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Skew lattices

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Abstract

In this paper mainly important properties of skew lattices and symmetric Lattices is obtained. A necessary and sufficient condition for skew lattice to be symmetric is obtained. Maximal element of a skew lattice is also obtained.

Key words : *Skew lattice, symmetric lattice.*

AMS(2000) subject classification : *20 M 10.*

1. Introduction

In general terms , a non-commutative lattice is an algebra (S, ,), where both and are associative ,Idempotent binary operations,connected by absorption laws.Pascal Jordon discussed about non-commutative lattices in 1949, paper[4], work within the scope of jordons approach has been carried out by Gerhardts in e.g[1] and [2]. Further the developments in non-commutative lattice is obtained by schein in[5] and are also obtained by schweigert [7] and [8]. In this paper mainly important properties of skew lattices and symmetric lattices is obtained .

By a skew lattice is meant an algebra (S,) where S is a non-empty set, both , are binary operations called the join and meet respectively,satisifies the following identities.

SL1: (x y) z = x (y z) and (x y) z =x (y z)

SL2: x x = x and x x = x

SL3: x (x y) = x (x y) and (x y) y= y = (x y) y

The identities found in SL1-SL3 are known as the associative law,the idempotent laws and absorption laws respectively.

Assuming the pair of laws ,the absorption laws are equivalent to the following pair of absorption equivalences.

SL31: x y =x if and only if x y = y , x y=y if and only if x y= x

Assuming SL3= x (x y) = x= x (x y) and (x y) y= y = (x y) y

Let x y =x then x y=(x y) y=y Conversely, if x y = y then x y = x (x y) = x

Now, let x y=y then x y = x (x y) = x if x y = x then x y= (x y) y = x if x y = x then x y= (x y) y =y clearly a skew lattice is infact a lattice if it also satisfies the commutative laws L4 : x y= y x and x y= y x Like a lattice, a skew lattice has a natural partial ordering defined by xy whenever x y= y x=y (or)

Equivalently x y= y x =x.

In result(1), it is obtained that on a skew lattice S x y x = (y x) (x y) for all x,y S. In result (4) it is obtained that on a skew lattice S if uAB Where A= and B= then u=(uau) (ubu) and (ubu) (uau)=(uau) (ubu)

A more general form of result (4) is obtained in Lemma(5);

Like a lattice, a skew lattice has a natural partial ordering defined by xy whenever x y = y =y x (or) equivalently x y= y x=x.

It is important to observe that if a skew lattice is a normal band, then x y (x y x) = (x y x) y x = x y x holds for any x,y S, which is obtained in Theorem(7);

In Theorem (10) it is observed that in any skew lattice S if y,z x then y z x and y z x. Congruence extension property for a skew lattice is obtained in Result(13).

It is important to examine that, whether a band can be made in to a skew lattice; this need not hold, for this an example is obtained in Example(14).

A necessary and sufficient for a skew lattice to be symmetric is obtained in Theorem(16). It is important to exercise whether a band can be embedded in any skew lattice,which is not true for this example is obtained in (17). Finally from Theorem(19) it is observed that an element m of a skew lattice S is maximal if and only if m S x is a lattice section of the underlying lattice s/in the skew lattice S.

First we startwith the following Priliminaries

Definition 1.1 : - By a skew lattice is meant an algebra (S,) where S with a pair of associative binary operations , that satisifies the absorption identities x (x y)x=(y x) x and x (x y)=x=(y x) x

Definition 1.2 : A skew lattice is symmetric if it is biconditionally commutative, that is for all pairs x and y: x y =y x if and only if x y= y x

Definition 1.3 : A semi group S is called a normal band if abca=acba for all a,b,c S.

Result 1 : For all x,y S ,we have x y x =(y x) (x y)

Proof : x y x [(y x) (x]=(y x) (x y)=(y x) (x y) and [(y x) (x y)] x y x =(y x) (x y) thus x y x commutes with (y x) (x y) which are inverse of each other and hence they are equal so that x y x =(y x) (x y)

Result 2 : Dually x y x = (y x) (x y) for x,y S

Result 3 : [(x y x) y x] (x y) (x y x)=
[(x y x) y x] (y x) (x y) [(x y) (x y) x)]

Proof : since all three belong to which is a rectangular band with respect to we have (y x) (x y) = x y x by using Result1 and also [(x y) (x y x)] [(x y x) y x] = [(x y) (x y x)] (x y x) [(x y x) (y x)]= x y x

Result 4 : If u A B where A= and B= then u= (uau) (ubu) where = and = and (u bu) (uau)= (uau) (ubu)

Proof : (uau) u, ubu u and since u A B = = we have u ab and hence uau (ab) a u (ba) a u =au a(ab)= a imply that = similarly = and therefore u= (uau) (ubu) The following is the more general form the result4

Lemma 5 : If u is lower bound of a,b where a A and bB and u AB= then u= a b=b a

Proof : We have u a and u b ,so that u ab. Since u(ab) = ub= u and (ab) u = au, now, u and ab which is a rectangular b and with respect to and u and ab commutative and hence they must be equal i.e u= ab similarly u= ba .

bf Remark 6 : For any x,y S ,we have x y x x x y x. Also x y x [(x y x) y x]=(x y x) (y x)

Theorem 7 : If S is normal band then x y(x y x)=(x yx) x y x

Proof : We have xy (x yx) (x yx) y x =(x y) (x yx) y x =x(x yx) yy x=x y x and (x yx) (y x) x y(x yx) =(x yx) y xy(x yx) =(x yx) x y y(x yx) =x y (x yx) ———————————— 1

Also (x yx) y x y (x yx) =(x yx) y y x (x yx) =(x yx) y x ——————————2

From 1 and 2

xy(x yx)= (x yx) (y x)= x y x

Corollary 8 : If (S,) is a normal band ,then x y= y x imply that x y=y x

Proof : Let x,y S be such that x y= y x by Lemma8 x y (x y x)=(x y x) y x and hence x y (y x)=(x y) y x imply that x y = y x

Remark 9 : If x y and either x y= y x or x y= y x,then x=y

Theorem 10 : In any skew lattice if y,z x then y z x and y z x

Proof : Let z y x = z y(y x=y)and x (z y) =z y (zx) (z y) x=z x (yx)=x(zx) and

Now x(z y)= x y(zx)=x(yx) Also if xy ,z then xyz and and x yz then x (y z)= (x y) z = x z=x, (y z) x= y x = x =¿x y z, x(y z)=(x y) z=y z and (y z) x=y(z x)= y z

Theorem 11 : If (S, ,) is a skew lattice then xy(x y) x x, in fact x y (x y) x = x y x and x(y x) y x = x y x

Proof : - we have (x y) (x y) x = x y x (x y) x = x y x x. and x (y x) y x =x (y x) x y x = x y x x

Theorem 12 : If (S,) is a rectangular band ,then for subsemigroup T of S and a T we have JT(a)=JS (a) T

Proof : - obviously JT(a) JS(a) T Let t JS(a) T so that t =x a y T and hence t at = x a y a x a y = x = (a y a x a) y = x a y. x a y = (x a y) 2 = x a y = t so that t JT(a) imply that JS(a) T = JT(a).

Result 13 : If mx m = m for some x, then x m x =x

Proof : - We have m= mx m = (x m) (m x) So that x m=x (m x) imply that x m x = x (m x) x = x x = x

DO bands enjoy C E P

x y = y x =¿ x y = y x x y (x y x) y x x y x y=(y x) (y x)

The following is an example which shows that not every band(S,.) can be made into skew lattice with . As and with any

. a b c d a a a a a b a b b a c a c c a d d d d d

Example 14 :

Let S=a, b, c, d: Define . On S by the composition table as follows

Clearly S can not be made into a skew lattice with a b as a.b=b.a a c since a.c=a=c.a hence bc=b and cb=c imply that bccb

Theorem 15 : In a skew-lattice (S, ,) we have a b=b a for all a,b S if and only if a b= b a and S is in fact a lattice.

Proof : Assume a b =b a for all a,b S then we have a b a=(b a) (a b)=(a b) (b a)=b a b so that a b= a b a b=b a b b=b a b and b a=b a b a= a b a a= a b a=b a b hence a b=b a

Dually a b=b a for all a,b S implies a b = b a for all a,b in S

Theorem 16 : Skew lattice (s, ,) is symmetric if and only is any non empty subset A of S whose elements one either all computer under else all elements commute under .must generate a sub lattice of S

Proof : Let S0 be the sub skew lattice generated by A and let a b= b a for all a,b A then any element of A* can be written as x=a1*a2*..an Where *= or *= By the above x y =(a1*a2*..an) (b1*b2*..bn) = y x hence x y= y x for all x,y A* and thus A* is the sub skew lattice generated A is a lattice

The following example shows that any band cannot be embedded in any skew lattice

Example 17 : Jx= f/f=x x , f is constant f/f:xx such that range of f=1,2, f is identity on the range of f is a band but this can not be imbedded in any skew lattice as f o g o f o h o f f og o h o f , h = constant a f og oh of =,fogofohof=fohof= hence this band cannot be embedded in any skew lattice

Result 18 : For any x,y S, x S x = y S/ y x is a sub skew lattice of S. Dually xy x is a sub skew lattice of S

Theorem 19 : An element m is a maximal if and only if m sx is a lattice section of the underlying lattice s/ in the skew lattice S

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