Phi(b_{12})^*$$
$$=\Phi(p_1^*\diamond p_2+p_1p_2^*)+\Phi(p_1^*\diamond b_{12}+p_1b_{12}^*)$$
$$+\Phi(a_{12}^*\diamond p_2+a_{12}p_2^*)+\Phi(a_{12}^*\diamond b_{12}+a_{12}b_{12}^*)$$
$$=\Phi(b_{12})+\Phi(a_{12})+\Phi(a_{12}^*b_{12}+a_{12}b_{12}^*).$$

This permits us to conclude that $\Phi(a_{12}+b_{12})=\Phi(a_{12})+\Phi(b_{12})$.
Similarly, we prove the case (ii) using the Property 2(ii) and the identity

$$(p_2+b_{21})^*\diamond(p_1+c_{21})+(p_2+b_{21})(p_1+c_{21})^*$$
$$=b_{21}+c_{21}+b_{21}^*c_{21}+b_{21}c_{21}^*.$$

□

Property 2.4. *For arbitrary elements* $b_{12}\in\mathcal{A}_{12}$ *and* $c_{21}\in\mathcal{A}_{21}$ *holds* $\Phi(b_{12}+c_{21})=\Phi(b_{12})+\Phi(c_{21})$.

Proof. From the surjectivity of Φ choose $f=f_{11}+f_{12}+f_{21}+f_{22}\in\mathcal{A}$ such that $\Phi(f)=\Phi(b_{12})+\Phi(c_{21})$. Hence, for an arbitrary element $d_{12}\in\mathcal{A}_{12}$, we have

$$\Phi(d_{12}^*\diamond f+d_{12}f^*)=\Phi(d_{12}^*\diamond b_{12}+d_{12}b_{12}^*)+\Phi(d_{12}^*\diamond c_{21}+d_{12}c_{21}^*)$$
$$=\Phi(d_{12}^*b_{12}+d_{12}b_{12}^*)+\Phi(c_{21}d_{12})$$

which implies that

$$\Phi(p_1^*\diamond(d_{12}^*\diamond f+d_{12}f^*)+p_1(d_{12}^*\diamond f+d_{12}f^*)^*)$$
$$=\Phi(p_1^*\diamond(d_{12}^*b_{12}+d_{12}b_{12}^*)+p_1(d_{12}^*b_{12}+d_{12}b_{12}^*)^*)$$
$$+\Phi(p_1^*\diamond(c_{21}d_{12})+p_1(c_{21}d_{12})^*)=\Phi(2d_{12}b_{12}^*+b_{12}d_{12}^*).$$

This results that $p_1^* \diamond (d_{12}^* \diamond f + d_{12} f^*) + p_1 (d_{12}^* \diamond f + d_{12} f^*)^* = 2 d_{12} b_{12}^* + b_{12} d_{12}^*$ which yields that $f_{11} = 0$, $f_{12} = b_{12}$ and $f_{22} = 0$. Thus, $\Phi(b_{12} + f_{21}) = \Phi(b_{12}) + \Phi(c_{21})$. Next, for an arbitrary element $d_{21} \in \mathcal{A}_{21}$, we have

$$\Phi(d_{21}^* \diamond (b_{12} + f_{21}) + d_{21}(b_{12} + f_{21})^*) = \Phi(d_{21}^* \diamond b_{12} + d_{21} b_{12}^*)$$
$$+\Phi(d_{21}^* \diamond c_{21} + d_{21} c_{21}^*) = \Phi(b_{12} d_{21}) + \Phi(d_{21}^* c_{21} + d_{21} c_{21}^*).$$

This implies that

$$\Phi(p_2^* \diamond (d_{21}^* \diamond (b_{12} + f_{21}) + d_{21}(b_{12} + f_{21})^*)$$
$$+p_2(d_{21}^* \diamond (b_{12} + f_{21}) + d_{21}(b_{12} + f_{21})^*)^*)$$
$$= \Phi(p_2^* \diamond (b_{12} d_{21}) + p_2 (b_{12} d_{21})^*)$$
$$+\Phi(p_2^* \diamond (d_{21}^* c_{21} + d_{21} c_{21}^*) + p_2 (d_{21}^* c_{21} + d_{21} c_{21}^*)^*)$$
$$= \Phi(2 d_{21} c_{21}^* + c_{21} d_{21}^*).$$

As a consequence, we have $p_2^* \diamond (d_{21}^* \diamond (b_{12} + f_{21}) + d_{21}(b_{12} + f_{21})^*) + p_2(d_{21}^* \diamond (b_{12} + f_{21}) + d_{21}(b_{12} + f_{21})^*)^* = 2 d_{21} c_{21}^* + c_{21} d_{21}^*$ which allows us to conclude that $f_{21} = c_{21}$. □

Property 2.5. *For arbitrary elements $a_{11} \in \mathcal{A}_{11}$, $b_{12} \in \mathcal{A}_{12}$, $c_{21} \in \mathcal{A}_{21}$ and $d_{22} \in \mathcal{A}_{22}$ hold:*

(i) $\Phi(a_{11} + b_{12} + c_{21}) = \Phi(a_{11}) + \Phi(b_{12}) + \Phi(c_{21})$,

(ii) $\Phi(b_{12} + c_{21} + d_{22}) = \Phi(b_{12}) + \Phi(c_{21}) + \Phi(d_{22})$.

Proof. From the surjectivity of Φ choose $f = f_{11} + f_{12} + f_{21} + f_{22} \in \mathcal{A}$ such that $\Phi(f) = \Phi(a_{11}) + \Phi(b_{12}) + \Phi(c_{21})$. Hence, by Properties 2.3 and 2.4
we have

$$\Phi(p_2^* \diamond f + p_2 f^*) = \Phi(p_2^* \diamond a_{11} + p_2 a_{11}^*) + \Phi(p_2^* \diamond b_{12} + p_2 b_{12}^*)$$
$$+\Phi(p_2^* \diamond c_{21} + p_2 c_{21}^*) = \Phi(b_{12} + b_{12}^*) + \Phi(c_{21}) = \Phi(b_{12} + b_{12}^* + c_{21})$$

which shows that $p_2^* \diamond f + p_2 f^* = b_{12} + b_{12}^* + c_{21}$. This results that $f_{12} = b_{12}$, $f_{21} = c_{21}$ and $f_{22} = 0$ which yields that

$$\Phi(f_{11} + b_{12} + c_{21}) = \Phi(a_{11}) + \Phi(b_{12}) + \Phi(c_{21}).$$

Next, for an arbitrary element $d_{12} \in \mathcal{A}_{12}$, we have

$$\Phi(d_{12}^* \diamond (f_{11} + b_{12} + c_{21}) + d_{12}(f_{11} + b_{12} + c_{21})^*)$$
$$= \Phi(d_{12}^* \diamond a_{11} + d_{12} a_{11}^*) + \Phi(d_{12}^* \diamond b_{12} + d_{12} b_{12}^*) + \Phi(d_{12}^* \diamond c_{21} + d_{12} c_{21}^*)$$

which implies that

$$\Phi(d_{12}^* f_{11} + d_{12}^* b_{12} + f_{11} d_{12} + c_{21} d_{12} + d_{12} b_{12}^*)$$
$$= \Phi(d_{12}^* a_{11} + a_{11} d_{12}) + \Phi(d_{12}^* b_{12} + d_{12} b_{12}^*) + \Phi(c_{21} d_{12}).$$

It follows that

$$\begin{aligned}&\Phi(p_1^* \diamond (d_{12}^* f_{11} + d_{12}^* b_{12} + f_{11} d_{12} + c_{21} d_{12} + d_{12} b_{12}^*)\\&+p_1(d_{12}^* f_{11} + d_{12}^* b_{12} + f_{11} d_{12} + c_{21} d_{12} + d_{12} b_{12}^*)^*)\\&= \Phi(p_1^* \diamond (d_{12}^* a_{11} + a_{11} d_{12}) + p_1(d_{12}^* a_{11} + a_{11} d_{12})^*)\\&+\Phi(p_1^* \diamond (d_{12}^* b_{12} + d_{12} b_{12}^*) + p_1(d_{12}^* b_{12} + d_{12} b_{12}^*)^*)\\&+\Phi(p_1^* \diamond (c_{21} d_{12}) + p_1(c_{21} d_{12})^*)\end{aligned}$$

which results

$$\begin{aligned}&\Phi(f_{11} d_{12} + d_{12}^* f_{11} + 2 d_{12} b_{12}^* + f_{11}^* d_{12} + b_{12} d_{12}^*)\\&= \Phi(d_{12}^* a_{11} + a_{11} d_{12} + a_{11}^* d_{12}) + \Phi(2 d_{12} b_{12}^* + b_{12} d_{12}^*).\end{aligned}$$

As consequence, we obtain

$$\begin{aligned}&\Phi(p_2^* \diamond (f_{11} d_{12} + d_{12}^* f_{11} + 2 d_{12} b_{12}^* + f_{11}^* d_{12} + b_{12} d_{12}^*)\\&+p_2(f_{11} d_{12} + d_{12}^* f_{11} + 2 d_{12} b_{12}^* + f_{11}^* d_{12} + b_{12} d_{12}^*)^*)\\&= \Phi(p_2^* \diamond (d_{12}^* a_{11} + a_{11} d_{12} + a_{11}^* d_{12}) + p_2(d_{12}^* a_{11} + a_{11} d_{12} + a_{11}^* d_{12})^*)\\&+\Phi(p_2^* \diamond (2 d_{12} b_{12}^* + b_{12} d_{12}^*) + p_2(2 d_{12} b_{12}^* + b_{12} d_{12}^*)^*)\end{aligned}$$

which yields that

$$\begin{aligned}&\Phi(2 d_{12}^* f_{11} + f_{11} d_{12} + f_{11}^* d_{12} + d_{12}^* f_{11}^*)\\&= \Phi(2 d_{12}^* a_{11} + a_{11} d_{12} + a_{11}^* d_{12} + d_{12}^* a_{11}^*).\end{aligned}$$

This shows that $2d_{12}^* f_{11} + f_{11} d_{12} + f_{11}^* d_{12} + d_{12}^* f_{11}^* = 2d_{12}^* a_{11} + a_{11} d_{12} + a_{11}^* d_{12} + d_{12}^* a_{11}^*$ which implies that $d_{12}^*(2f_{11} + f_{11}^*) = d_{12}^*(2a_{11} + a_{11}^*)$ and $(f_{11} + f_{11}^*) d_{12} = (a_{11} + a_{11}^*) d_{12}$. It follows that $2f_{11} + f_{11}^* = 2a_{11} + a_{11}^*$ and $f_{11} + f_{11}^* = a_{11} + a_{11}^*$ which shows that $f_{11} = a_{11}$.

Similarly, we prove the case (ii). □

Property 2.6. *For arbitrary elements $a_{11} \in \mathcal{A}_{11}$, $b_{12} \in \mathcal{A}_{12}$, $c_{21} \in \mathcal{A}_{21}$ and $d_{22} \in \mathcal{A}_{22}$ holds $\Phi(a_{11} + b_{12} + c_{21} + d_{22}) = \Phi(a_{11}) + \Phi(b_{12}) + \Phi(c_{21}) + \Phi(d_{22})$.*

Proof. From the surjectivity of Φ choose $f = f_{11} + f_{12} + f_{21} + f_{22} \in \mathcal{A}$ such that $\Phi(f) = \Phi(a_{11}) + \Phi(b_{12}) + \Phi(c_{21}) + \Phi(d_{22})$. By Properties 2.3(i), 2.4 and 2.5(i), we have

$$\begin{aligned}&\Phi(p_1^* \diamond f + p_1 f^*) = \Phi(p_1^* \diamond a_{11} + p_1 a_{11}^*) + \Phi(p_1^* \diamond b_{12} + p_1 b_{12}^*)\\&+\Phi(p_1^* \diamond c_{21} + p_1 c_{21}^*) + \Phi(p_1^* \diamond d_{22} + p_1 d_{22}^*) = \Phi(2a_{11} + a_{11}^*) + \Phi(b_{12})\\&+\Phi(c_{21} + c_{21}^*) = \Phi(2a_{11} + a_{11}^* + b_{12} + c_{21} + c_{21}^*).\end{aligned}$$

It follows that $p_1^* \diamond f + p_1 f^* = 2a_{11} + a_{11}^* + b_{12} + c_{21} + c_{21}^*$ which implies that $f_{11} = a_{11}$, $f_{12} = b_{12}$ and $f_{21} = c_{21}$. Also,

$$\begin{aligned}&\Phi(p_2^* \diamond f + p_2 f^*) = \Phi(p_2^* \diamond a_{11} + p_2 a_{11}^*) + \Phi(p_2^* \diamond b_{12} + p_2 b_{12}^*)\\&+\Phi(p_2^* \diamond c_{21} + p_2 c_{21}^*) + \Phi(p_2^* \diamond d_{22} + p_2 d_{22}^*) = \Phi(b_{12} + b_{12}^*)\\&+\Phi(c_{21}) + \Phi(2d_{22} + d_{22}^*) = \Phi(b_{12} + b_{12}^* + c_{21} + 2d_{22} + d_{22}^*).\end{aligned}$$

This results that $p_2^* \diamond f + p_2 f^* = b_{12} + b_{12}^* + c_{21} + 2d_{22} + d_{22}^*$ which yields that $f_{22} = d_{22}$. □

Property 2.7. *For arbitrary elements $a_{11}, b_{11} \in \mathcal{A}_{11}$ and $c_{22}, d_{22} \in \mathcal{A}_{22}$ hold:*

(i) $\Phi(a_{11} + b_{11}) = \Phi(a_{11}) + \Phi(b_{11})$,

(ii) $\Phi(c_{22} + d_{22}) = \Phi(c_{22}) + \Phi(d_{22})$.

Proof. From the surjectivity of Φ there exists $f = f_{11}+f_{12}+f_{21}+f_{22} \in \mathcal{A}$ such that $\Phi(f) = \Phi(a_{11}) + \Phi(b_{11})$. By Property 2.3(ii), for an arbitrary element $d_{21} \in \mathcal{A}_{12}$ we have

$$\begin{aligned}&\Phi(d_{21}^* \diamond f + d_{21} f^*) = \Phi(d_{21}^* \diamond a_{11} + d_{21} a_{11}^*) + \Phi(d_{21}^* \diamond b_{11} + d_{21} b_{11}^*)\\ &= \Phi(d_{21} a_{11}^*) + \Phi(d_{21} b_{11}^*) = \Phi(d_{21}(a_{11} + b_{11})^*).\end{aligned}$$

This implies that $d_{21}^* \diamond f + d_{21} f^* = d_{21}(a_{11} + b_{11})^*$ which results that $f_{11} = a_{11} + b_{11}$ and $f_{12} = f_{21} = f_{22} = 0$.

Similarly, we prove the case (ii). □

Property 2.8. *Φ is an additive mapping.*

Proof. The result is an immediate consequence of Properties 2.3, 2.6 and 2.7. □

Lemma 2.4. $\Phi(1_{\mathcal{A}}) = 1_{\mathcal{B}}$.

Proof. Choose $a \in \mathcal{A}$ such that $\Phi(a) = 1_{\mathcal{B}}$. Note that

$$\begin{aligned}\Phi(ia^*) &= \Phi(a^* \diamond i1_{\mathcal{A}} + a(i1_{\mathcal{A}})^*)\\ &= \Phi(a)^* \diamond \Phi(i1_{\mathcal{A}}) + \Phi(a)\Phi(i1_{\mathcal{A}})^*\\ &= \Phi(a)^*\Phi(i1_{\mathcal{A}}) + \Phi(i1_{\mathcal{A}})\Phi(a) + \Phi(a)\Phi(i1_{\mathcal{A}})^*\\ &= 2\Phi(i1_{\mathcal{A}}) + \Phi(i1_{\mathcal{A}})^*.\end{aligned}$$

This results that $\Phi(i(a^* - 1_{\mathcal{A}})) = \Phi(i1_{\mathcal{A}}) + \Phi(i1_{\mathcal{A}})^*$ which shows that $\Phi(i(a^* - 1_{\mathcal{A}}))$ is a self-adjoint element. It follows that

$$\begin{aligned}\Phi(3i(a^* - 1_{\mathcal{A}})) &= \Phi(a)^* \diamond \Phi(i(a^* - 1_{\mathcal{A}})) + \Phi(a)\Phi(i(a^* - 1_{\mathcal{A}}))^*\\ &= \Phi(a^* \diamond (i(a^* - 1_{\mathcal{A}})) + a(i(a^* - 1_{\mathcal{A}}))^*)\\ &= \Phi(i(a^{*2} - a^* + a^*a - a^2))\end{aligned}$$

which implies that

$$3(a^* - 1_{\mathcal{A}}) = a^{*2} - a^* + a^*a - a^2, \tag{2.1}$$

and by taking $*$ on both sides of the equation (2.1), we get

$$3(a - 1_{\mathcal{A}}) = a^2 - a + a^*a - a^{*2}. \tag{2.2}$$

Thus subtracting (2.2) from (2.1), we obtain

$$a^2 - a^{*2} = 2(a - a^*). \tag{2.3}$$

Next, note also that
$\Phi(a^* \diamond a + aa^*) = \Phi(a)^* \diamond \Phi(a) + \Phi(a)\Phi(a)^* = 31_{\mathcal{B}} = \Phi(3a)$
which shows that
$a^*a + a^2 + aa^* = 3a.$

It follows from the last equation that $a^2 - 3a$ is a self-adjoint element, that is, $(a^2 - 3a)^* = a^2 - 3a$. Hence

$$a^2 - a^{*2} = 3(a - a^*). \tag{2.4}$$

From (2.3) and (2.4) we can conclude $a = a^*$. Yet, we have

$$\begin{aligned} 31_{\mathcal{B}} &= \Phi(3a) \\ &= \Phi(a^* \diamond 1_{\mathcal{A}} + a1_{\mathcal{A}}^*) \\ &= \Phi(a)^* \diamond \Phi(1_{\mathcal{A}}) + \Phi(a)\Phi(1_{\mathcal{A}})^* \\ &= \Phi(a)^*\Phi(1_{\mathcal{A}}) + \Phi(1_{\mathcal{A}})\Phi(a) + \Phi(a)\Phi(1_{\mathcal{A}})^* \\ &= 2\Phi(1_{\mathcal{A}}) + \Phi(1_{\mathcal{A}})^*, \end{aligned}$$

and by taking $*$ on both sides of the above equation, we obtain

$$31_{\mathcal{B}} = 2\Phi(1_{\mathcal{A}})^* + \Phi(1_{\mathcal{A}}) \tag{2.5}$$

which leads to $\Phi(1_{\mathcal{A}})^* = \Phi(1_{\mathcal{A}})$. By substituting it in (2.5) we obtain $\Phi(1_{\mathcal{A}}) = 1_{\mathcal{B}}$. □

Lemma 2.5. *Φ preserves involution on the both sides.*

Proof. For an arbitrary element $a \in \mathcal{A}$ we have

$$\begin{aligned} &\Phi(a^* \diamond 1_{\mathcal{A}} + a1_{\mathcal{A}}^*) = \Phi(a)^* \diamond \Phi(1_{\mathcal{A}}) + \Phi(a)\Phi(1_{\mathcal{A}})^* \\ &= \Phi(a)^*\Phi(1_{\mathcal{A}}) + \Phi(1_{\mathcal{A}})\Phi(a) + \Phi(a)\Phi(1_{\mathcal{A}})^* = \Phi(a)^* + 2\Phi(a) \end{aligned}$$

which implies that $\Phi(a^* + 2a) = \Phi(a)^* + 2\Phi(a)$. As consequence, we obtain $\Phi(a^*) = \Phi(a)^*$. Since Φ^{-1} has the same characteristics of Φ, then Φ preserves involution on the both sides. □

Lemma 2.6. *The following hold:*

(i) $\Phi(i1_{\mathcal{A}})^2 = -1_{\mathcal{B}}$,

(ii) $\Phi(ia) = \Phi(i1_{\mathcal{A}})\Phi(a) = \Phi(a)\Phi(i1_{\mathcal{A}})$, *for all element* $a \in \mathcal{A}$.

Proof. (i) By Lemmas 2.4 and 2.5 we obtain

$$\begin{aligned}\Phi(1_{\mathcal{A}}) &= \Phi((i1_{\mathcal{A}})^* \diamond (i1_{\mathcal{A}}) + (i1_{\mathcal{A}})(i1_{\mathcal{A}})^*) \\ &= \Phi(i1_{\mathcal{A}})^* \diamond \Phi(i1_{\mathcal{A}}) + \Phi(i1_{\mathcal{A}})\Phi(i1_{\mathcal{A}})^* \\ &= -\Phi(i1_{\mathcal{A}})^2,\end{aligned}$$

which implies that $\Phi(i1_{\mathcal{A}})^2 = -1_{\mathcal{B}}$.
(ii) For every element $a \in \mathcal{A}$ with $a^* = a$, we have

$$\begin{aligned}\Phi(ia) &= \Phi(a^* \diamond (i1_{\mathcal{A}}) + a(i1_{\mathcal{A}})^*) = \Phi(a)^* \diamond \Phi(i1_{\mathcal{A}}) + \Phi(a)\Phi(i1_{\mathcal{A}})^* \\ &= \Phi(i1_{\mathcal{A}})\Phi(a),\end{aligned}$$

and by taking $*$ on both sides of the above equation, we conclude that

$$\Phi(ia) = \Phi(i1_{\mathcal{A}})\Phi(a) = \Phi(a)\Phi(i1_{\mathcal{A}}).$$

Thus, for an arbitrary element $a \in \mathcal{A}$, we can write it as $a = a_1 + ia_2$, where a_1 and a_2 are self-adjoint elements. As consequence, using identity (i) we obtain

$$\begin{aligned}\Phi(ia) = \Phi(ia_1 - a_2) &= \Phi(i1_{\mathcal{A}})\Phi(a_1) + \Phi(i1_{\mathcal{A}})^2\Phi(a_2) \\ &= \Phi(i1_{\mathcal{A}})(\Phi(a_1) + \Phi(i1_{\mathcal{A}})\Phi(a_2)) \\ &= \Phi(i1_{\mathcal{A}})(\Phi(a_1) + \Phi(ia_2)) \\ &= \Phi(i1_{\mathcal{A}})\Phi(a).\end{aligned}$$

□

Lemma 2.7. *$\Phi : \mathcal{A} \to \mathcal{B}$ preserves product.*

Proof. By Lemma 2.5, for two arbitrary self-adjoint elements $a, b \in \mathcal{A}$ we have

$$\begin{aligned}\Phi(2ab + ba) &= \Phi(a^* \diamond b + ab^*) = \Phi(a)^* \diamond \Phi(b) + \Phi(a)\Phi(b)^* \\ &= 2\Phi(a)\Phi(b) + \Phi(b)\Phi(a),\end{aligned}$$

which implies that

$$2\Phi(ab) + \Phi(ba) = 2\Phi(a)\Phi(b) + \Phi(b)\Phi(a).$$

Substituting a by b and b by a, in the above identity, we also get

$$2\Phi(ba) + \Phi(ab) = 2\Phi(b)\Phi(a) + \Phi(a)\Phi(b).$$

Multiplying the first identity by 2 and subtracting from the second identity, we obtain $\Phi(ab) = \Phi(a)\Phi(b)$. As consequence of this last result and the Lemma 2.6(i), for two arbitrary elements $a = a_1 + ia_2, b = b_1 + ib_2 \in \mathcal{A}$, where a_1, a_2, b_1, b_2 are self-adjoint elements, we obtain that

$$\begin{aligned}\Phi(ab) &= \Phi((a_1 + ia_2)(b_1 + ib_2)) \\ &= \Phi(a_1b_1 + ia_1b_2 + ia_2b_1 - a_2b_2) \\ &= \Phi(a_1)\Phi(b_1) + \Phi(a_1)\Phi(ib_2) + \Phi(ia_2)\Phi(b_1) + \Phi(ia_2)\Phi(ib_2) \\ &= (\Phi(a_1) + \Phi(ia_2))\Phi(b_1) + (\Phi(a_1) + \Phi(ia_2))\Phi(ib_2) \\ &= \Phi(a_1 + ia_2)\Phi(b_1 + ib_2) = \Phi(a)\Phi(b).\end{aligned}$$

□

Finalizing the proof of the Main Theorem, we conclude that Φ is an $*$-ring isomorphism, by Property 2.8 and the Lemmas 2.5 and 2.7.

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