Proyecciones Journal of Mathematics
Vol. 42, N^o 6, pp. 1435-1465, December 2023.
Universidad Católica del Norte
Antofagasta - Chile

10.22199/issn.0717-6279-5130

Characterizations of generalized Fuzzy γ^*-closed sets

Baby Bhattacharya
NIT Agartala, India
Birojit Das
Tripura University, India
G. Sree Anusha
Iit Hyderabad, India
Jayasree Chakraborty
NIT Agartala, India
Received : September 2021. Accepted : June 2023

Abstract

Generalized fuzzy open sets are playing a vital role in the study of fuzzy topological space as well as that of fuzzy bitopological space since its inception. More often, it is reported that fuzzy closed sets are always included in the family of generalized fuzzy closed sets. Very recently, it has appeared that fuzzy γ^*-open sets are incomparable with fuzzy open sets. This paper aims to present three different kinds of fuzzy generalized closed sets in the light of fuzzy γ^*-open set and associated closure operators with the terminologies- generalized fuzzy γ^*-closed set, γ^*-generalized fuzzy closed set and γ^*-generalized fuzzy γ^*-closed set and it is found that the relation between any two concepts is not necessarily linear. Also, the interrelationships among them are established along with suitable counter examples which are properly placed to make the paper self-sufficient.

Keywords: *Generalized fuzzy γ^*-closed set; γ^*-generalized fuzzy closed set; γ^*-generalized fuzzy γ^*-closed set.*

M. S. C. 2010: *54E55, 54A40.*

1. Introduction

The notion of fuzzy topological space was first introduced and investigated by Chang in 1968 [9] and soon later, the concept of generalized closed set was initialized by Levine in the year 1970 [19]. After that, many generalizations have been made in this direction in different environments and this generalization of closed sets is also extended in the field of L-fuzzy topological space for instance, we refer to [23]. Balasubramaniam and Sundaram [3] first proposed the idea of generalized closed set in fuzzy topological space and named it generalized fuzzy closed set. Later, Park and Park [21] characterized regular generalized closed set as a weaker form of generalized closed set in fuzzy topological space and recently S. Bhattacharya [7] defined generalized regular closed set, which is different from regular generalized closed set and explored various characterizations of this set in an ordinary topological space followed by B. Bhattacharya et al [6]. Soon after, Shi and Li [24] introduced the notion of compactness in L-fuzzy topological space. Moreover, Bhattacharya [4] has introduced the notion of fuzzy γ^*-open set in the sense of Andrijevic's γ-open sets [1] and for the first time, it is reported that fuzzy γ^*-open sets are incomparable with fuzzy open sets. Furthermore, Bhattacharya et al. [5] investigated the notion of γ^*-hyperconnectedness in fuzzy topological space which provides evidence for the existing generalized fuzzy γ^*-closed set. Also, Paul and Bhattacharya [22] introduced new types of functions via γ-open sets in fuzzy bitopological spaces. More works on this particular field may be seen in [14], [15], [16], [17], [23], [18], [24].

This paper aims to propose three different kinds of fuzzy generalized closed sets in the light of fuzzy γ^*-open set and associated closure operators with the terminologies- generalized fuzzy γ^*-closed set, γ^*-generalized fuzzy closed set and γ^*-generalized fuzzy γ^*-closed set respectively. Also, the interrelationships among those are established and verified along with suitable counter examples. Here, the notion of γ^*-generalized fuzzy γ^*-closed set is considered equivalent to the concept of generalized fuzzy γ^*-closed set proposed in [5]. Very often, it is observed that the relation between fuzzy closed set and generalized fuzzy closed set is linear in fuzzy topological space and for instance, one may remember that every fuzzy closed set is generalized fuzzy closed set [3]. In recent days, Das et al. [11], [12], [13] characterized generalized closed set in the light of $(i, j)^*$-fuzzy γ-open set in a given bitopological space. In this treatise, we shall investigate the interrelationships among γ^*-generalized fuzzy closed set, generalized fuzzy γ^*-closed set and γ^*-generalized fuzzy γ^*-closed set and provide logic to support that these concepts are independent of each other. Finally, all these concepts are used to introduce different forms of continuity and their decompositions along with some other results and applications.

Throughout this paper, we refer a fuzzy topological space due to [9] by fts which is traditionally denoted as (X, τ) and very rarely by X. A fuzzy subset is denoted by $\mu,\ \lambda,\ \beta,\ \nu$ and fuzzy topology by $\tau,\ \sigma,\ \eta,\ \chi$. Here, $0_X, 1_X$ denote the null set and X itself respectively. Again, a fuzzy point in X with support $x \in X$ and value p $(0 \leq p \leq 1)$ is denoted by x_p. Also, for a fuzzy subset λ in X, $FC(X)$ denotes the collection of all fuzzy closed sets in the fts and $FO(X)$ denotes the collection of all fuzzy open sets in fts. Furthermore, $1_X - \lambda$ represent the complement of the fuzzy set λ in X. Moreover, closure of μ is denoted by $cl(\mu)$ and fuzzy γ^*-closure of μ is denoted by $cl_{\gamma^*}(\mu)$.

Now, we shall have some definitions which are considered as ready references to explore our research work subsequently.

1.1. Definition [9]

A fuzzy topology τ on X is a collection of fuzzy subsets of I^X, such that
1) $0_X,\ 1_X \in \tau$,
2) if $\lambda, \mu \in \tau$, then $\lambda \wedge \mu \in \tau$ and
3) if $\mu_i \in \tau$, for each $i \in I$, then $\vee_{i \in I}\ \mu_i \in \tau$.
The pair (X, τ) is called a fuzzy topological space. For a fuzzy set λ in X:

(i) fuzzy closure of λ is the intersection of all fuzzy closed sets containing λ and is denoted simply by $cl(\lambda)$,

(ii) fuzzy interior of λ is the union of all fuzzy open sets contained in λ and is denoted simply by $int(\lambda)$.

1.2. Definition [9]

A fuzzy subset λ in a fuzzy topological space (X, τ) is said to be q-coincident with a fuzzy set μ, denoted by $\lambda q \mu$, if there exists a $x_p \in X$ such that $\lambda(x_p) + \mu(x_p) > 1_X$.
A fuzzy subset λ in a fts (X, τ) is called q-neighbourhood (in short, q-nbd) of x_p if there exists a fuzzy open set μ such that $x_p\ q\ \mu \leq \lambda$.

1.3. Definition [4]

A fuzzy subset λ of a fuzzy topological space (X, τ) is said to be fuzzy γ^*-open set if $\lambda \wedge \mu \in FPO(X)$ for each $\mu \in FPO(X)$, where $FPO(X)$ is the family of all fuzzy pre-open sets in X.
A fuzzy subset η is called fuzzy γ^*-closed set if its complement $1_X - \eta$ is a fuzzy γ^*-open set.
The family of all fuzzy γ^*-open sets is denoted by $F\gamma^*O(X)$ and the family of all fuzzy γ^*-closed sets is denoted by $F\gamma^*C(X)$. For a fuzzy set λ in X:

(i) fuzzy γ^*-closure of λ is the intersection of all fuzzy γ^*-closed sets containing λ and is denoted simply by $cl_{\gamma^*}(\lambda)$,

(ii) fuzzy γ^*-interior of λ is the union of all fuzzy γ^*-open sets contained in λ and is denoted simply by $int_{\gamma^*}(\lambda)$.

1.4. Definition [3]

A fuzzy subset λ in a fts (X, τ) is said to be generalized fuzzy closed (in short, gf closed) if $\lambda \leq \mu \Rightarrow cl(\lambda) \leq \mu$, whenever μ is a fuzzy open set. A fuzzy subset η is called generalized fuzzy open (in short gf open) if its complement $1_X - \eta$ is a gf closed set. The family of all gf closed set is denoted by $gf\ C(X)$.

1.5. Definition [3]

A map $f : X \to Y$ from fuzzy topological space (X, τ) into another fuzzy topological space (Y, σ) is called a generalized fuzzy continuous (in short, gf-continuous) function if the inverse image of every fuzzy closed set in Y is a gf closed set in (X, τ).

1.6. Definition [2]

Let (X, τ) be a fuzzy topological space. Then a fuzzy subset λ is said to be fuzzy regular open set in X if $\lambda = int(cl(\lambda))$.

1.7. Definition [6]

A fuzzy subset λ in a fts X is called generalized regular fuzzy closed (in short, grf-closed) set if $Rcl(\lambda) = \mu$, whenever $\lambda \leq \mu$ and μ is a fuzzy open set.

1.8. Definition [21]

A fuzzy subset λ in a fuzzy topological space (X, τ) is called regular generalized fuzzy closed (in short, rgf closed) set if $cl(\lambda) \leq \mu$ whenever $\lambda \leq \mu$ and μ is fuzzy regular open. A fuzzy subset λ is called regular generalized fuzzy open (in short, rgf- open) set if its complement $1_X - \lambda$ is a rgf closed set.

1.9. Definition [5]

A fuzzy subset μ of a fts X is said to be

(i) fuzzy γ^*-dense if $cl_{\gamma^*}(\mu) = 1_X$

(ii) fuzzy γ^*-nowhere dense if $int_{\gamma^*}cl_{\gamma^*}(\mu) = 0_X$.

1.10. Definition [5]

A fts (X, τ) is said to be fuzzy γ^*-hyperconnected space if every non-empty fuzzy γ^*-open set of X is a fuzzy γ^*-dense set in X.

1.11. Definition [5]

A fuzzy subset μ of a fts X is said to be fuzzy pre γ^*-open set (resp. fuzzy pre γ^*-closed set) if $\mu \leq int_{\gamma^*}cl_{\gamma^*}(\mu)$ (resp.$cl_{\gamma^*}int_{\gamma^*}(\mu) \leq \mu$).

2. Generalizations of Fuzzy γ^*-Closed Set

In this section, we introduce and study the conventional and basic properties of γ^*-generalized fuzzy closed set, generalized fuzzy γ^*-closed set and γ^*-generalized fuzzy γ^*-closed set along with few interrelationships between these new sets and already existing ones with suitable counter examples.

2.1. Definition

A fuzzy subset λ in a fts (X, τ) is said to be γ^*-generalized fuzzy closed (in short, γ^*-gf closed) set if $\lambda \leq \mu \Rightarrow cl(\lambda) \leq \mu$, whenever μ is a fuzzy γ^*-open set. A fuzzy set η is called γ^*-generalized fuzzy open (in short γ^*-gf open) set if its complement $1_X - \eta$ is a γ^*-gf closed set. The family of all γ^*-gf closed sets is denoted by γ^*-gf $C(X)$.

2.2. Definition

A fuzzy subset λ in a fts (X, τ) is said to be generalized fuzzy γ^*-closed (in short, $gf\gamma^*$-closed) set if for any fuzzy open set μ in X, $\lambda \leq \mu$ implies $cl_{\gamma^*}(\lambda) \leq \mu$. The family of all $gf\gamma^*$-closed sets is denoted $gf\gamma^*\ C(X)$.

2.3. Definition

A fuzzy subset λ in a fts (X, τ) is said to be γ^*-generalized fuzzy γ^*-closed (in short, γ^*-$gf\gamma^*$-closed) set if for any fuzzy γ^*-open set μ in X, $\lambda \leq \mu$ implies $cl_{\gamma^*}(\lambda) \leq \mu$. The family of all γ^*-$gf\gamma^*$-closed sets is denoted γ^*-$gf\gamma^*C(X)$.

2.4. Remark

Every fuzzy closed set is always a generalized fuzzy closed set but the converse may not be true in general. This result is already proved in [3]. However, we cite an example in this context for our convenience as well as that of the readers.

2.5. Example

We know that closure of any closed set is always in the same set. Hence, the necessary part is obvious. Now, for the converse, we consider a fts (X, τ) with $X = \{x\}$ and $\tau = \{0_X, 1_X, \{(x, 0.4)\}, \{(x, 0.7)\}, \{(x, 0.8)\}\}$. Then, we have $FO(X) = \{0_X, 1_X, \{(x, 0.7)\}, \{(x, 0.8)\}, \{(x, 0.4)\}\}$ and so, $FC(X) = \{0_X, 1_X, \{(x, 0.3)\}, \{(x, 0.2)\}, \{(x, 0.6)\}\}$. Then, we have $gfC(X) = \{(x, \alpha) : \alpha \leq 0.3, 0.4 < \alpha \leq 0.6, \alpha > 0.8\}$. Here, $\lambda_1 = \{(x, 0.9)\}$ is a gf closed set but not a fuzzy closed set.

2.6. Theorem

If λ_1 and λ_2 are any two $gf\gamma^*$-closed sets in a fts X, then $\lambda_1 \vee \lambda_2$ is also a $gf\gamma^*$-closed set in X.

Proof: Let us consider two $gf\gamma^*$-closed sets λ_1 and λ_2 and $\lambda_1 \vee \lambda_2 \leq \mu$, where μ is a fuzzy open set. Now, $\lambda_1, \lambda_2 \leq \mu$ and then, $cl_{\gamma^*}(\lambda_1) \leq \mu$, $cl_{\gamma^*}(\lambda_2) \leq \mu$. Therefore, $cl_{\gamma^*}(\lambda_1 \vee \lambda_2) = cl_{\gamma^*}(\lambda_1) \vee cl_{\gamma^*}(\lambda_2) \leq \mu$. Hence, $\lambda_1 \vee \lambda_2$ is a $gf\gamma^*$-closed set in X. $\square$

2.7. Theorem

Let λ and η be any two γ^*-gf closed sets in a fts (X, τ). Then, $\lambda \vee \eta$ is a γ^*-gf closed set in X.

Proof: Let $\lambda \vee \eta \leq \mu$ and μ is any fuzzy γ^*-open set. Then $\lambda \leq \mu$ and $\eta \leq \mu$ and thus $cl(\lambda), cl(\eta) \leq \mu$. Here, $cl(\lambda \vee \eta) \leq \mu$, since $cl(\lambda) \vee cl(\eta) = cl(\lambda \vee \eta)$. Therefore, $\lambda \vee \eta$ is a γ^*-gf closed set. □

2.8. Theorem

If λ and η are any two γ^*-$gf\gamma^*$-closed sets in a fts X, then $\lambda \vee \eta$ is a γ^*-$gf\gamma^*$-closed set in X.

Proof: Let us consider $\lambda \vee \eta \leq \mu$, where μ is any fuzzy γ^*-open set. Then $\lambda \leq \mu$ and $\eta \leq \mu$ and thus $cl_{\gamma^*}(\lambda)$, $cl_{\gamma^*}(\eta) \leq \mu$. Hence, $cl_{\gamma^*}(\lambda \vee \eta) \leq \mu$, since $cl_{\gamma^*}(\lambda) \vee cl_{\gamma^*}(\eta) = cl_{\gamma^*}(\lambda \vee \eta)$. Consequently, $\lambda \vee \eta$ is a γ^*-$gf\gamma^*$-closed set. □

2.9. Remark

Both the notions of fuzzy γ^*-closed and gf-closed sets are independent of each other. The following example provides the essential evidence.

2.10. Example

We consider a fts (X, τ) with $X = \{x\}$ and
$\tau = \{0_X, 1_X, \{(x, 0.4)\}, \{(x, 0.7)\}, \{(x, 0.8)\}\}$. and so,
$FC(X) = \{0_X, 1_X, \{(x, 0.2)\}, \{(x, 0.3)\}, \{(x, 0.6)\}\}$. Then, we have
$F\gamma^*O(X) = \{0_X, 1_X, \{(x, \alpha) : \alpha > 0.6,\ 0.3 < \alpha \leq 0.4\}\}$ and also,
$F\gamma^*C(X) = \{0_X, 1_X, \{(x, \alpha) : \alpha < 0.4,\ 0.6 \leq \alpha < 0.7\}\}$. Here, $gfC(X) = \{(x, \alpha) : \alpha \leq 0.3, 0.4 < \alpha \leq 0.6, \alpha > 0.8\}$. Clearly, the fuzzy set $\lambda_2 = \{(x, 0.65)\}$ is a fuzzy γ^*-closed set but not a gf-closed set. Again, by considering the same example as previously, the set $\lambda_3 = \{(x, 0.5)\}$ is a gf-closed set but not a fuzzy γ^*-closed set.

2.11. Theorem

Every fuzzy closed set is a γ^*-gf closed set in X.

Proof: We know that closure of any closed set is again a closed set. Thus, it is obvious. □

2.12. Remark

The converse of the above theorem may not be true in general and this result is verified in the following example.

2.13. Example

Consider the same fuzzy topology as in example 2.10. Here, γ^*-$gfC(X) = \{0_X, 1_X, \{(x, \alpha) : \alpha \leq 0.3 \text{ or } 0.4 < \alpha \leq 0.6\}\}$. We consider the fuzzy set $\lambda_4 = \{(x, 0.1)\}$, which is not a fuzzy closed set but it is a γ^*-gf closed set.

2.14. Remark

Both the notions of fuzzy γ^*-closed set and γ^*-gf closed set are independent of each other. The result is verified in the following example.

2.15. Example

Again, we consider the example 2.10. So, $\lambda_2 = \{(x, 0.65)\}$ is a fuzzy γ^*-closed set but not a γ^*-gf closed set by example 2.13. Also, $\lambda_5 = \{(x, 0.45)\}$ is a γ^*-gf closed set but it is not a fuzzy γ^*-closed set.

2.16. Remark

Both the notions of gf closed set and γ^*-gf closed set are completely independent of each other. This result is illustrated in the following examples.

2.17. Example

Again , we consider the example 2.10 and example 2.13. Here, $\lambda_1 = \{(x, 0.9)\}$ is a gf closed set but not a γ^*-gf closed set.

2.18. Example

We consider a fts (X, τ) with $X = \{x, y\}$ and
$\tau = \{0_X, 1_X, \{(x, 0.2), (y, 0.3)\}, \{(x, 0.4), (y, 0.4)\}, \{(x, 0.6), (y, 0.7)\}\}$. Then we have,
$FC(X) = \{0_X, 1_X, \{(x, 0.8), (y, 0.7)\}, \{(x, 0.6), (y, 0.6)\}, \{(x, 0.4), (y, 0.3)\}\}$.
Thus, we get $F\gamma^*O(X) = \{\{(x, \alpha), (y, \delta)\} : \alpha > 0.2, \delta > 0.7\}$ and also

$F\gamma^*C(X) = \{\{(x,\alpha),(y,\delta)\} : \alpha < 0.8, \delta < 0.3\}$. Here, $\mu_1 = \{(x, 0.2), (y, 0.3)\}$ is a γ^*-gf closed but not gf closed since $cl(\mu_1) = \mu_2$, which is not equal to μ_1, where $\mu_2 = \{(x, 0, 4), (y, 0.3)\}$.

2.19. Remark

A fuzzy closed set in a fts is independent of fuzzy γ^*-closed set. This result is proved in [4]. Nevertheless, we verify this again as below using an example for a clear idea.

2.20. Example

We consider the same topology as in example 2.18. Here, $\mu_3 = \{(x, 0.8), (y, 0.7)\}$ is a fuzzy closed set but not a fuzzy γ^*-closed set. On the other hand, $\mu_4 = \{(x, 0.1), (y, 0.1)\}$ is a fuzzy γ^*-closed set but not a fuzzy closed set.

2.21. Remark

Both the notions of fuzzy closed set and $gf\gamma^*$-closed set are independent of each other which is evident in the following examples.

2.22. Example

The above example 2.10 serves the purpose. Here, $gf\gamma^*C(X) = \{(x,\alpha) : \alpha < 0.4, 0.4 < \alpha < 0.7, \alpha > 0.8\}$. Here, $\lambda_1 = \{(x, 0.9)\}$ is $gf\gamma^*$-closed but not fuzzy closed.

2.23. Example

Let us consider example 2.18. We have, $\mu_2 = \{(x, 0.4), (y.0.3)\}$ is a fuzzy closed set. Now, $\mu_2 \leq \mu_5$ but $cl_{\gamma^*}(\mu_2) = 1_X$, which is not equal to μ_5, where $\mu_5 = \{(x, 0.4), (y, 0.4)\}$. So, μ_2 is not a $gf\gamma^*$-closed set.

2.24. Remark

Every fuzzy γ^*-closed set is gf γ^*-closed set but the converse may not be true in general. We illustrate this result in the following example.

2.25. Example

We know that the γ^*-closure of every fuzzy γ^*-closed set is again a fuzzy γ^*-closed set. Thus, the necessary condition is obvious. Now, for the converse part, we consider the example 2.10 to explain the claim. Clearly, $\lambda_1 = \{(x, 0.9)\}$ is a $gf\gamma^*$-closed set but not a fuzzy γ^*-closed set.

2.26. Remark

The notions of gf closed set and $gf\gamma^*$-closed set are independent of each other. The result is verified in the following examples.

2.27. Example

We consider a fts (X, τ) with $X = \{x\}$ and
$\tau = \{0_X, 1_X, \{(x, 0.4)\}, \{(x, 0.7)\}, \{(x, 0.8)\}\}$. Also,
$FC(X) = \{0_X, 1_X, \{(x, 0.3)\}, \{(x, 0.2)\}, \{(x, 0.6)\}\}$ Then, we have
$F\gamma^*O(X) = \{0_X, 1_X, \{(x, \alpha) : \alpha > 0.6,\ 0.3 < \alpha \leq 0.4\}\}$ and also,
$F\gamma^*C(X) = \{0_X, 1_X, \{(x, \alpha) : \alpha < 0.4,\ 0.6 \leq \alpha < 0.7\}\}$. Here, $gf\gamma^*C = \{(x, \alpha) : \alpha < 0.4, 0.4 < \alpha < 0.7, \alpha > 0.8\}$ and thus $gfC(X) = \{(x, \alpha) : \alpha \leq 0.3, 0.4 < \alpha \leq 0.6, \alpha > 0.8\}$. Thus, $\lambda_2 = \{(x, 0.65)\}$ is $gf\gamma^*$-closed set but not gf closed set.

2.28. Example

From example 2.18, we have $\mu_6 = \{(x, 0.3), (y, 0.3)\}$ is a gf closed set. Again, $\mu_6 \leq \mu_5$. But $cl_{\gamma^*}(\mu_6) = 1_X$, which is not equal to μ_5 where $\mu_5 = \{(x, 0.4), (y, 0.4)\}$. So, μ_6 is not a $gf\gamma^*$-closed set.

2.29. Remark

Both the notions of γ^*-gf closed set and $gf\gamma^*$-closed set are independent of each other. This claim is verified in the following examples.

2.30. Example

From example 2.13 and example 2.27, $\lambda_6 = \{(x, 0.85)\}$ is a $gf\gamma^*$-closed set but not a γ^*-gf closed set.

2.31. Example

We consider example 2.18 to prove our claim. Here, we have $\mu_6 = \{(x, 0.3), (y, 0.3)\}$ is γ^*-gf closed set but not $gf\gamma^*$-closed set.

2.32. Remark

The concepts of fuzzy closed set and γ^*-$gf\gamma^*$-closed set are independent from each other. This is verified in the following examples.

2.33. Example

From example 2.27 and considering the same topology, we have $\gamma^* gf\gamma^* C(X) = \{(x, \alpha) : \alpha < 0.4, 0.4 < \alpha < 0.7\}$. Now, $\lambda_4 = \{(x, 0.1)\}$ is γ^*-$gf\gamma^*$-closed set but not fuzzy closed set.

2.34. Example

We consider example 2.18 to explain this result. Here, $\mu_2 = \{(x, 0.4), (y, 0.3)\}$ is fuzzy closed set. Now, $\mu_2 \leq \mu_7$. But, $cl_{\gamma^*}(\mu_2) = 1_X$, which is not equal to μ_7, where $\mu_7 = \{(x, 0.4), (y, 0.75)\}$. Hence, μ_2 is not a γ^*-$gf\gamma^*$-closed set.

2.35. Remark

Every fuzzy γ^*-closed set is γ^*-$gf\gamma^*$-closed set but the converse may not be true in general. This result is verified in the following example.

2.36. Example

Again, we consider example 2.27 and example 2.33. Here, $\lambda_3 = \{(x, 0.5)\}$ is γ^*-$gf\gamma^*$-closed but not a fuzzy γ^*-closed set.

2.37. Proposition

If λ is a gf closed set in a fts X and if $cl(\lambda) \leq cl_{\gamma^*}(\lambda) \leq \mu$, where μ is a fuzzy open set in X, then λ is a $gf\gamma^*$-closed set therein.

The proof is easy and hence left for the readers.

2.38. Theorem

If λ is a $gf\gamma^*$-closed set in a fts X and $\lambda \leq \delta \leq cl_{\gamma^*}(\lambda)$, then δ is a $gf\gamma^*$-closed set in X.

Proof: Let us suppose that $\eta \leq \mu$, where μ is fuzzy open set in X. Now as $\lambda \leq \eta$ and λ is a $gf\gamma^*$-closed set in X, so $cl_{\gamma^*}(\lambda) \leq \mu$. But $cl_{\gamma^*}(\eta) \leq \ cl_{\gamma^*}(\lambda)$ since $\eta \leq cl_{\gamma^*}(\lambda)$. Therefore, $cl_{\gamma^*}(\eta) \leq \mu$ and hence η is a $gf\gamma^*$-closed set in X. □

2.39. Remark

Both the notions of gf closed set and γ^*-$gf\gamma^*$-closed set are independent of each other.

2.40. Example

From example 2.27 and example 2.33, we have $\lambda_6 = \{(x, 0.85)\}$ is a gf closed set but not a γ^*-$gf\gamma^*$-closed set. Now, from example 2.27 we have, $\lambda_7 = \{(x, 0.95)\}$ is a γ^*-$gf\gamma^*$-closed set but not a gf-closed set.

2.41. Remark

Both the notions of γ^*-gf closed set and γ^*-$gf\gamma^*$-closed set are independent of each other.

2.42. Example

We consider example 2.13 and 2.33 to prove our claim. Here, $\lambda_2 = \{(x, 0.65)\}$ is a γ^*-$gf\gamma^*$-closed set but not a γ^*-gf closed set.

2.43. Example

We consider example 2.18, where $\mu_5 = \{(x, 0.4), (y, 0.4)\}$ is a γ^*-gf closed set but not a γ^*-$gf\gamma^*$-closed set.

2.44. Remark

The notions of gf γ^*-closed set and γ^*-gf γ^*-closed set are independent of each other. This result is verified in the following example.

2.45. Example

We consider example 2.27 and example 2.33 to explain this result. Here, $\lambda_1 = \{(x, 0.9)\}$ is a gf γ^*-closed set but not γ^*-gf-γ^*-closed set.

2.46. Remark

Both the notions of gf-γ^*-closed set and rgf-closed set (or grf-closed set) are independent of each other. This result is demonstrated in the following examples.

2.47. Example

Let us consider a fts (X, τ) such that $X = \{x\}$ and $\tau = \{0_X, 1_X, \{(x, 0.2)\}, \{(x, 0.4)\}, \{(x, 0.9)\}\}$. Then we have $FC(X) = \{0_X, 1_X, \{(x, 0.1)\}, \{(x, 0.6)\}, \{(c, 0.8)\}\}$, Also, the collection of all fuzzy regular open sets that is $FRO(X) = \{0_X, 1_X, \{(x, 0.4)\}\}$ and that of fuzzy regular closed sets is $FRC(X) = \{0_X, 1_X, \{(x, 0.6)\}\}$. Thus we have, $F\gamma^*O(X) = \{0_X, 1_X, \{(x, \alpha) : 0.1 < \alpha \leq 0.4 \text{ or } \alpha > 0.8\}\}$ and $F\gamma^*C(X) = \{0_X, 1_X, \{(x, \alpha) : \alpha < 0.2 \text{ or } 0.6 \leq \alpha < 0.9\}\}$.
So, $gf\gamma^*C(X) = \{0_X, 1_X, \{(x, \alpha) : \alpha < 0.2 \text{ or } \alpha > 0.4\}\}$. Here, $\{(x, 0.15)\}$ is a $gf\gamma^*$-closed set but it is neither a rgf-closed set nor a grf-closed set.

2.48. Remark

The concepts of γ^*-gf closed set and rgf-closed set (or grf-closed set) are totally independent of each other. This result is verified through the following example.

2.49. Example

We consider the same topology as in the example 2.47. Here, γ^*-$gfC(X) = \{0_X, 1_X, \{(x, \alpha) : \alpha \leq 0.1 \text{ or } 0.4 < \alpha \leq 0.8\}\}$. Now, $\lambda_9 = \{(x, 0.7)\}$ is a γ^*-gf closed set but it is neither a rgf-closed set nor a grf-closed set. Again, $\lambda_{10} = \{(x, 0.9)\}$ is a rgf-closed set and also a grf-closed set but it is not a γ^*-gf closed set.

2.50. Remark

The notions of γ^*-$gf\gamma^*$-closed set and rgf-closed set (or grf-closed set) are independent of each other. The result is demonstrated in the following example.

2.51. Example

We again consider example 2.47 to verify the result. Here, γ^*-$gf\gamma^*C(X) = \{0_X, 1_X, \{(x,\alpha) : \alpha < 0.2 \text{ or } 0.4 < \alpha \leq 0.9\}\}$. Now,$\lambda_9 = \{(x, 0.7)\}$ is a γ^*-$gf\gamma^*$-closed set but it is neither a rgf-closed set nor a grf-closed set. Again, $\lambda_1 = \{(x, 0.95)\}$ is a rgf-closed set and also a grf-closed set but it is not a γ^*-$gf\gamma^*$-closed set.

2.52. Remark

Both the notions of γ^*-closed set and grf-closed (or rgf-closed) set are independent of each other. This fact is demonstrated in the following example.

2.53. Example

We consider a fts (X, τ) such that $X = \{x\}$ and
$\tau = \{0_X, 1_X, \{(x, 0.2)\}, \{(x, 0.3)\}, \{(x, 0.6)\}\}$. Then we have
$FC(X) = \{0_X, 1_X, \{(x, 0.8)\}, \{(x, 0.7)\}, \{(x, 0.4)\}\}$. Also,
$FRO(X) = \{0_X, 1_X, \{(x, 0.3)\}, \{(x, 0.4)\}\}$ and
$FRC(X) = \{0_X, 1_X, \{(x, 0.6)\}, \{(x, 0.7)\}\}$.
So, we have $F\gamma^*O(X) = \{(x,\alpha) : \alpha \leq 0.3, 0.4 < \alpha \leq 0.6, \alpha > 0.8\}$ and $F\gamma^*C(X) = \{(x,\alpha) : \alpha < 0.2, 0.4 \leq \alpha < 0.6, \alpha \leq 0.7\}$. Also, $RGF(X) = \{(x,\alpha) : 0.3 < \alpha \leq 0.4 \text{ and } \alpha > 0.6\}$ and $GRF(X)\{(x,\alpha) : 0.3 < \alpha \leq 0.4$ and $\alpha > 0.6\}$. Here, $\{(x, 0.5)\}$ is fuzzy γ^*-closed but neither rgf-closed nor grf-closed set. Again, $\{(x, 0.35)\}$ is rgf-closed as well as grf-closed set but not a fuzzy γ^*-closed set.

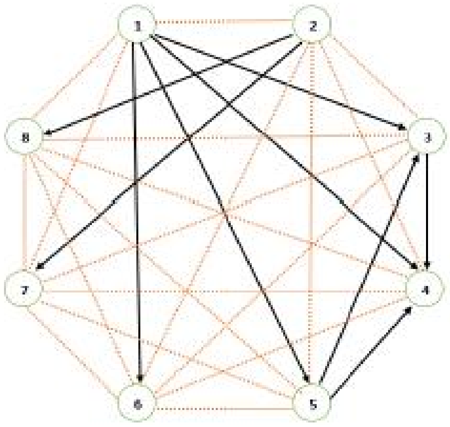

Figure 1 Interrelationships Between Various Closed Sets

Now, we shall depict these interrelationships in the form a graph. In figure 1, the dotted line denote the relations which are independent of each other whereas the one sided relations are represented via directed edges where,
1) fuzzy closed set
2) fuzzy γ^*-closed set
3) gf closed set
4) γ^*-gf closed set
5) $gf\gamma^*$-closed set
6) γ^*-$gf\gamma^*$-closed set
7) rgf-closed set
8) grf-closed set.

2.54. Theorem [5]

Every fuzzy γ^*-nowhere dense set is γ^*-$gf\gamma^*$-closed set in fuzzy topological space (X, τ).

Proof: Let us suppose λ be any fuzzy γ^*-nowhere dense set in a fts (X, τ). Then, $int_{\gamma^*}cl_{\gamma^*}(\lambda) = 0_X$ and there doesnot exist any fuzzy γ^*-open set in between λ and $cl_{\gamma^*}(\lambda)$.Also, we consider a fuzzy γ^*-open set μ such that $\lambda \leq \mu$. Obviously $cl_{\gamma^*}(\lambda) \leq \mu$. Hence, λ is a $\gamma^*gf\gamma^*$-closed set. □

2.55. Theorem [5]

Let (X, τ) be a fts. Then the following conditions are equivalent.

(i) (X, τ) is fuzzy γ^*-hyperconnected.

(ii) Every fuzzy pre γ^*-open set is fuzzy γ^*-dense set.

2.56. Proposition

Every fuzzy pre γ^*-closed set is a $\gamma^*gf\gamma^*$-closed set in a γ^*-hyperconnected space.

Proof: Using theorem 2.6 and proposition 2.37, the above result can easily be verified. □

3. Related Fuzzy Continuous Functions

In this section, we define various fuzzy continuous functions and establish their interrelationships with suitable counter examples. Also, we investigate the composition relation and finally, propose few results based on the notion of fuzzy γ^*-hyperconnectedness.

3.1. Definition [9]

A function $f : X \to Y$ from a fts (X, τ) to another fts (Y, σ) is said to be fuzzy continuous if the inverse image of every fuzzy open set in Y is a fuzzy open set in X.

3.2. Definition [3]

A function $f : X \to Y$ from a fts (X, τ) to another fts (Y, σ) is said to be generalized fuzzy continuous (in short, gf continuous) if the inverse image of every fuzzy closed set in Y is a gf closed set in X.

3.3. Definition [4]

A function $f : X \to Y$ from a fts (X, τ) to another fts (Y, σ) is said to be fuzzy γ^*-continuous if the inverse image of every fuzzy closed set in Y is a fuzzy γ^*-closed set in X.

3.4. Definition

Any function $f : X \to Y$ from a fts (X, τ) to another fts (Y, σ) is said to be γ^*-generalized fuzzy continuous (in short, γ^*-gf continuous) if the inverse image of every fuzzy closed set in Y is a γ^*-gf closed set in X.

3.5. Definition

Any function $f : X \to Y$ from a fts (X, τ) to another fts (Y, σ) is said to be generalized fuzzy γ^*-continuous (in short, $gf\gamma^*$-continuous) if the inverse image of every fuzzy closed set in Y is a $gf\gamma^*$-closed set in X.

3.6. Definition

Any function $f : X \to Y$ from a fts (X, τ) to another fts (Y, σ) is said to be γ^*-generalized fuzzy γ^*-continuous (in short, γ^*-$gf\gamma^*$-continuous) if the inverse image of every fuzzy closed set in Y is a γ^*-$gf\gamma^*$-closed set in X.

3.7. Theorem [3]

Every fuzzy continuous function is a gf continuous function.

3.8. Remark

A gf continuous function may not be a fuzzy continuous function. This result is already proved in [3], yet, we illustrate this result in the following example.

3.9. Example

Suppose two fts (X, τ) and (Y, σ) with $X = \{x\}$ and $Y = \{y\}$ along with $\tau = \{0_X, 1_X, \{(x, 0.4)\}, \{(x, 0.7)\}, \{(x, 0.8)\}\}$, and $\sigma = \{0_Y, 1_Y, \{(y, 0.9)\}\}$. Then, we have $gfC(X) = \{(x, \alpha) : \alpha \leq 0.3, 0.4 < \alpha \leq 0.6, \alpha > 0.8\}$.
Consider a function $f : X \to Y$ such that $f(x) = y$. Obviously, f is a gf continuous function. Take $\beta_1 = \{(y, 0.9)\}$ then $f^{-1}(\beta_1) = \lambda_1$, where $\lambda_1 = \{(x, 0.9)\}$, which is not a fuzzy closed set. Therefore, f is not a fuzzy continuous function.

3.10. Remark

The notions of fuzzy continuous function and fuzzy γ^*-continuous function are independent to each other. Again, this result is considered from [4] and yet an evidence is presented for a better understanding. Thus, the result is demonstrated in the following example.

3.11. Example

Consider three fts (X, τ), (Y, σ) and (Z, χ) with $X = \{x, y\}$,$Y = \{a, b\}$, $Z = \{p, q\}$ and $\tau = \{0_X, 1_X, \{(x, 0.2), (y, 0.3)\}, \{(x, 0.4), (y, 0.4)\},$
$\{(x, 0.6), (y, 0.7)\}\}$, $\sigma = \{0_Y, 1_Y, \{(a, 0.8), (b, 0.7)\}\}$,
$\chi = \{0_Z, 1_Z, \{(p, 0.1), (q, 0.1)\}\}$. Then,
$FC(X) = \{0_X, 1_X, \{(x, 0.8), (y, 0.7)\}, \{(x, 0.6), (y, 0.6)\}, \{(x, 0.4), (y, 0.3)\}\}$.
Then, we have $F\gamma^*O(X) = \{\{(x, \alpha), (y, \delta)\} : \alpha > 0.2, \delta > 0.7\}$ and also $F\gamma^*C(X) = \{\{(x, \alpha), (y, \delta)\} : \alpha < 0.8, \delta < 0.3\}$. Define two functions $f : X \to Y$ and $g : X \to Z$ such that $f(x) = a$, $f(y) = b$, $g(x) = p$ and $g(y) = q$.
It can be easily verified that f is a fuzzy continuous function and g is a fuzzy γ^*-continuous function and let $\nu_1 = \{(a, 0.8), (b, 0.7)\}$ such that $f^{-1}(\nu_1) = \mu_3$, where $\mu_3 = \{(x, 0.8), (y, 0.7)\}$, which is not a fuzzy γ^*-closed

set in X. Thus, f is not a fuzzy γ^*-continuous function. Again, let $\nu_2 = \{(p.0.1), (q, 0.1)\}$ such that $g^{-1}(\nu_2) = \mu_4$, where $\mu_4 = \{(x.0.1), (y, 0.1)\}$, which is not a fuzzy closed set in X. So, g is not a fuzzy continuous function.

3.12. Remark

Both the notions of fuzzy γ^*-continuous function and gf-continuous function are independent from each other. This result is verified in the following example.

3.13. Example

Let us consider three fts (X, τ), (Y, σ) and (Z, χ) with $X = \{x\}$,$Y = \{a\}$, $Z = \{p\}$ and $\tau = \{0_X, 1_X, \{(x, 0.4)\}, \{(x, 0.7)\}, \{(x, 0.8)\}\}$,
$\sigma = \{0_Y, 1_Y, \{(a, 0.65)\}\}$, $\chi = \{0_Z, 1_Z, \{(p, 0.5)\}\}$. We define two functions $f : X \to Y$ and $g : X \to Z$ such that $f(x) = a$ and $g(x) = p$. From example 3.9 and example 3.11, it is clear that f is a fuzzy γ^*-continuous function and g is a gf continuous function respectively and we consider $\beta_2 = \{(a, 0.65)\}$ such that $f^{-1}(\beta_2) = \lambda_2$, where $\lambda_2 = \{(x, 0.65)\}$, which is not a gf closed set in X. Thus, f is not a gf-continuous function. Again, we have $\beta_3 = \{(p, 0.5)\}$ such that $g^{-1}(\beta_3) = \lambda_3$, which is not a fuzzy γ^*-closed set in X. So, g is not a fuzzy γ^*-continuous function.

3.14. Theorem

Every fuzzy continuous function is a γ^*-gf continuous function in a fts X.

Proof: One can easily verify from the definitions and hence left for the readers. □

3.15. Remark

A γ^*-gf continuous function may not be a fuzzy continuous function. This result is illustrated in the following example.

3.16. Example

Let us consider two fts (X, τ) and (Y, σ) with $X = \{x\}$ and $Y = \{y\}$, $\tau = \{0_X, 1_X, \{(x, 0.4)\}, \{(x, 0.7)\}, \{(x, 0.8)\}\}$. $\sigma = \{0_Y, 1_Y, \{(y, 0.1)\}\}$. Then, γ^*-$gfC(X) = \{0_X, 1_X, \{(x, \alpha)\} : \alpha \leq 0.3 \text{ or } 0.4 < \alpha \leq 0.6\}\}$. We define a function $f : X \to Y$ such that $f(x) = y$. Obviously, f is a γ^*-gf continuous function and we consider $\beta_4 = \{(y, 0.1)\}$, $f^{-1}(\beta_4) = \lambda_4$, where $\lambda_4 = \{(x, 0.1)\}$, which is not a fuzzy closed set. Therefore, f is not a fuzzy continuous function.

3.17. Remark

Both the concepts of fuzzy γ^*-continuous function and γ^*-gf continuous function are independent of each other. The result is verified in the following example.

3.18. Example

We consider three fts (X, τ), (Y, σ) and (Z, χ) with $X = \{x\}$,$Y = \{a\}$, $Z = \{p\}$ and $\tau = \{0_X, 1_X, \{(x, 0.4)\}, \{(x, 0.7)\}, \{(x, 0.8)\}\}$, $\sigma = \{0_Y, 1_Y, \{(a, 0.65)\}$, $\chi = \{0_Z, 1_Z, \{(p, 0.45)\}$. We define two functions $f : X \to Y$ and $g : X \to Z$ such that $f(x) = a$ and $g(x) = p$. From example 3.11 and example 3.16, it is evident that f is fuzzy γ^*-continuous function and g is a γ^*-gf continuous function respectively. Now, $f^{-1}(\beta_2) = \lambda_2$, where $\beta_2 = \{(a, 0.65)\}$ and $\lambda_2 = \{(x, 0.65)\}$, while later is not a γ^*-gf closed set in X. As a consequence, f is not a γ^*-gf continuous function. Again, we consider $\beta_5 = \{(p.0.45)\}$ such that $g^{-1}(\beta_5) = \lambda_5$, where $\lambda_5 = \{(x, 0.45)\}$, which is not a fuzzy γ^*-closed set in X. So, g is not a fuzzy γ^*-continuous function.

3.19. Remark

The concepts of gf continuous function and γ^*-gf fuzzy continuous function are independent of each other. This result is illustrated in the following examples.

3.20. Example

Let us consider two fts (X, τ) and (Y, σ) with $X = \{x\}$ and $Y = \{y\}$, $\tau = \{0_X, 1_X, \{(x, 0.4)\}, \{(x, 0.7)\}, \{(x, 0.8)\}\}$ and $\sigma = \{0_Y, 1_Y, \{(y, 0.9)\}\}$. We define a function $f : X \to Y$ such that $f(x) = y$. From example 3.9, it is obvious that f is a gf continuous function. Now, $f^{-1}(\beta_1) = (\lambda_1)$, where $\beta_1 = \{(y, 0.9)\}$ and $\lambda_1 = \{(x, 0.9)\}$, while latter is not a γ^*-gf closed set. Therefore, f is not a γ^*-gf continuous function.

3.21. Example

Let us consider two fts (X, τ) and (Y, σ) with $X = \{x, y\}$ and $Y = \{a, b\}$ such that $\tau = \{0_X, 1_X, \{(x, 0.2), (y, 0.3)\}, \{(x, 0.4), (y, 0.4)\}, \{(x, 0.6), (y, 0.7)\}\}$ $\sigma = \{0_Y, 1_Y, \{(a, 0.2), (b, 0.3)\}\}$. Define a function $f : X \to Y$ such that $f(x) = a$ and $f(y) = b$. From example 2.18, it is clear that, f is a γ^*-gf continuous function. Consider $\nu_4 = \{(a, 0.2), (b, 0.3)\}$. Then, $f^{-1}(\nu_4) = \mu_2$, where $\mu_2 = \{(x, 0.2), (y, 0.3)\}$, which is not a gf closed set. Consequently, f is not a gf continuous function.

3.22. Remark

Both the notions of fuzzy continuous function and $gf\gamma^*$-continuous function are independent of each other. This result is verified in the following examples.

3.23. Example

Let us consider two fts (X, τ) and (Y, σ) with $X = \{x\}$ and $Y = \{y\}$, $\tau = \{0_X, 1_X, \{(x, 0.4)\}, \{(x, 0.7)\}, \{(x, 0.8)\}\}$. $\sigma = \{0_Y, 1_Y, \{(y, 0.9)\}\}$. Here, $gf\gamma^*C = \{0_X, 1_X, \{(x, \alpha)\} : \alpha < 0.4, 0.4 < \alpha < 0.7, \alpha > 0.8\}$. We define a function $f : X \to Y$ such that $f(x) = y$. Clearly, f is a gf γ^*-continuous function. Now, $f^{-1}(\beta_1) = \lambda_1$ as in example 3.9, which is not a fuzzy closed set . Thus, f is not a fuzzy continuous function.

3.24. Example

Let us consider two fts (X, τ) and (Y, σ) with $X = \{x, y\}$ and $Y = \{a, b\}$ such that $\tau = \{0_X, 1_X, \{(x, 0.2), (y, 0.3)\}, \{(x, 0.4), (y, 0.4)\}, \{(x, 0.6), (y, 0.7)\}\}$ and $\sigma = \{0_Y,$
$1_Y, \{(a, 0.4), (b, 0.3)\}\}$. We define a function $f : X \to Y$ such that $f(x) = a$ and $f(y) = b$. Here, f is a fuzzy continuous function and we consider $\nu_3 = \{(a, 0.4), (b, 0.3)\}$ such that $f^{-1}(\nu_3) = \mu_2$, where $\mu_2 = \{(x, 0.4), (y, 0.3)\}$, which is not a $gf\gamma^*$-closed set as in example 3.23. Therefore, f is not a $gf\gamma^*$-continuous function.

3.25. Theorem

Every fuzzy γ^*-continuous function is a $gf\gamma^*$-continuous function in X.

Proof: Proof can be done by usual techniques and hence omitted. □

3.26. Remark

The converse of the above theorem may not be true in general. This is verified in the following example.

3.27. Example

Let us consider two fts (X,τ) and (Y,σ) with $X = \{x\}$ and $Y = \{y\}$, $\tau = \{0_X, 1_X, \{(x,0.4)\}, \{(x,0.7)\}, \{(x,0.8)\}\}$ and $\sigma = \{0_Y, 1_Y, \{(y,0.9)\}\}$. We define a function $f : X \to Y$ such that $f(x) = y$. From example 3.23, it is clear that, f is a $gf\gamma^*$-continuous function. Now, $f^{-1}(\beta_1) = \lambda_1$ as in example 3.9, which is not a fuzzy γ^*-closed set. Therefore, f is not a fuzzy γ^*-continuous function.

3.28. Remark

The notions of gf continuous function and $gf\gamma^*$-continuous function are independent of each other. This result is verified in the following examples.

3.29. Example

Let us consider two fts (X,τ) and (Y,σ) with $X = \{x\}$ and $Y = \{y\}$, $\tau = \{0_X, 1_X, \{(x,0.4)\}, \{(x,0.7)\}, \{(x,0.8)\}\}$ and $\sigma = \{0_Y, 1_Y, \{(y,0.65)\}\}$. We define a function $f : X \to Y$ such that $f(x) = y$. From example 3.23, it is obvious that, f is a $gf\gamma^*$-continuous function. Now, $f^{-1}(\beta_2) = \lambda_2$ as in example 3.11, which is not a gf closed set. Thus, f is not a gf continuous function.

3.30. Example

We consider two fts (X,τ) and (Y,σ) with $X = \{x,y\}$ and $Y = \{a,b\}$ such that $\tau = \{0_X, 1_X, \{(x,0.2),(y,0.3)\}, \{(x,0.4),(y,0.4)\}, \{(x,0.6),(y,0.7)\}\}$ and $\sigma = \{0_Y, 1_Y, \{(a,0.3),(b,0.3)\}\}$. We define a function $f : X \to Y$ such that $f(x) = a$ and $f(y) = b$. Obviously, f is a gf continuous function and we consider $\nu_5 = \{(a,0.3),(b,0.3)\}$ such that $f^{-1}(\nu_5) = \mu_5$, where $\mu_5 = \{(x,0.3),(y,0.3)\}$, which is not a $gf\gamma^*$-closed set. Thus, f is not a $gf\gamma^*$-continuous function.

3.31. Remark

Both the notions of γ^*-gf continuous and $gf\gamma^*$-continuous functions are completely independent of each other. This is illustrated in the following consecutive examples.

3.32. Example

Let us consider two fts (X, τ) and (Y, σ) with $X = \{x\}$ and $Y = \{y\}$, $\tau = \{0_X, 1_X, \{(x, 0.4)\}, \{(x, 0.7)\}, \{(x, 0.8)\}\}$ and $\sigma = \{0_Y, 1_Y, \{(y, 0.85)\}\}$. We define a function $f : X \to Y$ such that $f(x) = y$. From example 3.23, it is evident that, f is a $gf\gamma^*$-continuous function and we consider $\beta_6 = \{(y, 0.85)\}$ such that $f^{-1}(\beta_5) = \lambda_5$, where $\lambda_5 = \{(x, 0.85\}$, which is not a γ^*-gf closed set. Therefore, f is not a γ^*-gf continuous function.

3.33. Example

We consider two fts (X, τ) and (Y, σ) with $X = \{x, y\}$ and $Y = \{a, b\}$ such that $\tau = \{0_X, 1_X, \{(x, 0.2), (y, 0.3)\}, \{(x, 0.4), (y, 0.4)\}, \{(x, 0.6), (y, 0.7)\}\}$ and $\sigma = \{0_Y, 1_Y, \{(a, 0.3), (b, 0.3)\}\}$. We define a function $f : X \to Y$ such that $f(x) = a$ and $f(y) = b$. Here, f is a γ^*-gf continuous function. Now, $f^{-1}(\nu_5) = \mu_5$ as in example 3.30, which is not a $gf\gamma^*$-closed set. So, f is not a $gf\gamma^*$-continuous function.

3.34. Remark

The notions of fuzzy continuous function and $\gamma^* gf\gamma^*$-continuous function are completely independent of each other. This result is demonstrated in the following examples.

3.35. Example

We consider two fts (X, τ) and (Y, σ) with $X = \{x\}$ and $Y = \{y\}$, $\tau = \{0_X, 1_X, \{(x, 0.4)\}, \{(x, 0.7)\}, \{(x, 0.8)\}\}$ and $\sigma = \{0_Y, 1_Y, \{(y, 0.1)\}\}$. Then we have, $\gamma^* gf\gamma^* C(X) = \{0_X, 1_X, \{(x, \alpha)\} : \alpha < 0.4, 0.4 < \alpha < 0.7\}$. We define a function $f : X \to Y$ such that $f(x) = y$. Obviously, f is a γ^*-$gf\gamma^*$-continuous function. Now, $f^{-1}\{(y, 0.1)\} = \{(x, 0.1)\}$ which is not a fuzzy closed set. Therefore, f is not a fuzzy continuous function.

3.36. Example

Let us consider two fts (X, τ) and (Y, σ) with $X = \{x, y\}$ and $Y = \{a, b\}$ such that $\tau = \{0_X, 1_X, \{(x, 0.2), (y, 0.3)\}, \{(x, 0.4), (y, 0.4)\}, \{(x, 0.6), (y, 0.7)\}\}$ and $\sigma = \{0_Y, 1_Y, \{(a, 0.4), (b, 0.3)\}\}$. We define a function $f : X \to Y$ such that $f(x) = a$ and $f(y) = b$. Obviously f is a fuzzy continuous function. Now, $f^{-1}(\nu_3) = \mu_2$, where $\nu_3 = \{(a, 0.4), (b, 0.3)\}$ and $\mu_2 = \{(x, 0.4), (y, 0.3)\}$, which is not a γ^*-$gf\gamma^*$-closed set as in example 3.15. Therefore, f is not a γ^*-$gf\gamma^*$-continuous function.

3.37. Theorem

Let $f : X \to Y$ be a function from a fts (X, τ) to another fts (Y, σ). Then the following statements are equivalent:

(i) The function f is a $gf\gamma^*$-continuous function.

(ii) The inverse image of every fuzzy open set in Y is a $gf\gamma^*$-open set in X.

Proof: The logic behind the proof is very clear and hence left for the readers.

Now, we give another composition without any proof. □

3.38. Theorem

Let $f : X \to Y$ be a function from a fts (X, τ) to another fts (Y, σ) and $g : Y \to Z$ be a function from a fts (Y, σ) to another fts (Z, χ). If f is a $gf\gamma^*$-continuous function and g is any fuzzy continuous function then their composition $g \circ f$ is a $gf\gamma^*$-continuous function.

3.39. Remark

The composition of any two $gf\gamma^*$-continuous functions may not be a $gf\gamma^*$-continuous function. This result is verified in the following example.

3.40. Example

Let us consider three fts (X, τ), (Y, σ) and (Z, χ) with $X = \{x\}$, $Y = \{y\}$, $Z = \{z\}$, $\tau = \{0_X, 1_X, \{(x, 0.2)\}, \{(x, 0.4)\}, \{(x, 0.7)\}\}$,
$\sigma = \{0_Y, 1_Y, \{(y, 0.7)\}, \{(y, 0.1)\}\}$ and $\chi = \{0_Z, 1_Z, \{(z, 0.6)\}, \{(z, 0.8)\}\}$. We define two functions $f : X \to Y$ and $g : Y \to Z$ in such a way that $f(x) = y$, $g(y) = z$. Here, f and g are both $gf\gamma^*$-continuous function. Now, $\delta_1 = \{(z, 0.4)\}$ is fuzzy closed in Z and $(g \circ f)^{-1}(\delta_1) = f^{-1}(g^{-1}(\delta_1)) = f^{-1}(\beta_7) = \lambda_8$ where $\beta_7 = \{(y, 0.4)\}$ and $\lambda_8 = \{(x, 0.4)\}$. But, λ_8 is not a $gf\gamma^*$-closed set in X. Thus, the composition is not a $gf\gamma^*$-continuous function.

3.41. Remark

The composition of any two γ^*-gf continuous functions may not be a γ^*-gf continuous function. This result is illustrated in the following example.

3.42. Example

Let us consider three fts (X,τ), (Y,σ) and (Z,χ) with $X=\{x\}$, $Y=\{y\}$, $Z=\{z\}$, $\tau=\{0_X,1_X,\{(x,0.2)\},\{(x,0.5)\},\{(x,0.7)\}\}$,
$\sigma=\{0_Y,1_Y,\{(y,0.25)\},\{(y,0.5)\},\{(y,0.75)\}$ and $\chi=\{0_Z,1_Z,\{(z,0.9)\}\}$. We define two functions $f:X\to Y$ and $g:Y\to Z$ in such a way that $f(x)=y$, $g(y)=z$. Here, f and g are both γ^*-gf continuous function. Now, $\delta_2=\{(z,0.1)\}$ is fuzzy closed in Z and $(g\circ f)^{-1}(\delta_2)=f^{-1}(g^{-1}(\delta_2))=f^{-1}(\beta_4)=\lambda_4$, where $\beta_4=\{(y,0.1)\}$ and $\lambda_4=\{(x,0.1)\}$. But, λ_4 is not a γ^*-gf closed set in X. Thus, the composition is not a γ^*-gf continuous function.

3.43. Remark

The composition of any two γ^*-$gf\gamma^*$-continuous functions may not be a γ^*-$gf\gamma^*$-continuous function. This result is demonstrated in the following example.

3.44. Example

We consider the same three fts's as considered in example 3.18. We define two functions $f:X\to Y$ and $g:Y\to Z$ such that $f(x)=y$ and $g(y)=z$. Here, f and g are both γ^*-$gf\gamma^*$-continuous function.Now, $\delta_2=\{(z,0.1)\}$ is fuzzy closed in Z and $(g\circ f)^{-1}(\delta_2)=f^{-1}(g^{-1}(\delta_2))=f^{-1}(\beta_4)=\lambda_4$ as in example 3.18. But, λ_4 is not a γ^*-$gf\gamma^*$-closed set in X. Thus, the composition is not a γ^*-$gf\gamma^*$-continuous function.

3.45. Theorem

Every fuzzy γ^*-continuous function is a γ^*-$gf\gamma^*$-continuous function.

Proof: Easy and hence omitted. □

3.46. Remark

The converse of the above theorem may not be true in general. This is verified in the following example.

3.47. Example

Let us consider two fts (X,τ) and (Y,σ) with $X = \{x\}$ and $Y = \{y\}$ $\tau = \{0_X, 1_X, \{(x, 0.4)\}, \{(x, 0.7)\}, \{(x, 0.8)\}\}$. $\sigma = \{0_Y, 1_Y, \{(y, 0.5)\}\}$. Then we have, $\gamma^* gf\gamma^* C(X) = \{0_X, 1_X, \{(x,\alpha)\} : \alpha < 0.4, 0.4 < \alpha < 0.7\}$. We define a function $f : X \to Y$ such that $f(x) = y$. Obviously, f is a γ^*-$gf\gamma^*$-continuous function. Now, $f^{-1}(\beta_3) = \lambda_3$, as in example 3.11, which is not a fuzzy γ^*-closed set. Therefore, f is not a fuzzy γ^*-continuous function.

3.48. Remark

Both the notions of gf continuous function and γ^*-$gf\gamma^*$-continuous function are completely independent of each other. Our claim is illustrated in the following example.

3.49. Example

We consider three fts (X,τ), (Y,σ) and (Z,χ) with $X = \{x\}$,$Y = \{a\}$, $Z = \{p\}$ and $\tau = \{0_X, 1_X, \{(x, 0.4)\}, \{(x, 0.7)\}, \{(x, 0.8)\}\}$, $\sigma = \{0_Y, 1_Y, \{(a, 0.85)\}$, $\chi = \{0_Z, 1_Z, \{(p, 0.35)\}$. We define two functions $f : X \to Y$ and $g : X \to Z$ such that $f(x) = a$ and $g(x) = p$. From example 3.9 and example 3.47, it is clear that, f is a gf continuous function and g is a γ^*-$gf\gamma^*$-continuous function respectively and we consider $\beta_6 = \{(a, 0.85)\}$ such that $f^{-1}(\beta_6) = \lambda_6$, where $\lambda_6 = \{(x, 0.85)\}$, which is not a γ^*-$gf\gamma^*$-closed set in X. Thus, f is not a γ^*-$gf\gamma^*$-continuous function. Again, we consider $\beta = \{(p, 0.35)\}$ such that $g^{-1}(\beta) = \lambda$, where $\lambda = \{(x, 0.35)\}$, which is not a gf closed set in X. So, g is not a gf-continuous function.

3.50. Remark

Both the notions of γ^*-gf continuous function and γ^*-$gf\gamma^*$-continuous function are completely independent of each other. This is verified in the following examples.

3.51. Example

Let us consider two fts (X,τ) and (Y,σ) with $X = \{x, y\}$ and $Y = \{a, b\}$ such that $\tau = \{0_X, 1_X, \{(x, 0.2), (y, 0.3)\}, \{(x, 0.4), (y, 0.4)\}, \{(x, 0.6), (y, 0.7)\}\}$, $\sigma = \{0_Y, 1_Y, \{(a, 0.4), (b, 0.4)\}\}$. We define a function $f : X \to Y$ such that $f(x) = a$ and $f(y) = b$. From example 3.16, it is obvious that, f is a γ^*-gf continuous function and we consider $\nu_6 = \{(a, 0.4), (b, 0.4)\}$ such that

$f^{-1}(\nu_6) = \mu_5$, where $\mu_5 = \{(x, 0.4), (y, 0.4)\}$, which is not a γ^*-$gf\gamma^*$-closed set from example 3.47. Therefore, f is not a γ^*-$gf\gamma^*$-continuous function.

3.52. Example

Let us consider two fts (X, τ) and (Y, σ) with $X = \{x\}$ and $Y = \{y\}$ $\tau = \{0_X, 1_X, \{(x, 0.4)\}, \{(x, 0.7)\}, \{(x, 0.8)\}\}$, $\sigma = \{0_Y, 1_Y, \{(y, 0.65)\}\}$. We define a function $f : X \to Y$ such that $f(x) = y$. From example 3.47, it is evident that, f is a γ^*-$gf\gamma^*$-continuous function. Now, $f^{-1}(\beta_2) = \lambda_2$ as in example 3.11, which is not a γ^*-gf closed set. Thus, f is not a γ^*-gf continuous function.

3.53. Remark

The notions of gf γ^*-continuous and γ^*-$gf\gamma^*$-continuous function are independent of each other. This result is verified in the following example.

3.54. Example

Let us consider two fts (X, τ) and (Y, σ) with $X = \{x\}$ and $Y = \{y\}$ $\tau = \{0_X, 1_X, \{(x, 0.4)\}, \{(x, 0.7)\}, \{(x, 0.8)\}\}$, $\sigma = \{0_Y, 1_Y, \{(y, 0.9)\}\}$. We define a function $f : X \to Y$ such that $f(x) = y$. From example 3.23, it is clear that, f is a gf γ^*-continuous function. Now, $f^{-1}(\beta_1) = \lambda_1$ as in example 3.9, which is not a γ^*-$gf\gamma^*$-closed set. Thus, f is not a γ^*-$gf\gamma^*$-continuous function.

Interested readers can go for examples to show that both the notions of rgf-continuous function (respectively, grf-continuous function) and gf γ^*-continuous functions are independent of each other.

3.55. Remark

The notions of γ^*-gf continuous function and rgf-continuous (or grf-continuous) function are totally independent of each other. The fact is demonstrated via the following example.

3.56. Example

Let us consider three fts (X, τ), (Y, σ) and (Z, χ) with $X = \{x\}$, $Y = \{y\}$ and $Z = \{z\}$ $\tau = \{0_X, 1_X, \{(x, 0.2)\}, \{(x, 0.4)\}, \{(x, 0.9)\}\}$. $\sigma =$

$\{0_Y, 1_Y, \{(y, 0.7)\}\}$ and $\chi = \{0_Z, 1_Z, \{(z, 0.9)\}\}$ along with the fuzzy regular open sets as $FRO(X) = \{0_X, 1_X, \{(x, 0.4)\}\}$ and that of fuzzy regular closed sets as $FRC(X) = \{0_X, 1_X, \{(x, 0.6)\}\}$. We define a function $f : X \to Y$ and $g : X \to Z$ such that $f(x) = y$ and $g(x) = z$ respectively. From example 3.23 and example 2.47, it is obvious that f is a γ^*-gf continuous function and g is a rgf-continuous function respectively. Here, $f^{-1}(\beta_9) = \lambda_9$ is not a rgf-closed set, where $\beta_9 = \{(y, 0.7)\}$ and $\{(x, 0.7)\}$. Thus, f is not a rgf-continuous function. Again, we consider $\beta_{10} = \{(z, 0.9)\}$ such that $g^{-1}(\beta_{10}) = \lambda_{10}$, where $\lambda_{10} = \{(x, 0.9)\}$, which is not a γ^*-gf closed set in X. So, g is not a γ^*-gf continuous function.

3.57. Remark

The concepts of γ^*-$gf\gamma^*$-continuous function and rgf-continuous (or grf-continuous) function are independent of each other. The result is demonstrated in the following example.

3.58. Example

Let us consider three fts (X, τ), (Y, σ) and (Z, χ) with $X = \{x\}$, $Y = \{y\}$ and $Z = \{z\}$ $\tau = \{0_X, 1_X, \{(x, 0.2)\}, \{(x, 0.4)\}, \{(x, 0.9)\}\}$. $\sigma = \{0_Y, 1_Y, \{(y, 0.7)\}\}$ and $\chi = \{0_Z, 1_Z, \{(z, 0.95)\}\}$.
We define a function $f : X \to Y$ and $g : X \to Z$ such that $f(x) = y$ and $g(x) = z$ respectively. From example 3.47 and example 3.56, it is obvious that, f is a γ^*-$gf\gamma^*$-continuous function and g is a rgf-continuous function respectively. $f^{-1}(\beta_9) = \lambda_9$ is not a rgf-closed set, where $\beta_9 = \{(y, 0.7)\}$ and $\{(x, 0.7)\}$. Thus, f is not a rgf-continuous function. Again, we consider $\beta_{11} = \{(z, 0.95)\}$ such that $g^{-1}(\beta_{11}) = \lambda_{11}$, where $\lambda_{10} = \{(x, 0.95)\}$, which is not a γ^*-$gf\gamma^*$-closed set in X. So, g is not a γ^*-$gf\gamma^*$-continuous function.

Note: The above two remarks, are also valid for showing the independence of grf-continuous function with γ^*-gf continuous function and γ^*-$gf\gamma^*$-continuous function.

Finally, we depict these interrelationships between various continuities discussed in the form graph, which is similar to figure 1. The dotted line denote the relations which are independent of each other whereas the one sided relations are represented via directed edges, where the notations imply

the following:
1) fuzzy continuous function
2) fuzzy γ^*-continuous function
3) gf-continuous function
4) γ^*-gf continuous function
5) $gf\gamma^*$-continuous function
6) γ^*-$gf\gamma^*$-continuous function
7) rgf-continuous function
8) grf-continuous function.

4. Conclusion

In this paper, we defined new generalizations fuzzy γ^*-closed sets and cited their interrelationships with already existing fuzzy closed sets. Generally, every fuzzy closed set is a generalized fuzzy closed set but here, we observed that there is a case of non-linearity in their relations. Such an abnormality is also valid in continuity and thus provided valid inferences for our claim. Furthermore, we investigated the notion of regular generalized fuzzy closed set and generalized regular fuzzy closed set in our treatise. This study may be extended to find the equivalences of the reported three generalized closed sets. Moreover, every fuzzy pre closed set is a generalized fuzzy closed set in a fuzzy hyperconnected space [5]. Therefore, in the future endeavours, we shall consider this concept as a base to extend our work in the direction of ω-topological space [10]. Finally, we contemplate that this work is relevant in the study of parallel circuit of electric networks or in parallel topology.

References

[1] D. Andrijevic, "On topology generated by pre-open sets", *Mate Bech*, vol. 39, pp. 367-376, 1987.

[2] K. K. Azad, "On Fuzzy Semicontinuity, Fuzzy Almost Continuity and Fuzzy Weakly Continuity", *Journal of Mathematical Analysis and Applications*, vol. 82, pp. 14-32, 1981.

[3] G. Balasubramanian and P. Sundaram, "On Some Generalizations of Fuzzy Continuous Functions", *Fuzzy Sets and Systems*, vol. 86, pp. 93-100, 1997.

[4] B. Bhattacharya, "Fuzzy Independent Topological Spaces Generated by Fuzzy γ -Open Sets and Their Applications", *Afrika Mathematika,* vol. 28, no. 5, pp. 909-928, 2017.

[5] B. Bhattacharya, J. Chakraborty and A. Paul, "γ -Hyperconnectedness in Fuzzy Topological Spaces", *Journal of Analysis,* vol. 28, pp. 9-19, 2020.

[6] B. Bhattacharya and J. Chakraborty, "Generalized Regular Fuzzy Closed Sets and Their Applications", *The Journal of Fuzzy Mathematics,* vol. 23, no. 1, pp. 227-239, 2015.

[7] S. Bhattacharya, "On Generalized Regular Closed Sets", *International Journal of Contemporary Mathematical Sciences,* vol. 6, no. 3, pp. 145-152, 2011.

[8] J. Cao, M. Ganster and I. Reilly, "On Generalized Closed Sets", *Topology and its Applications,* vol. 123, pp. 37-46, 2002.

[9] C. L. Chang, "Fuzzy Topological Spaces", *Journal of Mathematical Analysis and Applications*, vol. 24, pp. 182-190, 1968.

[10] A. R. Choudhury, A. Mukherjee and M. K. Bose, "Hyperconnectedness and extremal disconnectedness in (a) topological spaces", *Hacettepe Journal of Mathematics and Statistics,* vol. 44, no. 2, pp. 289-294, 2015.

[11] B. Das, B. Bhattacharya, J. Chakaraborty and B.C. Tripathy, "Generalized fuzzy closed sets in a fuzzy bitopological space via γ-open sets", *Afrika Mathematika,* vol. 32, no. 3, pp. 333-345, 2020.

[12] B. Das, B. Bhattacharya, J. Chakaraborty, S. A. Ganapathiraju, and A. Paul, "A new type of generalized closed set via γ-open set in a fuzzy bitopological space", *Proyecciones (Antofagasta, On line),* vol. 38, no. 3, pp. 511-536, 2019. doi: 10.22199/issn.0717-6279-2019-03-0033

[13] B. Das, J. Chakraborty, G. Paul, and B. Bhattacharya, "A new approach for some applications of generalized fuzzy closed sets", *Computational and Applied Mathematics,* vol. 40, no. 2, pp. 1-14, 2020.

[14] B. Das, B. Bhattacharya and A. K. Saha, "On Infi-topological spaces", *Journal of Fuzzy Mathematics,* vol. 25, no. 2, pp. 437-448, 2017.

[15] B. Das, B. Bhattacharya and A. K. Saha, "Some remarks on fuzzy infi topological spaces", *Proyecciones (Antofagasta, On line)*, vol. 40, no. 2, pp. 399-415, 2021. doi: 10.22199/issn.0717-6279-2021-02-0024

[16] B. Das and B. Bhattacharya, "On (i, j) generalized fuzzy γ-closed set in fuzzy bitopological Spaces", *Soft Computing for Problem Solving*, pp. 661-673, 2019.

[17] B. Das, B. Bhattacharya and J. Chakraborty, "A stronger form of generalized closed set via ij-γ-open sets", *Afrika Mathematika*, vol. 33 75, pp. 1-16, 2022, doi: 10.1007/s13370-022-01006-8

[18] A. Dutta and B.C. Tripathy, "On fuzzy b-θ open sets in fuzzy topological space", *Journal of Intelligent and Fuzzy Systems*, vol. 32, no. 1, pp. 137-139, 2017.

[19] N. Levine, "Generalised Closed Sets in Topology", *Rend. Circ. Math. Palermo*, vol. 19, pp. 89-96, 1970.

[20] N. Palaniappan and K. Chandrasekhar Rao, "Regular Generalized Closed Sets", *Kyungpook Mathematical Journal*, vol. 33, no. 2, pp. 211-219, 1993.

[21] J. H. Park and J. K. Park "On Regular Generalized Fuzzy Closed Sets and Generalization of Fuzzy Continuous Functions", *Indian J. Pure Appl. Math.*, vol 34, no. 7:1013-1024, 2003.

[22] A. Paul and B. Bhattacharya, "New Types of Functions and Separation Axioms by γ-Open Sets in Fuzzy Bitopological Spaces", *Malaysian Journal of Mathematical Sciences*, vol. 9, no. 3, pp. 397-407, 2015.

[23] A. A. Ramadan, E. H. Elkordy and M. El-Dardery, "L-Fuzzy Approximation Spaces and L-Fuzzy Topological Spaces", *Iranian Journal of Fuzzy Systems*, vol. 13, no. 1, pp. 115-129, 2016. doi: 10.22111/IJFS.2016.2292

[24] F. G. Shi and R. X. Li, "Compactness in L-Fuzzy Topological Spaces", *Hacettepe Journal of Mathematics and Statistics*, vol. 40, no. 6, pp. 767-774, 2011.

[25] B.C. Tripathy, S. Debnath, "γ-open sets and γ-continuous mappings in fuzzy bitopological spaces", *Journal of Intelligent and Fuzzy Systems*, vol. 24, no. 3, pp. 631-635, 2013.

[26] B.C. Tripathy, S. Debnath, "On fuzzy b-locally open sets in bitopological spaces", *Songklanakarin Journal of Science and Technology*, vol. 37, no. 1, pp. 93-96, 2015.

Baby Bhattacharya
Department of Mathematics,
NIT Agartala
Punjab,
India
e-mail: babybhatt75@gmail.com

Birojit Das
Dr. Ambedkar Centre of Excellence,
Tripura University
Punjab,
India
e-mail: dasbirojit@gmail.com

G. Sree Anusha
Department of Mathematics,
Iit Hyderabad
Punjab,
India
e-mail: sreeanusha17@gmail.com

and

Jayasree Chakraborty
Department of Mathematics,
NIT Agartala
Punjab,
India
e-mail: chakrabortyjayasree1@gmail.com