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Semirings of graphs: homomorphisms and applications in network problems

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Abstract

This paper deals with studying some algebraic structures of the graphs as an attempt to visualize abstract mathematics. We have used some binary graph operations to investigated the algebraic structures of graphs with examples. This work emphasizes specifically the construction of semigroup or monoid and semiring, and their properties. This manuscript also aims to give a focused introduction of a class of homomorphism on the semiring of graphs. Some instances of real-life decision problems are consequently discussed. This article is also in a nascent stage of relating number theory and graph theory through mappings.

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1. Introduction

The close association of graph theory and abstract algebra have been extensively studied in the literatures. For details on the graph of semirings, we refer to [10, 11]. Study of semiring goes back to at least 1934 [18], when Vandiver introduce the notion which says that a universal algebra with two associative binary operations, where one of them distributes over the other is called semiring. This definition relaxes the requirement of neutral elements. Some authors consider semirings to posses neutral elements with respect to both the binary operations defined on it (see [6]). In some literatures, the authors consider algebraic structures with two binary operations, addition and multiplication such that the structure is additively commutative monoid, multiplicatively semigroup and multiplication distributes over addition from both left and right, called "hemirings" (see [1]). It is worth mentioning that this nomenclature is not 100% settled yet. Needless to say semirings found their full place in mathematics even before 1934 (e.g., the semirings of positive elements in ordered rings) and even more so after (e.g., various applications in theoretical computer science and algorithm theory). For background basic and more advanced properties, historical remarks and further references on semirings (see, for example, [2, 4, 5, 7]).

In this article, we use algebraic graph operations like, *union* $\cup$, *intersection* $\cap$ and *join* ∇, wherein self loops and multiple edges of the graphs are dropped. So to say by graph we mean a simple and undirected graph. The definitions of *union* and *join* coincide with the one used by Mokhov [13]. He calls overlay $+$ and connect$\rightarrow$ which deals with directed graphs that satisfy various algebraic properties and subsequently applied to working with graphs in Haskell. Being motivated by these works, we develop an independent approach to study semiring structures and their various properties. An approach to graph theory in an algebraic setting has also been found attempted by Bustamante [3], where the graph operation called the linking between two graphs G and G', which is akin to what we call *join* ∇ in this paper, and an algebraic structure called "Link Algebra" is analogous to the semiring $(S, \cup, \nabla, (\emptyset, \emptyset))$. Study of graphs in algebraic settings have also been investigated by Umbrey and Rahman in 2020 [15, 16, 17].

There are many papers in the literature that deal with the semring of matrices e.g; Kishka, Z. M. G., et al [9] studies the matrix of matrices over semiring. An adjacency structure or, an incidence structure of a graph G

stores all the information of G, hence the specification of any large or complicated graph can be communicated to computers in these matrix forms. In this line, we see that a set of certain specific graphs forms an algebraic structure, namely semiring, that is if S is a semiring, then the collection $M_n(S)$ of all $n \times n$ is again a semiring, where n is a positive integer. The corresponding addition and multiplication in $M_n(S)$ is given by the usual law of matrix addition and multiplication, respectively. Treating the graphs as algebraic elements has several advantages due to the excellent visual properties of the graphs. Therefore, the graphs in this article are considered potential elements in visualizing abstract mathematics.

A semiring S is said to be mono-semiring if $xy = x + y$ for all $x, y \in S$. Additive, and multiplicative identities coincide in the case of mono-semirings. We marked that no semiring (except mono-semiring) in which additive and multiplicative identities coincide in literature. But in this article, we present a semiring of graphs (other than mono-semiring) in which additive and multiplicative identities are the same. Incorporating graphs into abstract algebraic structures would also open a new avenue to investigate more such un-explored properties of abstract algebra in future work.

2. Preliminary

A graph G is an ordered pair (V, E), where $V = V(G)$ is a set of abstract objects (known as the set of vertices) and $E = E(G)$ is a set of unordered pair of objects in V (known as the set of edges). An edge is also called an arc or line. A graph is directed if the edges are directed by arrows, indicating that the relationship represented by the edge only applies from one vertex to the other, but not the other way around. On the other hand, a graph whose edges are not directed is called an undirected graph. The set of directed edges of $E \in V \times V$, where E is a binary relation on V and $V \times V$ is the Cartesian product of V. The set of undirected edges of G is $E \in [V]^2$, where $[V]^2$ is the set of 2-element subsets of V. Here, E can be considered as a symmetric binary relation on V. If a graph G' is a subgraph of G'', then it is abbreviated as $G' \subseteq G''$ for convenience. For a graph G, the set of all its possible subgraphs is denoted by S' or $P(G)$ unless and otherwise stated.

The union of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is defined as the graph $G = G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$. The join ∇ of two graphs G and

H is a graph formed by the union of edge sets of G and H (disregarding self-loops and multiple edges), and by connecting each vertex of G to each vertex of H. It is denoted by $G\nabla H = (V(G)\cup V(H), E(G)\cup E(H)\cup\{(u,v) : u \in V(G), v \in V(H)\} \setminus \{(a,a) : a \in V(G) \cap V(H)\})$. We define another binary operation denoted by $\cap$ called *intersection*, such that $G_1 \cap G_2$ denotes the merging of the vertices and the edges of $G_1 = (V_1, E_1)$ to those of $G_2 = (V_2, E_2)$ which are identical and remaining distinct vertices and edges are removed and it is defined and denoted by $G_1 \cap G_2 = (V_1 \cap V_2, E_1 \cap E_2)$. The chromatic number of a graph G is the smallest number of colors required to color the vertices of G such that no two adjacent vertices get the same color, and it is denoted by $\chi(G)$. A Semiring is usually defined to be a non-empty set S together with two binary operations namely addition and multiplication denoted by $(S, +, \cdot)$ such that $(S, +)$ is monoid and $(S, \cdot)$ is semigroup, where addition and multiplication are connected by distributivity. Apart from this, the additive identity is multiplicatively absorbing.

There are several papers in the literatures which deal with more general concept of a semiring which neither require additive identity nor multiplicative absorbing. Here, we restrict ourselves to the formal definition of semirings defined by Vandiver in [18] wherein, the requirement of neutral elements and absorbing property is omitted.

For our purpose, parallel edges connecting any two vertices are merged, and loops are ignored. Henceforth, we will denote the set of all simple undirected graphs by S unless and otherwise stated.

3. Semiring Structures on Graphs and their Properties

Theorem 3.1. *If S is the set of all graphs, then $(S, \cup, \nabla)$ is a semiring.*

Proof. First, we claim that $(S, \cup)$ is a semigroup. Since S is the set of all graphs, for any graphs $G_1,\ G_2 \in S$, it follows that $G_1 \cup G_2 \in S$. The graphs are associative under $\cup$. This associativity inherits from the associativity of union of edge sets and vertex sets respectively. Hence it follows that $(S, \cup)$ is a semigroup. Next, we claim that $G_1\nabla(G_2\nabla G_3) = (G_1\nabla G_2)\nabla G_3$.

The symbols in bold viz., $\subseteq$, $\cup$ and $\cap$ distinguish them from the usual set's subset, union and intersection, respectively.

From the associativity of union of sets, it is clear that $V(G_1\nabla(G_2\nabla G_3)) = V((G_1\nabla G_2)\nabla G_3)$. Also, $e \in E(G_1\nabla(G_2\nabla G_3))$ if and only if $e \in E(G_1)$ or ($e \in E(G_2)$ or $e \in E(G_3)$) or $e = (u, v)$, where $u \in V(G_1), v \in V(G_2)$ or $u \in V(G_1), v \in V(G_3)$ or $u \in V(G_2), v \in V(G_3)$. Then the following cases arise:

Case I: If $e \in E(G_1)$ or $e \in E(G_2)$ or $e \in E(G_3)$, then it is easy to show that every edge e in $G_1\nabla(G_2\nabla G_3)$ is also an edge of $(G_1\nabla G_2)\nabla G_3$ and vice-versa.

Case II: If $e = (u, v)$, where $u \in V(G_1), v \in V(G_2)$ or $u \in V(G_1), v \in V(G_3)$ or $u \in V(G_2), v \in V(G_3)$, then the following sub-cases arise.
If $e = (u, v)$, where $u \in V(G_1), v \in V(G_2)$, then e is an edge of $G_1\nabla G_2$, and hence is an edge of $(G_1\nabla G_2)\nabla G_3$. Also, $u \in V(G_1), v \in V(G_2)$ imply that $u \in V(G_1)$, $v \in V(G_2\nabla G_3)$ and, thus, e is an edge of $G_1\nabla(G_2\nabla G_3)$. Similar arguments hold true for the remaining cases, viz., $e = (u, v)$, where $u \in V(G_1), v \in V(G_3)$ and $e = (u, v)$, where $u \in V(G_2), v \in V(G_3)$. Thus, the graphs $G_1\nabla(G_2\nabla G_3)$ and $(G_1\nabla G_2)\nabla G_3$ have the same sets of vertices and edges, and consequently, $G_1\nabla(G_2\nabla G_3)$=$(G_1\nabla G_2)\nabla G_3$. Now, we show that ∇ distributes over $\cup$. The distributivity of ∇ over $\cup$ has to be proved in two parts, namely vertex part and edge part. But the distributivity of ∇ over $\cup$ on vertex set inherits from the associativity of union of sets.

$$
\begin{array}{ll}
\text{Let} & e \in E(G_1\nabla(G_2 \cup G_3)) \\
\Longrightarrow & e \in E(G_1) \text{ or } e \in E(G_2) \text{ or } e \in E(G_3) \text{ or} \\
& e = (u, v), \text{ where } u \in V(G_1), v \in V(G_2) \cup V(G_3) \\
\Longrightarrow & e \in E(G_1) \text{ or } e \in E(G_2) \text{ or } e \in E(G_3) \text{ or} \\
& e = (u, v), \text{ where } u \in V(G_1), v \in V(G_2) \text{ or } V(G_3).
\end{array}
$$

In all the above cases, it is easy to observe that every edge of $G_1\nabla(G_2\cup G_3)$ also belongs to $(G_1\nabla G_2)\cup(G_1\cup G_3)$ and vice-versa, and consequently, $G_1\nabla(G_2 \cup G_3)$=$(G_1\nabla G_2) \cup (G_1 \cup G_3)$, which is the left distributivity ∇ over $\cup$. The right distributivity holds analogously. □

REMARK 1. *Theorem 3.1 is a counterexample in which the empty graph $(\emptyset, \emptyset)$ is neutral element for both the operations, where S is not mono-semiring.*

Proposition 3.2. *Let G be a graph with n vertices and S' be the set of all subgraphs of G such that $(S', \cup, \nabla)$ is a semiring, then G is a complete graph.*

Proof. Let S' be the set of all subgraphs of the graph G, then by definition of the operation ∇, $G\nabla G$ is a complete graph and $G \subseteq G\nabla G$. But, by closure property of the semiring S', $G\nabla G \subseteq G$. Thus, $G = G\nabla G$ is a complete graph. □

REMARK 2. *We note that the operation $\cup$ is commutative and idempotent. Whereas, the operation ∇ is commutative but non-idempotent on S in general. Further, it is not difficult to verify that the structure $(S, \cup, \cap)$ is an idempotent semiring.*

Example 3.1. *Let G be a complete graph, and S_k be the set of all complete subgraphs of G. Then (S_k, ∇) is a commutative and an idempotent monoid with the identity $(\emptyset, \emptyset)$ and the absorbing element G. Further, we observe that for all $G_1, G_2 \in S_k$, $G_1 \subseteq G_2$ if and only if $G_1 \cup G' = G_2$ and $G''\nabla G_1 = G_2$ for some G' and G'' in S_k. Let us see the following illustrations, for instance:*

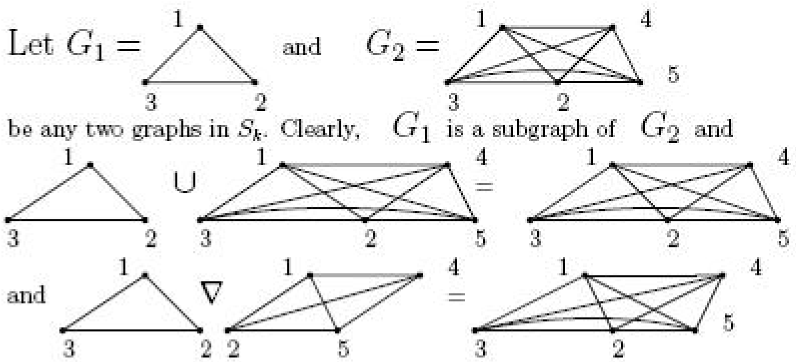

Figure 1: *Graphs satisfying algebraic identities*

Here, it is easy to see that the semiring $(S_k, \cup, \nabla, \subseteq)$ is an ordered semiring.

Definition 3.1. *An undirected graph with n distinct nodes is called a labeled n-node graph. A labeled 1-node graph contains only one vertex, i.e., the vertex itself is the only possible graph (if loops are not allowed). A* *labeled 2-node graph contains two vertices, so the number of a distinct 2-node graph is* 2. *Similarly, an* n*-node graph contains* n *distinct vertices that may have upto* $\frac{n(n-1)}{2}$ *edges (if the loops are not allowed). So, the number of a possible distinct* n*-node graph is* $2^{\frac{n(n-1)}{2}}$. *For a fixed* n*, consider the set of all possible labeled* n*-node graphs (with each graph having same set of vertices),* $S^{'} = \{G_1, G_2, G_3, ..., G_{2^{\frac{n(n-1)}{2}}}\}$.

Proposition 3.3. *If* $S^{'}$ *is the set of all* $2^{\frac{n(n-1)}{2}}$ *different* n*-node graphs, then* $(S^{'}, \cup, \nabla)$ *is a semiring. In particular,* $S^{'}$ *is a subsemiring of* S.

Proposition 3.4. *If* G *is a complete graph and* $P(G)$ *is the set of all possible subgraphs of* G*, then* $(P(G), \cup, \cap)$ *is an idempotent semiring.*

The proofs of the above propositions are straightforward and so omitted.

REMARK 3. *Let* $S'(G)$ *be the set of all subgraphs of a simple undirected graph* G*. For the ease of notation; we denote* S' *to mean* $S'(G)$ *when the associated graph* G *is understood. Let* $P(S')$ *be the power set of* S'*. Let* $H \in S'$ *be such that* S'_H *denotes the set of all those subgraphs of* H *whose vertex set is* $V(H)$*, and* $S'(H)$ *be the set of all subgraphs of* H*, then we have* $S'_H \subseteq S'(H)$*. It is easy to see that* $(S'_H, \cup, (V(H), \emptyset))$ *is a monoid. Let* $H_i, H_j \in S'_H$*, then* $H_i \cap H_j \in S'_H$*, i.e., the operation '*$\cap$*' is closed in* S'_H*. This is because of the hypothesis that* H_i *and* H_j *are subgraphs of* H*, whose vertex sets are same, i.e.,* $V(H)$*. It is also easy to see that the operation* $\cap$ *distributes over* $\cup$*. Hence* $(S'_H, \cup, \cap, (V(H), \emptyset), H)$ *is a semiring. But the operation '*∇*' in general is not closed in* S'_H*, so the structure* $(S'_H, \cup, \nabla, (V(H), \emptyset), H)$ *is not a semiring. However, when* H *is a complete graph, the operation '*∇*' is closed in* S'_H*, and consequently, the structure* $(S'_H, \cup, \nabla, (V(H), \emptyset), H)$ *is a semiring (the operation '*∇*' distributes over the operation '*$\cup$*'). When* G *is a complete graph, we see that* $(S'_G, \cup, \nabla)$ *is a subsemiring of* $(S', \cup, \nabla)$*. Let* $G_g \in S'_G$ *and* $G_s \in S'$*, then* $G_g \nabla G_s$ *is a subgraph of* G *whose vertex set is* $V(G)$*. Hence* $G_g \nabla G_s \in S'_G$*. Similarly,* $G_s \nabla G_g \in S'_G$*. Therefore,* S'_G *is an ideal of* S'*. Proceeding likewise by considering a complete subgraph* H_k *of* G*, such that* S'_{H_k} *is the set of all those subgraphs of* H_k *whose vertex set is* $V(H_k)$*, and* $S'(H_k)$*, the set of all subgraphs of* H_k*. Then* $(S'_{H_k}, \cup, \nabla)$ *is a subsemiring of* $(S'(H_k), \cup, \nabla)$*. Also,* S'_{H_k} *is an ideal of* $S'(H_k)$ *under the additive operation '*$\cup$*' and the multiplicative operation '*∇*'.*

4. Homomorphisms

This section is an introductory note on the homomorphisms of the semirings built on graphs. We discuss homomorphisms on different domains of graphs. A homomorphism from semiring of non-negative integer to semiring of graphs could be a potential tool for interchanging some properties of number theory and graph theory. Matrix operations analogous to the graph union and intersection are also defined because of the possible requirements of matrix forms of graphs for computational purposes. However, we left the implementations of such matrix operations as a future research problem. Finally, some artificial problems are discussed.

4.1. Homomorphism on Semiring of Non-negative Integers to Semiring of Graphs.

Definition 4.1. *Suppose,* $(S, +, \cdot, 0, 1)$ *and* $(S', \oplus, \otimes, 0', 1')$ *are two semirings. Then a map* $f : S \to S'$ *is said to be a semiring homomorphism if for all* $a, b \in S$, $f(a + b) = f(a) \oplus f(b)$, $f(a.b) = f(a) \otimes f(b)$, $f(0) = 0'$ *and* $f(1) = 1'$*. Note that* $f(1) = 1'$ *and* $f(0) = 0'$ *are disregarded when we consider the semirings in which the multiplicative and additive identities are not considered or not defined.*

Example 4.1. *Let* $P(\mathbf{Z}_0)$ *be the power set of the set of non-negative integers* $\mathbf{Z}_0$*. Then* $(\mathbf{Z}_0, +, \cdot, 0, 1)$ *and* $(P(\mathbf{Z}_0), \cup, \cap, \emptyset, \mathbf{Z}_0))$ *are semirings, where the operations on* $\mathbf{Z}_0$ *are usual addition and multiplication, while the operations on* $P(\mathbf{Z}_0)$ *are usual set union and intersection. We note that if a mapping* $\phi : \mathbf{Z}_0 \to P(\mathbf{Z}_0)$ *is a semiring homomorphism, then* $\phi(n) = \mathbf{Z}_0$ *for all* $n \geq 1$*. Since* ϕ *is a homomorphism,* $\phi(0) = \emptyset$ *and* $\phi(1) = \mathbf{Z}_0$*. Now, for all* $n \geq 1$, $\phi(n) = \phi(1 + \ldots + 1) = \phi(1) \cup \ldots \cup \phi(1) = \phi(1) = \mathbf{Z}_0$.

Example 4.2. *Let* $m + n = max\{m, n\}$ *and* $m.n = min\{m, n\}$ *for all* $m, n \in \mathbf{Z}_0$*, where* $\mathbf{Z}_0$ *is the set of non-negative integers. Let* $[n]$ *be a subset of* $\mathbf{Z}_0$ *encoded as* $[n] = \{0, 1, 2, ..., n - 1\}$*, where* $[0] = \{\}$ *or,* $\emptyset$*;* $[1] = \{0\}$*;* $[2] = \{0, 1\}$*;* $[3] = \{0, 1, 2\}$*, and so on. Recalling the definition of* $[V]^2$*, we have that* $[[0]]^2 = \emptyset$, $[[1]]^2 = \emptyset$, $[[2]]^2 = \{\{0, 1\}\}$, $[[3]]^2 = \{\{0, 1\}, \{0, 2\}, \{1, 2\}\}$*, etc.*

Define a mapping $\phi : \mathbf{Z}_0 \longrightarrow P(\mathbf{Z}_0)$ *by* $\phi(n) = [n] = \{0, 1, 2, ..., n - 1\}$*. Then we see in the following that* ϕ *is a semiring homomorphism. Let* $m, n \in \mathbf{Z}_0$ *and without loss of generality, assume* $m \leq n$*. Then* $\phi(m+n) = \{0, 1, 2, ..., m+n-1\} = \{0, 1, 2, ..., m-1\} \cup \{0, 1, 2, ..., n-1\} = \phi(m) \cup \phi(n)$.

Similarly, $\phi(m \cdot n) = \{0, 1, 2, ..., m \cdot n - 1\} = \{0, 1, 2, ..., m - 1\} \cap \{0, 1, 2, ..., n - 1\} = \phi(m) \cap \phi(n)$. *Also, by definition of* ϕ, *we have* $\phi(0) = \emptyset$. *Hence* ϕ *is a semiring homomorphism.*

Note. We consider that a graph is infinite if the corresponding vertex set is an infinite set, and denote the graph by G.

Let E be a symmetric binary relation on $\mathbf{Z}_0$. Then for all $a, b \in \mathbf{Z}_0$, aEb implies that bEa, i.e., $(a, b) \in E, \Longrightarrow (b, a) \in E$. Hence, the combined (a, b) and (b, a) can be considered as an unordered pair or a 2-element subset $\{a, b\} \in E$. We can consider that for each $n \in \mathbf{Z}_0$, $([n], E_n)$ is a graph, where $E_n = [[n]]^2 \cap E$ such that for all $m, n \in \mathbf{Z}_0$, $m \leq n$ implies that $([m], E_m) \subseteq ([n], E_n)$. As a consequence, we have the following proposition.

Proposition 4.1. *Let* E *be a symmetric binary relation on* $\mathbf{Z}_0$, *and* S_E *be the set of all simple undirected graphs with vertex sets* $[n]$ *for all* $n \in \mathbf{Z}_0$, *and edge sets* $E_n = [[n]]^2 \cap E$. *Then* $(S_E, \cup, \cap)$ *forms an ordered semiring, and hence is a subsemiring of the semiring of graphs* $(S, \cup, \cap)$.

Proof. Let $G_1, G_2 \in S_E$ be arbitrary such that $G_1 = ([m], E_m)$ and $G_2 = ([n], E_n)$. Without loss of generality, we assume that $m \leq n$. Then, $G_1 \subseteq G_2$, and hence, $G_1 \cup G_2 = G_2 \in S_E$ and $G_1 \cap G_2 = G_1 \in S_E$. Thus, S_E is closed. Since $S_E \subseteq S$, the distributivity of $\cap$ over $\cup$, and the associativity of S_E is inherited from that of S. Moreover, every pair of elements of S_E are comparable under the subgraph relation $\subseteq$, and the operations $\cup$ and $\cap$ preserve this order relation. Thus, $(S_E, \cup, \cap)$ forms an ordered semiring, and hence is a subsemiring of the semiring of graphs $(S, \cup, \cap)$. □

Proposition 4.2. *Let* E *be a symmetric binary relation on* $\mathbf{Z}_0$, *then* $f : (\mathbf{Z}_0, +, \cdot) \longrightarrow (S_E, \cup, \cap)$ *defined by* $f(n) = ([n], E_n)$, *where* $m + n = max\{m, n\}$, $m \cdot n = min\{m, n\}$ *and* $E_n = [[n]]^2 \cap E$ *is a semiring homomorphism.*

Proof. We have to prove the following axioms for f to be a semiring homomorphism

- $f(m + n) = f(m) \cup f(n)$
- $f(m.n) = f(m) \cap f(n)$ and
- $f(0) = (\emptyset, \emptyset)$.

Let $m, n \in \mathbf{Z}_0$. When $m = n$, then $f(m+n) = f(m) \cup f(n)$ and $f(m.n) = f(m) \cap f(n)$ trivially hold. If $m < n$, then $[m] \subset [n]$. Consequently, $([m], E_m) \subseteq ([n], E_n)$. Therefore, $f(m) \cup f(n) = ([m], E_m) \cup ([n], E_n) = ([n], E_n) = f(n) = f(m+n)$. Also, $f(m) \cap f(n) = ([m], E_m) \cap ([n], E_n) = ([m], E_m) = f(m) = f(m \cdot n)$. Similarly, the proof also holds for $m > n$. Also, $f(0) = (\emptyset, \emptyset)$. $\square$

Note that $\mathbf{Z}_0$ and S_E are both semirings with unity ∞ (infinity) and infinite graph G (say), respectively such that $f(\infty) = G$. The kernel of the semiring homomorphism $f : \mathbf{Z}_0 \longrightarrow S$ is $kerf = \{m \in \mathbf{Z}_0 : f(m) = (\emptyset, \emptyset)\} = \{0\}$, and hence f is one-one. The image of the semiring homomorphism $f : \mathbf{Z}_0 \longrightarrow S_E$ is $\{f(m) : m \in \mathbf{Z}_0\}$. Note that the $kerf$ is a two sided ideal of S. The image of f is a subsemiring. Since for each $G \in S_E$, there exists $n \in \mathbf{Z}_0$, such that $G = ([n], E_n)$, and hence $f(n) = ([n], E_n) = G$. Thus, f is surjective. Moreover, f is injective also, hence f is an isomorphism.

Remark 4. *Since f is isomorphic, many properties of S_E can be studied in $\mathbf{Z}_0$. Let us see some instances of this fact. For any $a, b \in \mathbf{Z}_0$, $a \leq b$ or, $b \leq a$ which implies that $f(a) \subseteq f(b)$ or, $f(b) \subseteq f(a)$ for all $f(a), f(b) \in S_E$. Thus, the order relation in $\mathbf{Z}_0$ is preserved in S_E by isomorphism f. The relation $\leq$ is linear in $\mathbf{Z}_0$, that is, $(\mathbf{Z}_0, \leq)$ is a chain or, the structure $(\mathbf{Z}_0, +, \cdot, \leq)$ is a chain semiring. Consequently, $(S_E, \cup, \cap, \subseteq)$ is also a chain semiring. Now, it is obvious that for all $G_1, G_2, G_3 \in S$, if $G_1 \subseteq G_2$ then $G_1 \cup G_3 \subseteq G_2 \cup G_3$, and $G_1 \cap G_3 \subseteq G_2 \cap G_3$ or, $G_3 \cap G_1 \subseteq G_3 \cap G_2$. In particular, if $G_1, G_2, ..., G_n$; $H_1, H_2, ..., H_n$ are elements of S, satisfying the condition that $G_i \subseteq H_i$ for $i = 1, 2, ..., n$, then $\bigcup\limits_{i=0}^{n} G_i \subseteq \bigcup\limits_{i=0}^{n} H_i$ and $\bigcap\limits_{i=0}^{n} G_i \subseteq \bigcap\limits_{i=0}^{n} H_i$. Consequently, the semiring S_E is positive (since $(\emptyset, \emptyset) \subseteq G$ for each $G \in S_E$), and a zerosumfree (since any positive partially ordered semiring is zerosumfree).*

Example 4.3. *Let E be a relation on $\mathbf{Z}_0$ defined by mEn if and only if either $m|n$ or $n|m$ for all $m, n \in \mathbf{Z}_0$. Then the elements of the set S_E are given by $f(0) = (\emptyset, \emptyset)$, $f(1) = (\{0\}, \emptyset)$ (a graph of isolated vertex 0), $f(2) = (\{0, 1\}, \{0, 1\})$, $f(3) = (\{0, 1, 2\}, \{\{0, 1\}, \{0, 2\}, \{1, 2\}\})$,... and so on. It can be easily verified $f(0) \subseteq f(1) \subseteq f(2) \subseteq f(3) \subseteq ...$ That is, the set of graphs are linearly ordered.*

Again, if $E = \{(m, n) : gcd(m, n) = 1$, i.e., m and n are co-primes$\}$, then $f(0) = (\emptyset, \emptyset)$, $f(1) = (\{0\}, \emptyset)$, $f(2) = (\{0, 1\}, \{0, 1\})$,

$f(3) = (\{0,1,2\}, \{\{0,1\},\{1,2\}\})$, *etc.*

Remark 5. *Considering such kind of examples may motivate us to study further some properties of number theory in terms of abstract algebra via graph theory and vice-versa.*

4.2. Homomorphism between Graph Semirings to the Power Set of Semiring of Graphs.

Consider a simple graph G, and S', the set of all subgraphs of G. Let $P(S')$ be the power set of S' and for any two subsets $A,\ B \in P(S')$, define:

$$A \sqcup B = \begin{cases} A, & \text{if} \quad B = \emptyset; \\ B, & \text{if} \quad A = \emptyset; \\ \{G_i \cup G_j :\ G_i \in A, G_j \in B\}, & \text{otherwise.} \end{cases}$$

and

$$A \sqcap B = \begin{cases} \emptyset, & \text{when} \quad A = \emptyset \text{ or } B = \emptyset; \\ \{G_i \cap G_j :\ G_i \in A, G_j \in B\}, & \text{otherwise.} \end{cases}$$

Consider that S'_G is the collection of all those subgraphs of G each of whose vertex set is $V(G)$, and $P(S'_G)$ is the power set of S'_G. Then it is not hard to see that $(S'_G, \cup, \cap)$ and $(P(S'_G) \setminus \{\emptyset\}, \sqcup, \sqcap)$ are semirings. The multiplicative identities in S'_G and $P(S'_G)$ are G and S'_G, respectively. The additive identity of S'_G is $(V(G), \emptyset)$ while, $\{(V(G), \emptyset)\}$ is the additive identity of $P(S'_G) \setminus \{\emptyset\}$. Define a mapping $f : (S'_G, \cup, \cap) \to (P(S'_G) \setminus \{\emptyset\}, \sqcup, \sqcap)$ by $f(H) = S'_H$ for all $H \in S'_G$ and S'_H is the set of all those subgraphs of H each of whose vertex set is $V(H)$ (in fact, $V(H) = V(G)$). Then the following proposition 4.3 is a consequence of this definition:

Remark 6. *Although, it is nowhere mentioned in literature, the additive identity of a subsemiring may be different from that of the semiring (if exist) as we noted from the above discussion. Clearly,* $(S'_G, \cup, \cap)$ *is a subsemiring of* $(S', \cup, \cap)$*, but the additive identity of* S' *is an empty graph* $(\emptyset, \emptyset)$.

Proposition 4.3. *The mapping* $f : (S'_G, \cup, \cap) \longrightarrow (P(S'_G) \setminus \{\emptyset\}, \sqcup, \sqcap)$ *is a homomorphism (*f *is defined above).*

Proof. By definition of f, we have
$f((V(G),\emptyset)) = \{(V(G),\emptyset)\}$ and $f(G) = S'_G$. Let $H_1, H_2 \in S'_G$ be two arbitrary subgraphs of G (i.e., both of whose vertex sets are $V(G)$). Then $f(H_1 \cup H_2) = S'_{H_1 \cup H_2}$ =Set of all subgraphs of $H_1 \cup H_2$ whose vertex set is $V(H_1 \cup H_2)$ or, V(G). Let S'_{H_1} be the set of all subgraphs of H_1 whose vertex set is $V(G)$, and S'_{H_2}, the set of all subgraphs of H_2 whose vertex set is $V(G)$, then definition of $\sqcup$ suffices that $S'_{H_1} \sqcup S'_{H_2}$ =Set of all subgraphs of $H_1 \cup H_2$ whose vertex set is $V(H_1 \cup H_2)$ or, $V(G)$. First, we claim that $S'_{H_1 \cup H_2} = S'_{H_1} \sqcup S'_{H_2}$. A graph $G_1 \in S'_{H_1 \cup H_2}$ implies that $V(G_1) = V(H_1 \cup H_2) = V(H_1) \cup V(H_2) = V(G)$. Since G_1 is a subgraph of $H_1 \cup H_2$, for any $e \in E(G_1)$ implies that $e \in E(H_1 \cup H_2) = E(H_1) \cup E(H_2)$, i.e., $e \in E(H_1)$ or $e \in E(H_2)$. Thus, $G_1 = H'_1 \cup H'_2$ for some $H'_1 \in S'_{H_1}$ and $H'_2 \in S'_{H_2}$, which implies that $G_1 \in S'_{H_1} \sqcup S'_{H_2}$, thus we get $S'_{H_1 \cup H_2} \subseteq S'_{H_1} \sqcup S'_{H_2}$. Conversely, for any graph $G_1 \in S'_{H_1} \sqcup S'_{H_2}$, there exists H'_1 and H'_2 in S'_{H_1} and S'_{H_2}, respectively such $G_1 = H'_1 \cup H'_2$, where $H'_1 \subseteq H_1$ and $H'_2 \subseteq H_2$, i.e., $H'_1 \cup H'_2 \subseteq H_1 \cup H_2$ or, $G_1 \subseteq H_1 \cup H_2$, which means that $G_1 \in S'_{H_1 \cup H_2}$, and thus we get $S'_{H_1} \sqcup S'_{H_2} \subseteq S'_{H_1 \cup H_2}$. Hence the claim, i.e., $S'_{H_1 \cup H_2} = S'_{H_1} \sqcup S'_{H_2}$. Consequently, $f(H_1 \cup H_2) = S'_{H_1 \cup H_2} = S'_{H_1} \sqcup S'_{H_2} = f(H_1) \sqcup f(H_2)$.

By using a similar arguments, we see that $f(H_1 \cap H_2) = S'_{H_1 \cap H_2} = S'_{H_1} \sqcap S'_{H_2} = f(H_1) \sqcap f(H_2)$. Hence f is a semiring homomorphism. □

Let a non-empty $A' \subseteq S'$, and consider $H_i \in S'$ and $G_i \in A'$ such that if $H_i \subseteq G_i$ for all G_i. Then $H'_j \subseteq G_i$ for all $H'_j \in S'_{H_i}$ and $G_i \in A'$. Consequently, $S'_{H_i} \sqcap A' = S'_{H_i}$. Also, if $G_i \subset H_i$ for all $G_i \in A'$, then we get $S'_{H_i} \sqcap A' \subseteq S'_{H_i}$. Suppose, neither $H_i \subseteq G_i$ nor $G_i \subset H_i$, then also we get $S'_{H_i} \sqcap A' \subseteq S'_{H_i}$.

Remark 7. *More generally, $(S', \cup, \cap, (\emptyset, \emptyset), G)$ and $(P(S'), \sqcup, \sqcap, \emptyset, S')$ are semirings. Define $f : S' \to P(S')$ by $f(H) = P(S'_H)$, where $P(S'_H)$ is the power set of S'_H, and S'_H is the set of all subgraphs of H each of whose vertex set is $V(H)$, then f is a semiring homomorphism. The range (image) of f is a subset of $P(S')$ denoted by $f(S') = \{P(S'_H) \in P(S') : P(S'_H) = f(H)$ for some $H \in S'\}$. Let $S'(H)$ be the set of all subgraphs of H; then we have $S'_H \subseteq S'(H)$.*

Proposition 4.4. *Consider a map $f : S' \to P(S')$ such that for all $H \in S'$, define $f(H) = S'(H)$, where $S'(H)$ is the set of all subgraphs of H. Then f is a one-one homomorphism and $(f(S'), \sqcup, \sqcap)$ is an ideal of $(P(S'), \sqcup, \sqcap)$.*

Proof. It is defined that $f(H) = S'(H)$, where $S'(H)$ is the set of all subgraphs of H. We get $f((\emptyset,\emptyset)) = \{(\emptyset,\emptyset)\}$ and $f(G) = S'$, where $(\emptyset,\emptyset)$ and G are additive identity and multiplicative identity, respectively in S'. For any $H_1, H_2 \in S$, $f(H_1 \cup H_2) = S'(H_1 \cup H_2) = S'(H_1) \sqcup S'(H_2) = f(H_1) \sqcup f(H_2)$. Similarly, we get $f(H_1 \cap H_2) = f(H_1) \sqcap f(H_2)$. Therefore, f is a semiring homomorphism. Also, it is observed that for all $H_1, H_2 \in S'$, $H_1 \subseteq H_2$ implies that $f(H_1) \subseteq f(H_2)$. Hence f preserves the semiring operations and the order relations too. By definition, f maps an arbitrary graph H of S' to the set $S'(H)$, which exclusively contains all subgraphs of H. But, such a set is unique for each graph H, hence f is one-one. Since $f((\emptyset,\emptyset)) = \{(\emptyset,\emptyset)\} \in f(S')$, $f(S') \neq \emptyset$. Let $f(H_1), f(H_2) \in f(S')$, then $f(H_1) \sqcup f(H_2) = S'_{H_1} \sqcup S'_{H_2} = S'_{H_1 \cup H_2} = f(H_1 \cup H_2) \in f(S')$ (since $H_1 \cup H_2 \in S' \Longrightarrow f(H_1 \cup H_2) \in f(S')$). Let $f(H_1) \in f(S')$ and $A' \in P(S')$. If for a graph $H_i \in S'$, there exists a graph $G_i \in A'$ such that $H_i \subseteq G_i$, then $H'_j \subseteq G_i$ for all $H'_j \in S'_{H_i}$, hence by definition of $\sqcap$, we get $S'_{H_i} \sqcap A' = S'_{H_i} = f(H_i) \in f(S')$, that is, $f(H_i) \sqcap A' \in f(S')$. Also, if $G_i \subset H_i$, then $S'_{H_i} \sqcap A' \subseteq S'_{H_i}$, which implies that $f(H_i) \sqcap A' \subseteq f(H_i) \in f(S')$. Even if the graphs H_i and G_i are such that neither $H_i \subseteq G_i$ nor $G_i \subset H_i$, it doesn't create any problem to see that $f(H_i) \sqcap A' \subseteq f(H_i)$, which is in $f(S')$. Hence $f(S')$ is the right ideal of $P(S')$. Similarly, $f(S')$ holds good as the left ideal of $P(S')$. □

Remark 8. *Alternatively, the homomorphism f being one-one can also be shown as follows. Let $H, H' \in S'$ be arbitrary such that $\{h_1, h_2, ..., h_m\}$ and $\{h'_1, h'_2, ..., h'_m\}$ are the sets of all subgraphs of H and H', respectively. Let $f(H) = f(H')$, which implies that the set of all the subgraphs of H =the set of all the subgraphs of H', that is, $\{h_1, h_2, ..., h_m\} = \{h'_1, h'_2, ..., h'_m\}$. Therefore, $h_1 \cup h_2 \cup ... \cup h_m = h'_1 \cup h'_2 \cup ... \cup h'_m$ or $H = H'$. Hence f is one-one.*

Proposition 4.5. *Let G be a graph and S be the set of all subgraphs of G. Let S' be the power set of S; define a map $g : S' \longrightarrow S$ by $g(S'_i) = \cup_{G' \in S'_i} G'$, for all $S'_i \in S'$. Then g is a homomorphism.*

Proof. Since $\{(\emptyset,\emptyset)\} \in S'$, we get $g(\{(\emptyset,\emptyset)\}) = (\emptyset,\emptyset) \in S$ and $g(S)$ =the union of all the graphs of S=G. Let $S'_p = \{G_1, G_2, ..., G_p\}$ and $S'_q = \{H_1, H_2, ..., H_q\}$. Now,

$$\begin{aligned}
g(S'_p \sqcup S'_q) &= g[\{G_1, G_2, ..., G_p\} \sqcup \{H_1, H_2, ..., H_q\}] \\
&= g[(G_1 \cup H_1), (G_1 \cup H_2), ..., (G_1 \cup H_q), ..., (G_p \cup H_p)] \\
&= (G_1 \cup H_1) \cup (G_1 \cup H_2) \cup ... \cup (G_1 \cup H_q) \cup ... \cup (G_p \cup H_1) \\
&\quad \cup (G_p \cup H_2) \cup ... \cup (G_p \cup H_q) \\
&= (G_1 \cup G_2 \cup ... \cup G_p) \cup (H_1 \cup H_2 \cup ... \cup H_q) \\
&= g(S'_p) \cup g(S'_q).
\end{aligned}$$

Also,

$$\begin{aligned}
g(S'_i \sqcap S'_j) &= g[\{G_1, G_2, ..., G_p\} \sqcap \{H_1, H_2, ..., H_q\}] \\
&= g\{G_1 \cap H_1, G_1 \cap H_2, ..., G_1 \cap H_q, ..., G_p \cap H_1, G_p \cap H_2), \\
&\quad ..., G_p \cap H_p\} \\
&= (G_1 \cap H_1) \cup (G_1 \cap H_2) \cup ... \cup (G_1 \cap H_q) \cup ... \cup (G_p \cap H_1) \\
&\quad \cup (G_p \cap H_2) \cup ... \cup (G_p \cap H_q) \\
&= (G_1 \cup G_2 \cup ... \cup G_p) \cap (H_1 \cup H_2 \cup ... \cup H_p) \text{ (since } \cap \\
&\quad \text{distributes over } \cup) \\
&= g(S'_i) \cap g(S'_j).
\end{aligned}$$

□

4.3. Homomorphism of Semigroup of Graphs

A complete graph K_n with n vertices can be expressed as a join graph of its two complete subgraphs, namely, K_i and K_j with i and j vertices, respectively. In other words, if K_n is a complete graph with n vertices, then there exist $K_i \subseteq K_n$ and $K_j \subseteq K_n$ such that $K_n = K_i \nabla K_j$.

Proposition 4.6. *Let (S, ∇) and (S_k, ∇) be two semigroups of graphs, where each graph of S_k is complete and S is the set of all simple undirected graphs. Define $f : S \to S_k$ by $f(G_i) = K_{\chi(G_i)}$, where $\chi(G_i)$ is the chromatic number of G_i. Then f is a semigroup homomorphism.*

Proof. Let $G_p, G_q \in S$ be any two non-empty graphs with p and q vertices. Note that for empty graphs, the case is trivial. Now,

$$\begin{aligned}
f(G_p \nabla G_q) &= f(G_r), \text{ where } 1 \leq p, q \leq r \leq p+q \\
&= K_{\chi(G_r)} = K_{\chi(G_s)} \nabla K_{\chi(G_t)}, \text{ where } 1 \leq s, t \leq r \quad \text{Without} \\
&= f(G_s) \nabla f(G_t).
\end{aligned}$$

loss of generality, let $p = s$ and $q = t$ (since $1 \leq p, q \leq r$ and $1 \leq s, t \leq r$). Then from Equation 4.3, we get $f(G_p \nabla G_q) = f(G_p) \nabla f(G_q)$ for all $G_p, G_q \in S$. □

Corollary 4.7. *The homomorphism f may not be an injective morphism.*

Proof. Let $f(G_i) = f(G_j)$, then $K_{\chi(G_i)} = K_{\chi(G_j)}$, which implies that $\chi(G_i) = \chi(G_j)$, i.e., the graphs G_i and G_j have the same chromatic numbers, but the chromatic numbers of G_i and G_j in no ways guarantee that the graph G_i is the same as the graph G_j. In other words, it is possible that $G_i \neq G_j$ for some non-negative integers i and j. Hence f may not be an injective morphism. $\square$

5. Graph Union and Intersection in Matrix Form

For a given symmetric binary relation E on $\mathbf{Z}_0$, let S_E be the set of simple undirected graphs with each vertex set $[n_i]$ for all $n_i \in \mathbf{Z}_0$. Let M_E be the set of all adjacency matrices of the graphs of S_E, and let a matrix $[a_{ij}]_{n\times n}$ in M_E corresponds to the graph $G = ([n], E_n) = (\{0, 1, ..., n-1\}, E)$ with n vertices. Then the addition $\oplus$ and multiplication $\otimes$ on M_E is defined as follows.

For all $[a_{ij}]_{m\times m}, [b_{ij}]_{n\times n} \in M, [a_{ij}]_{m\times m} \oplus [b_{ij}]_{n\times n} = [c_{ij}]_{p\times p};\ p = max\{m, n\}$

$$c_{ij} = \begin{cases} max\{a_{ij}, b_{ij}\}, & \text{if } i, j \leq min\{m, n\}; \\ a_{ij}, & \text{if } n < i, j \leq m; \\ b_{ij}, & \text{if } m < i, j \leq n. \end{cases}$$

Also, $[a_{ij}]_{m\times m} \otimes [b_{ij}]_{n\times n} = [d_{ij}]_{q\times q}$; $q = min\{m, n\}$ and $d_{ij} = min\{a_{ij}, b_{ij}\}$.
Let us consider two graphs $G_1 = ([4], E_1)$ and $G_2 = ([3], E_2)$, where
$E_1 = \{(a, b) :\ a, b \text{ are coprimes in } [4]\}$ and $E_2 = \{(p, q) :\ p, q \text{ are coprime in } [3]\}$.
That is, the vertex sets are $[4] = \{0, 1, 2, 3\}$ and $[3] = \{0, 1, 2\}$. The following are illustrations.

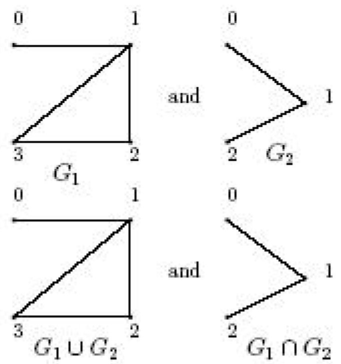

Figure 2: Co-prime Graphs, and their Union and Intersection

Then the adjacency matrices of the graphs $G_1 \cup G_2$ and $G_1 \cap G_2$ are

$$A(G_1 \cup G_2) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \text{ and } A(G_1 \cap G_2) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Also, we have $A(G_1) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} = [a_{ij}]_{4\times 4}$, and $A(G_2) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = [b_{ij}]_{3\times 3}$. By using the definition of the operations $\oplus$ and $\otimes$ on M, we get $[a_{ij}]_{4\times 4} \oplus [b_{ij}]_{3\times 3} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$ and $[a_{ij}]_{4\times 4} \otimes [b_{ij}]_{3\times 3} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

6. Some Artificial Exemplar Problems

By definition of homomorphism g, each graph $G_i \in S$ is considered to be generated by the union of all the graphs of S_i'. That is, the pre-image $S_i' \in S'$ has a homomorphic image $G_i \in S$ or, the union of all the graphs of S_i' is the same as the graph $G_i \in S$. Hence the chromatic number of the union of all the graphs of S_i' is the same as the chromatic number of $G_i \in S$. As a consequence of this definition, the following examples follow.

Example 6.1. *Let G be a locality, where each individual has some food habits, which may be influenced by the other people he/she is associated with. A particular food habit in that locality will be considered the best if it is common to most people. By considering each people in G as a node and any pair of nodes are connected if and only if the corresponding pair of individuals have different food habits, the locality G can be appropriately considered as a graph. Suppose that a Registered Dietitian*

(RD) plans to promote common minimum best food habits in the locality through community outreach programs. The challenge before the RD is; how the whole people of that locality be chosen for the most effective implementation of their mission? There are various options for the RD. Select all those with a common food habit or some of them with a common food habit or, non of them with a common food habit; choose some people where all of them have a common food habit or some of them have a common food habit or none of them have a common food habit and so forth. Otherwise, the RD can also quit its mission by not considering the people of that locality, and in this case, the graph G becomes an empty graph. The task of the selection process is the same as finding all the subgraphs of G. Let the selected groups be in a set S. The RD further plans to schedule different batches for imparting nutritional awareness batch-wise to different groups of people. Now, the RD has new challenges in accommodating different groups of people in different batches as to how many groups of people are to be taken at a time in a single batch? The options are: choose all the groups of people of S at a time (let this choice be denoted by S, the whole set) or choose some groups at a time and so on; otherwise, the RD has also the option of doing away with the scheduling of batches, and in this case, the choice will be denoted by an empty set $\emptyset$. The act of this choice is the same as finding the power set of S, and let it be S'. The problem is to find the group with highest percentage of common food habits in a single batch with k minimum different food habits. This problem can be solved by defining a homomorphism $g : S' \longrightarrow S$ by $g(S'_i) = \cup_{G' \in S'_i} G'$, for all $S'_i \in S'$. For instance, if $\chi(g(S'_1)) = \chi(g(S'_2)) = ... = \chi(g(S'_p)) = k$, where $S'_1, S'_2, ..., S'_p \in S'$ and $g(S'_1), g(S'_2), ..., g(S_p) \in S$. Then the required number of people in a single batch with k number of food habits=$max\{|V(g(S'_1))|, |V(g(S'_2))|, ..., |V(g(S'_p))|\}$, where $|V(g(S'_i))|$ is the number of vertices in the graph $(g(S'_i))$.

The following example is an attempt to visualize the problem for more clarity.

Example 6.2. *Let us consider six different groups of people, namely, $G_1, G_2, ..., G_6$ based on certain criterion. Each node represents an individual and each edge indicates that the pair of individuals have different food habits (or no common food habits). Supposing that considering all six groups at a time for a training (or meeting) is not feasible for the RD. Also, it may be supposed that conducting the meetings for each group separately at different times is more time consuming and have more other constraints.*

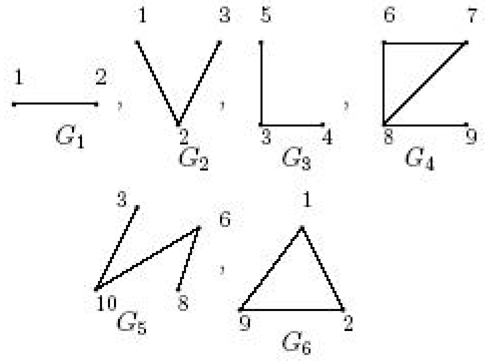

Figure 3: *Six Different Group of People to be Trained*

Without loss of generality, let us assume that the best option for the RD is to schedule three batches, namely, $S'_1 = \{G_1, G_2, G_3\}$, $S'_2 = \{G_1, G_4, G_5\}$ *and* $S'_3 = \{G_2, G_4, G_5, G_6\}$. *Then, we have the following outcomes.*

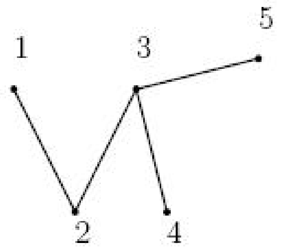

Figure 4: $g(S'_1) = G_1 \cup G_2 \cup G_3$

Here, the chromatic number of the graph is, $\chi(g(S'_1)) = 2$, *which is the minimum required food habits (or aspects) on which the people are to be trained. The number of people to be trained is* 5. *The percentage of the pair of people with common food habits (or required same training) can be calculated as follows. Recall that a pair of people will have same food habits (or required same training) if they are non-adjacent. In other words, any two vertices can be assigned the same color if they are disjoint (in terms of definition of chromatic number). Here, the maximum number of edges the graph* $g(S'_1)$ *can have is* 10 *and the actual number of the edges of* $g(S'_1)$ *is* 4. *Therefore, the percentage of non-adjacent pair of people is* $\{\frac{10-4}{10}\} \times 100\% = 60\%$, *which is the degree of agreement or commonness of the people in regard to their food habits.*
Similarly, we have

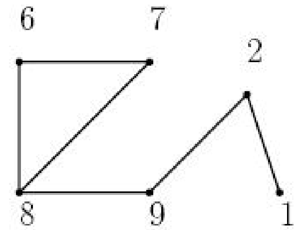

Figure 5: $g(S'_2) = G_1 \cup G_4 \cup G_5$

Here, the chromatic number of the graph is, $\chi(g(S'_2)) = 3$, *which is the minimum required food habits (or aspects) on which the people are to be trained. The number of people to be trained is* 6. *The percentage of non-adjacent pair of people*$=\{\frac{15-6}{15}\} \times 100\% = 60\%$.

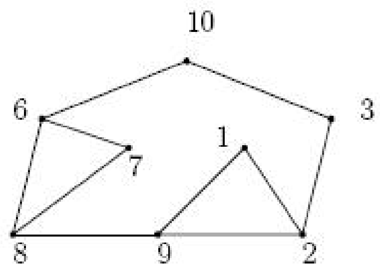

Figure 6: $g(S'_3) = G_2 \cup G_4 \cup G_5 \cup G_6$

Here, the chromatic number of the graph is, $\chi(g(S'_3)) = 3$, *which is the minimum required food habits (or aspects) on which the people are to be trained. The number of people to be trained is* 8. *The percentage of non-adjacent pair of people*$=\{\frac{28-10}{28}\} \times 100\% = 64.28\%$.
Thus, we can conclude that if the RD has to select the group with the highest degree of agreement or commonness in regards to their food habits, then the group S'_3 *must be selected, and in that case the training required to be imparted on atleast three different aspects (which is the minimum colors required to color the graph representing the network of people).*

Example 6.3. *Local security threats to the state and local government can come in various ways and from different motives, individuals or, organizations. Let us consider a town or a locality for a security review.*

Suppose that if a suspicious individual (suspected of illegal activities) is spotted in a locality, then that individual is connected to each neighbors living within a certain radius (as deemed fit) by an edge. Means, if A is a suspected individual in a locality, and P, Q, S, etc. are neighbors living within a certain radius from A, there must be an edge between A to all of these neighbors. The following are some illustrations of such networks.

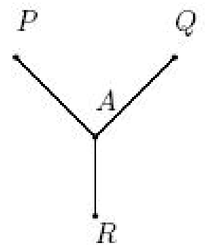

Figure 7: *A simple illustrative network.*

This graph will be a 2-chromatic graph, irrespective of the number of neighbors, unless there is an edge between any two neighbors. Any two neighbors will be connected by an edge if there are any chances of clashes or anything that may resort to law and order problems in the locality due to the two neighbors under consideration. Similarly, if there are two suspected individuals, namely, A and B, in a colony, and P, Q, R, S, etc. are neighbors, then there must be an edge between A and B, and both of this suspected individuals must be connected to every other neighbor by an edge as shown in figure (8). Further, if we consider that A, B, and C are suspected individuals and neighbors are P, Q, R, etc. and if there is no security threat from the external or from among the neighbors, then the figure (9) is the required graph of chromatic number 4.

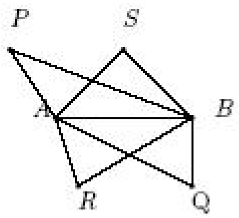

Figure 8: *3-chromatic*

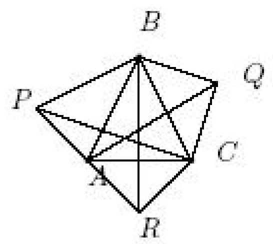

Figure 9: *4-chromatic*

In each of these three instances, the number of edges in the respective graphs will increase with the increase in security threats or potential law and order problems. With an increase in the edges, the chromatic numbers of those graphs will also increase. Hence each graph's chromatic number can be appropriately approximated as the minimum number of security personals to be deployed in that locality to monitor the situation. These motivations will consequently lead to the application of proposition 4.6.
An intelligence report on security review of a region S suggests that the number of suspected individuals and other potentially illegal activities identified at various locations/localities are from zero to n, and the number of localities reviewed in the region is also n. That is to say, if G_0 is a security network of a locality where no suspected individuals or, potential security threat in any form is found; G_1 is the security network of another locality where one suspected individual is found; G_2 is a security network of a locality where two suspected individual is found, and likewise; G_n is a network of locality where n (the highest number of suspected) individual of illegal activities, or other potential security threats are found. Note that among n localities in the region, there are some localities where no particular suspected individuals of illegal activities are recorded, but still not completely free from security threats due to some potential clashes or misunderstandings among the neighbors. On the other hand, there are also some localities where particular suspected individuals and threats from the neighbors are simultaneously observed. Considering a set S, where each of its element is a security network represented by a graph. That is, $S = \{G_0, G_1, ..., G_n\}$, where each vertex set viz., $V(G_0), (G_1), ..., (G_n)$ are distinct, and no two vertex sets have a common vertex. Let $S_k = \{K_0, K_1, ..., K_m\}$; $m \geq n$, where each $K_j \in S_k$ represents a group of personals taken at a time such that an edge for all j connects all the security personal in K_j, and hence

K_j is a complete graph. No single security personal can be in more than one group at a time; that is, no two graphs in S_k will have a common vertex. Thus, the minimum number of security personals to be deployed in a locality, denoted by graph $G_i \in S$ is the chromatic number of G_i. Hence any act of deployment of the security personals in the region S can be taken as a function $f : S \to S_k$ defined by $f(G_i) = K_{\chi(G_i)}$. In other words, the minimum number of security personals required in the locality that requires security network $G_i \in S$ is $\chi(G_i)$. Also, since the function f in this particular example is one-one, the set of networks of the security personals to be hired at a time for a minimum-security coverage of the region is $f(S) \subseteq S_k$. Note that in all localities, no direct external threats are considered in the security reviews.

7. Conclusion and Future Direction

This paper introduces a new direction of studying network problems in a generalized algebraic context (semiring). Though some instances of possible applications of the rules of semiring are highlighted, the emphasis is more on theoretical prospects of algebraic structures on graphs. In the line of this work, we will also explore lattices and Boolean algebra of graphs in the future. And their possible applications in logic circuit designing and cryptography by transforming Boolean functions into simple graphs called truth graphs or reduced truth graphs (this work is in progress). Apart from this, semiring homomorphisms discussed here will be used in dealing with social network problems like Facebook, etc.(this work is also in progress).

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