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Weak forms of double fuzzy α-continuous multifunctions

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Abstract

In this paper, we introduce and study the concept of double fuzzy upper (lower) almost α-continuous multifunctions and double fuzzy upper (lower) weakly α-continuous multifunctions.

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1. Introduction

The fuzzy concept has overrun almost all branches of mathematics since the definition of the concept by Zadeh [30]. Fuzzy sets have applications in many fields such as information [5] and control [3]. The theory of fuzzy topological spaces was defined and developed in the first time by Chang [5] and since then various notions in general topology have been generalized to Chang's fuzzy topological spaces. Resently, Tripathy and Debnath [27, 28, 29] studied various notions in fuzzy bitopological spaces. It has been developed in many directions. Šostak [26] also published a survey article of the developed areas of fuzzy topological spaces. The topologistes used to call Chang's fuzzy topology by "topology" and Kubiak-Šostak's fuzzy topology by "fuzzy topology" where L is any an appropriate lattice. In [3], Atanassove introduced the idea of intuitionistic fuzzy sets, then Coker [6, 7], introduced the concept of intuitionistic fuzzy topological spaces. On the other hand, as a generalization of fuzzy topological spaces Samanta and Mondal [23], introduced the concept of intuitionistic gradation of openness. In 2005, the term intuitionistic is ended by Garcia and Rodabaugh [14]. They proved that the term intuitionistic is unsuitable in mathematics and applications and they replaced it by double. Many other topologies (see [4, 9, 10, 11, 12, 13, 15, 18, 19, 20, 21, 31]) studied various notions in double fuzzy topological space.

2. Preliminaries

Throughout this paper, let X be a non-empty set, I the unit interval [0, 1], $I_0 = (0, 1]$ and $I_1 = [0, 1)$. The family of all fuzzy sets on X is denoted by I^X. By $\bar{0}$ and $\bar{1}$, we denote the smallest and the greatest fuzzy sets on X. For a fuzzy set $\lambda \in I^X, \bar{1} - \lambda$ denotes its complement. Given a function $f : I^X \longrightarrow I^Y$ and its inverse $f^{-1} : I^Y \longrightarrow I^X$ are defined by $f(\lambda)(y) = \bigvee_{f(x)=y} \lambda(x)$ and $f^{-1}(\mu)(x) = \mu(f(x))$, for each $\lambda \in I^X, \mu \in I^Y$ and $x \in X$, respectively. All other notations are standard notations of fuzzy set theory.

Definition 2.1. *[1] Let $F : (X, \tau) \to (Y, \sigma)$, then F is called a fuzzy multifunction if, and only if $F(x) \in I^Y$ for each $x \in X$. The degree of membership of y in $F(x)$ is denoted by $F(x)(y) = G_F(x, y)$ for any $(x, y) \in X \times Y$. The domain of F, denoted by $dom(F)$ and the range of F, denoted by $rng(F)$ for any $x \in X$ and $y \in Y$, are defined by: $dom(F)(x) =$*

$\bigvee_{y\in Y} G_F(x,y)$ *and* $rng(F)(x) = \bigvee_{x\in X} G_F(x,y)$.

Definition 2.2. *[1] Let $F : (X,\tau) \to (Y,\sigma)$ be a fuzzy multifunction. Then F is called:*

1. *normalized if, and only if for each $x \in X$, there exists $y_0 \in Y$ such that $G_F(x,y_0) = \overline{1}$.*
2. *a crisp if, and only if $G_F(x,y_0) = \overline{1}$ for each $x \in X$ and $y \in Y$.*

Definition 2.3. *[1] Let $F : (X,\tau) \to (Y,\sigma)$ be a fuzzy multifunction. Then $F(\lambda)(y) = \bigvee_{x\in X}(G_F(x,y) \wedge \lambda(x))$.the lower inverse of $\mu \in I^Y$ is an L-fuzzy set $F^l(\mu) \in I^X$ defined by $F^l(\mu)(x) = \bigvee_{y\in Y}(G_F(x,y) \wedge \mu(y))$. the upper inverse of $\mu \in I^Y$ is an L-fuzzy set $F^u(\mu) \in I^X$ defined by $F^u(\mu)(x) = \bigwedge_{y\in Y}(G^c_F(x,y) \vee \mu(y))$.*

2. **Definition 2.4.** *[1] Let $F : (X,\tau) \to (Y,\sigma)$ be a fuzzy multifunction. Then*

1. *$F(\lambda_1) \le F(\lambda_2)$ if $\lambda_1 \le \lambda_2$.*
2. *$F^l(\mu_1) \le F^l(\mu_2)$ and $F^u(\mu_1) \le F^u(\mu_2)$ if $\mu_1 \le \mu_2$.*
3. *$F^l(\mu^c) \le (F^u(\mu))^c$.*
4. *$F^u(\mu^c) \le (F^l(\mu))^c$.*
5. *$F(F^u(\mu)) \le \mu$ if F is a crisp.*
6. *$F^u(F(\lambda)) \ge \lambda$ if F is a crisp.*

Definition 2.5. *[1] Let $F : (X,\tau) \to (Y,\sigma)$ and $H : (Y,\sigma) \to (Z,\eta)$ be two fuzzy multifunctions. Then the composition $H \circ F$ is defined by $((H \circ F)(x))(z) = \bigvee_{y\in Y}(G_F(x,y) \wedge G_H(y,z))$.*

Theorem 2.6. *[1] Let $F : (X,\tau) \to (Y,\sigma)$ and $H : (Y,\sigma) \to (Z,\eta)$ be two fuzzy multifunctions. Then we have the following*

1. *$H \circ F = F(H)$.*
2. *$(H \circ F)^u = F^u(H^u)$.*
3. *$(H \circ F)^l = F^l(H^l)$.*

Theorem 2.7. *[1] Let $F : (X, \tau) \to (Y, \sigma)$ be a fuzzy multifunction. Then*

1. $(\bigcup\limits_{i\in\Gamma} F_i)(\lambda) = \bigvee\limits_{i\in\Gamma} F_i(\lambda)$.
2. $(\bigcup\limits_{i\in\Gamma} F_i)^l(\mu) = \bigvee\limits_{i\in\Gamma} F_i^l(\mu)$.
3. $(\bigcup\limits_{i\in\Gamma} F_i)^u(\mu) = \bigvee\limits_{i\in\Gamma} F_i^u(\mu)$.

Definition 2.8. *A fuzzy point x_t in X is a fuzzy set taking value $t \in I_0$ at x and zero elsewhere, $x_t \in \lambda$ if and only if $t \leq \lambda(x)$. $x \in X$ such that $\lambda(x) + \mu(x) > 1$. Otherwise $\lambda\overline{q}\mu$.*

Definition 2.9. *[7] A double fuzzy topology on X is a pair of maps $\tau, \tau^\star : I^X \to I$, which satisfies the following properties:*

1. $\tau(\lambda) \leq \overline{1} - \tau^\star(\lambda)$ *for each* $\lambda \in I^X$.
2. $\tau(\lambda_1 \wedge \lambda_2) \geq \tau(\lambda_1) \wedge \tau(\lambda_2)$ *and* $\tau^\star(\lambda_1 \wedge \lambda_2) \leq \tau^\star(\lambda_1) \vee \tau^\star(\lambda_2)$ *for each* $\lambda_1, \lambda_2 \in I^X$.
3. $\tau(\bigvee\limits_{i\in\Gamma} \lambda_i) \geq \bigwedge\limits_{i\in\Gamma} \tau(\lambda_i)$ *and* $\tau^\star(\bigvee\limits_{i\in\Gamma} \lambda_i) \leq \bigvee_{i\in\Gamma} \tau^\star(\lambda_i)$ *for each* $\lambda_i \in I^X, i \in \Gamma$.

The triplet $(X, \tau, \tau^\star)$ is called a double fuzzy topological space.

Remark 2.10. *Let (X, τ) be a smooth fuzzy topological space. Then for each $r \in I_0$, $\tau_r = \{\mu \in I^X : \tau(\mu) \geq r\}$ is Chang's fuzzy topology on X.*

Definition 2.11. *[8] Let $(X, \tau, \tau^\star)$ be a double fuzzy topological space. For each $\lambda \in I^X$, $r \in I_0$, an operator $C_{\tau,\tau^\star} : I^X \times I_0 \to I^X$ is defined as follows: $C_{\tau,\tau^\star}(\lambda, r, s) = \wedge\{\mu : \mu \geq \lambda, \tau(\overline{1} - \mu) \geq r\}$. For $\lambda, \mu \in I^X$ and $r, s \in I_0$, it satisfies the following conditions:*

1. $C_{\tau,\tau^\star}(\overline{0}, r, s) = \overline{0}$.
2. $\lambda \leq C_{\tau,\tau^\star}(\lambda, r, s)$.
3. $C_{\tau,\tau^\star}(\lambda, r, s) \vee C_{\tau,\tau^\star}(\mu, r, s) = C_{\tau,\tau^\star}(\lambda \vee \mu, r, s)$.
4. $C_{\tau,\tau^\star}(\lambda, r, s) \leq C_{\tau,\tau^\star}(\lambda, s)$ *if* $r \leq s$.
5. $C_{\tau,\tau^\star}(C_{\tau,\tau^\star}(\lambda, r, s), r, s) = C_{\tau,\tau^\star}(\lambda, r, s)$.

Proposition 2.12. *[8] Let $(X, \tau, \tau^\star)$ be a double fuzzy topological space. For each $\lambda \in I^X$, $r \in I_0$, an operator $I_{\tau,\tau^\star} : I^X \times I_0 \to I^X$ is defined as follows: $I_{\tau,\tau^\star}(\lambda, r, s) = \vee\{\mu : \mu \leq \lambda, \tau(\mu) \geq r\}$. For $\lambda, \mu \in I^X$ and $r, s \in I_0$, it satisfies the following conditions:*

1. $I_{\tau,\tau^\star}(\overline{1} - \lambda, r, s) = \overline{1} - C_{\tau,\tau^\star}(\lambda, r, s)$.
2. $I_{\tau,\tau^\star}(\overline{1}, r, s) = \overline{1}$.
3. $\lambda \geq I_{\tau,\tau^\star}(\lambda, r, s)$.
4. $I_{\tau,\tau^\star}(\lambda, r, s) \wedge I_{\tau,\tau^\star}(\mu, r, s) = I_{\tau,\tau^\star}(\lambda \wedge \mu, r, s)$.
5. $I_{\tau,\tau^\star}(\lambda, r, s) \geq I_{\tau,\tau^\star}(\lambda, s)$, *if* $r \leq s$.
6. $I_{\tau,\tau^\star}(I_{\tau,\tau^\star}(\lambda, r, s), r, s) = I_{\tau,\tau^\star}(\lambda, r, s)$.

Definition 2.13. *Let $(X, \tau, \tau^\star)$ be a double fuzzy topological space. For each $\lambda \in I^X$, $r \in I_0$* and *$s \in I_1$,*

1. *λ is called (r, s)-fuzzy α-open if $\lambda \leq I_{\tau,\tau^\star}(C_{\tau,\tau^\star}(I_{\tau,\tau^\star}(\lambda, r, s), r), r)$.*
2. *λ is called (r, s)-fuzzy semi-open if $\lambda \leq C_{\tau,\tau^\star}(I_{\tau,\tau^\star}(\lambda, r, s), r, s)$.*
3. *λ is called (r, s)-fuzzy preopen if $\lambda \leq I_{\tau,\tau^\star}(C_{\tau,\tau^\star}(\lambda, r, s), r, s)$.*
4. *λ is called (r, s)-fuzzy β-open if $\lambda \leq C_{\tau,\tau^\star}(I_{\tau,\tau^\star}(C_{\tau,\tau^\star}(\lambda, r, s), r, s), r, s)$.*
5. *λ is called (r, s)-fuzzy α-closed if $\overline{1} - \lambda$ is (r, s)-fuzzy α-open.*
6. *λ is called (r, s)-fuzzy semiclosed if $\overline{1} - \lambda$ is (r, s)-fuzzy semiopen.*
7. *$\alpha C_{\tau,\tau^\star}(\lambda, r, s) = \wedge\{\mu \in I^X : \mu \geq \lambda$ and μ is (r, s)-fuzzy α-closed$\}$ is called (r, s)-fuzzy α-closure of λ.*
8. *$\alpha I_{\tau,\tau^\star}(\lambda, r, s) = \vee\{\mu \in I^X : \mu \leq \lambda$ and μ is (r, s)-fuzzy α-open$\}$ is called (r, s)-fuzzy α-interior of λ.*
9. *$SC_{\tau,\tau^\star}(\lambda, r, s) = \wedge\{\mu \in I^X : \mu \geq \lambda$ and μ is (r, s)-fuzzy semiclosed$\}$ is called (r, s)-fuzzy semiclosure of λ.*
10. *$SI_{\tau,\tau^\star}(\lambda, r, s) = \vee\{\mu \in I^X : \mu \leq \lambda$ and μ is (r, s)-fuzzy semiopen$\}$ is called (r, s)-fuzzy semiinterior of λ.*

$f : (X, \tau, \tau^\star) \to (Y, \sigma, \sigma^\star)$ be a function. Then

Definition 2.14. *[24] Let $F : (X, \tau, \tau^\star) \to (Y, \sigma, \sigma^\star)$ be a fuzzy multifunction. Then F is called:*

1. *double fuzzy upper α-continuous at a fuzzy point $x_t \in dom(F)$ if $x_t \in F^u(\mu)$ for each $\mu \in I^Y$, $\sigma(\mu) \geq r$ and $\sigma^\star(\mu) \leq s$, there exists an (r, s)-fuzzy α-open set $\lambda \in I^X$ and $x_t \in \lambda$ such that $\lambda \wedge dom(F) \leq F^u(\mu)$.*
2. *double fuzzy lower α-continuous at a fuzzy point $x_t \in dom(F)$ if $x_t \in F^u(\mu)$ for each $\mu \in I^Y$, $\sigma(\mu) \geq r$ and $\sigma^\star(\mu) \leq s$, there exists an (r, s)-fuzzy α-open set $\lambda \in I^X$ and $x_t \in \lambda$ such that $\lambda \leq F^l(\mu)$.*
3. *double fuzzy upper (lower) α-continuous if it is double fuzzy upper (lower) α-continuous at every $x_t \in dom(F)$.*

Proposition 2.15. *[24] If F is normalized, then F is fuzzy upper α-continuous at a fuzzy point $x_t \in dom(F)$ if, and only if $x_t \in F^u(\mu)$ for each $\mu \in I^Y$ and $\sigma(\mu) \geq r$, there exists $\lambda \in I^X$, $\tau(\lambda) \geq r$, $\tau^\star(\lambda) \leq s$ and $x_t \in \lambda$ such that $\lambda \leq F^u(\mu)$.*

3. Double fuzzy almost α-continuous multifunctions

Definition 3.1. *Let $F : (X, \tau, \tau^\star) \to (Y, \sigma, \sigma^\star)$ be a fuzzy multifunction. Then F is called:*

1. *double fuzzy upper almost α-continuous at a fuzzy point $x_t \in dom(F)$ if $x_t \in F^u(\mu)$ for each $\mu \in I^Y$, $\sigma(\mu) \geq r$ and $\sigma^\star(\mu) \leq s$, there exists an (r, s)-fuzzy α-open set $\lambda \in I^X$ and $x_t \in \lambda$ such that $\lambda \wedge dom(F) \leq F^u(SC_{\tau,\tau^\star}(\mu, r, s))$.*
2. *fuzzy lower almost α-continuous at a fuzzy point $x_t \in dom(F)$ if $x_t \in F^u(\mu)$ for each $\mu \in I^Y$, $\sigma(\mu) \geq r$ and $\sigma^\star(\mu) \leq s$, there exists an (r, s)-fuzzy α-open set $\lambda \in I^X$ and $x_t \in \lambda$ such that $\lambda \leq F^l(SC_{\tau,\tau^\star}(\mu, r, s))$.*
3. *fuzzy upper (lower) almost α-continuous if it is fuzzy upper (lower) almost α-continuous at every $x_t \in dom(F)$.*

Proposition 3.2. *If F is normalized, then F is fuzzy upper almost α-continuous at a fuzzy point $x_t \in dom(F)$ if, and only if $x_t \in F^u(\mu)$ for each $\mu \in I^Y$, $\sigma(\mu) \geq r$ and $\sigma^\star(\mu) \leq s$, there exists $\lambda \in I^X$, $\tau(\lambda) \geq r$, $\tau^\star(\lambda) \leq s$ and $x_t \in \lambda$ such that $\lambda \leq F^u(SC_{\tau,\tau^\star}(\mu, r, s))$.*

Theorem 3.3. *For a function $F : (X, \tau, \tau^\star) \to (Y, \sigma, \sigma^\star)$, the following statements are equivalent:*

1. *F is fuzzy lower almost α-continuous.*
2. *$F^l(\mu)$ is (r,s)-fuzzy α-open for any μ is r-fuzzy regular open.*
3. *$F^u(\mu)$ is (r,s)-fuzzy α-closed for any μ is (r,s)-fuzzy regular closed.*
4. *$F^l(\mu) \leq \alpha I_{\tau,\tau^\star}(F^l(SC_{\sigma,\sigma^\star}(\mu,r,s)),r,s)$ if $\sigma(\mu) \geq r$ and $\sigma^\star(\mu) \leq s$.*
5. *$\alpha C_{\tau,\tau^\star}(F^u(SI_{\sigma,\sigma^\star}(\mu,r,s)),r,s) \leq F^u(\mu)$ if $\sigma(\overline{1}-\mu) \geq r$ and $\sigma^\star(\overline{1}-\mu) \leq s$.*

Proof. (1)$\Rightarrow$(2): Let $x_t \in dom(F)$, $\mu \in I^Y$, μ be (r,s)-fuzzy regular open and $x_t \in F^l(\mu)$. Then there exist (r,s)-fuzzy α-open set $\lambda \in I^X$ and $x_t \in \lambda$ such that $\lambda \leq F^l(SC_{\sigma,\sigma^\star}(\mu,r,s)) = F^l(\mu)$. Since $\lambda \leq I_{\tau,\tau^\star}(C_{\tau,\tau^\star}(I_{\tau,\tau^\star}(\lambda,r,s),r,s),r,s) \leq I_{\tau,\tau^\star}(C_{\tau,\tau^\star}(I_{\tau,\tau^\star}(F^l(\mu),r,s),r,s),r,s)$. Therefore, we obtain $F^l(\mu) \leq I_{\tau,\tau^\star}(C_{\tau,\tau^\star}(I_{\tau,\tau^\star}(F^l(\mu),r,s),r,s),r,s)$. Then, $F^l(\mu)$ is (r,s)-fuzzy α-open.
(2) $\Rightarrow$(3): Let $\mu \in I^Y$ and μ be (r,s)-fuzzy regular closed hence by (2), $F^l(\overline{1}-\mu) = \overline{1} - F^u(\mu)$ is (r,s)-fuzzy α-open. Then $F^u(\mu)$ is (r,s)-fuzzy α-closed.
(3) $\Rightarrow$ (4): Let $\mu \in I^Y$, $\sigma(\mu) \geq r$, $\sigma^\star(\mu) \leq s$ and $x_t \in F^l(\mu)$. Then we have $x_t \in F^l(SC_{\sigma,\sigma^\star}(\mu,r,s)) = \overline{1} - [F^u(\overline{1} - SC_{\sigma,\sigma^\star}(\mu,r,s))]$. Since $\overline{1} - SC_{\sigma,\sigma^\star}(\mu,r,s)$ is (r,s)-fuzzy regular closed. Then $F^u(\overline{1} - SC_{\sigma,\sigma^\star}(\mu,r,s))$ is (r,s)-fuzzy α-closed and $F^l(SC_{\sigma,\sigma^\star}(\mu,r,s))$ is (r,s)-fuzzy α-open. Hence $x_t \in \alpha I_{\tau,\tau^\star}(F^l(SC_{\sigma,\sigma^\star}(\mu,r,s)),r,s)$. Then, we obtain
$F^l(\mu) \leq \alpha I_{\tau,\tau^\star}(F^l(SC_{\sigma,\sigma^\star}(\mu,r,s)),r,s)$.
(4) $\Rightarrow$(5): Let $\mu \in I^Y$, $\sigma(\overline{1}-\mu) \geq r$ and $\sigma^\star(\overline{1}-\mu) \leq s$. Then $\overline{1} - F^u(\mu) = F^l(\overline{1}-\mu) \leq \alpha I_{\tau,\tau^\star}(F^l(SC_{\sigma,\sigma^\star}(\overline{1}-\mu,r,s)),r,s) = \alpha I_{\tau,\tau^\star}(F^l(\overline{1} - SI_{\sigma,\sigma^\star}(\mu,r,s)),r,s) = \alpha I_{\tau,\tau^\star}(\overline{1} - [F^u(SI_{\sigma,\sigma^\star}(\mu,r,s))],r,s) =$
$\overline{1} - \alpha C_{\tau,\tau^\star}(F^u(SI_{\sigma,\sigma^\star}(\mu,r,s)),r,s)$. Then $\alpha C_{\tau,\tau^\star}(F^u(SI_{\sigma,\sigma^\star}(\mu,r,s)),r,s) \leq F^u(\mu)$.
(5) $\Rightarrow$(4) This follows from that fact that $\overline{1} - F^l(\mu) = F^u(\overline{1}-\mu)$ for every $\mu \in I^Y$.
(4) $\Rightarrow$ (1) Let $x_t \in dom(F)$, $\mu \in I^Y$,$\sigma(\mu) \geq r$ and $\sigma^\star(\mu) \leq s$ with $x_t \in F^l(\mu)$ by (4), $F^l(\mu) \leq \alpha I_{\tau,\tau^\star}(F^l(SC_{\sigma,\sigma^\star}(\mu,r,s)),r,s)$. Since $x_t \in \alpha I_{\tau,\tau^\star}(F^l(SC_{\sigma,\sigma^\star}(\mu,r,s)),r,s) = \lambda$ (say) and hence λ is (r,s)-fuzzy α-open such that $\lambda \leq F^l(SC_{\sigma,\sigma^\star}(\mu,r,s))$. Then F is fuzzy lower almost α-continuous. $\square$

In view of Theorem 3.3, we formulate the following result without proof.

Theorem 3.4. *For a function $F : (X, \tau, \tau^\star) \to (Y, \sigma, \sigma^\star)$, the following statements are equivalent:*

1. *F is fuzzy upper almost α-continuous.*
2. *$F^u(\mu)$ is (r, s)-fuzzy α-open for any $\mu \in I^Y$ is (r, s)-fuzzy regular open.*
3. *$F^l(\mu)$ is (r, s)-fuzzy α-closed for any $\mu \in I^Y$ is (r, s)-fuzzy regular closed.*
4. *$F^u(\mu) \leq \alpha I_{\tau,\tau^\star}(F^u(SC_{\sigma,\sigma^\star}(\mu, r, s)), r, s)$ if $\sigma(\mu) \geq r$ and $\sigma^\star(\mu) \leq s$.*
5. *$\alpha C_{\tau,\tau^\star}(F^l(SI_{\sigma,\sigma^\star}(\mu, r, s)), r, s) \leq F^u(\mu)$ if $\sigma(\overline{1} - \mu) \geq r$ and $\sigma^\star(\overline{1} - \mu) \leq s$.*

Corollary 3.5. *Let $F : (X, \tau, \tau^\star) \to (Y, \sigma, \sigma^\star)$ be a fuzzy multifunction. Then we have the following*

1. *F is normalized implies F is fuzzy upper almost α-continuous at $x_t \in dom(F)$ if, and only if $x_t \in F^u(\mu)$ for each $\mu \in I^Y$ is (r, s)-fuzzy regular open there exists an (r, s)-fuzzy α-open set $\lambda \in I^X$ and $x_t \in \lambda$ such that $\lambda \leq F^u(\mu)$.*
2. *F is fuzzy lower almost α-continuous at $x_t \in dom(F)$ if, and only if $x_t \in F^l(\mu)$ for each $\mu \in I^Y$ is (r, s)-fuzzy regular open there exists an (r, s)-fuzzy α-open set $\lambda \in I^X$ and $x_t \in \lambda$ such that $\lambda \leq F^l(\mu)$.*

Proposition 3.6.

1. *Every fuzzy upper α-continuous multifunction is fuzzy upper almost α-continuous.*
2. *Every fuzzy lower α-continuous multifunction is fuzzy lower almost α-continuous.*

The following example shows that the converses of the Proposition 3.6 are not true.

Example 3.7. *Let $X = \{x_1, x_2\}$, $Y = \{y_1, y_2, y_3\}$ and $F : (X, \tau, \tau^\star) \to (Y, \sigma, \sigma^\star)$ be a fuzzy multifunction defined by $G_F(x_1, y_1) = 0.1$, $G_F(x_1, y_2) = \overline{1}$, $G_F(x_1, y_3) = 0.3$, $G_F(x_2, y_1) = 0.5$, $G_F(x_2, y_2) = 0.1$, and $G_F(x_2, y_3) = \overline{0}$. Define fuzzy topologies $\tau : I^X \to I$ and $\sigma : I^Y \to I$ as follows:*

$$\tau(\lambda) = \begin{cases} 1 & \text{if } \lambda = \overline{0} \text{ or } \overline{1} \\ \frac{1}{2} & \text{if } \lambda \in \overline{0.5} \\ 0 & \text{otherwise} \end{cases} \qquad \tau^\star(\lambda) = \begin{cases} 0 & \text{if } \lambda = \overline{0} \text{ or } \overline{1} \\ \frac{1}{2} & \text{if } \lambda \in \overline{0.5} \\ 1 & \text{otherwise} \end{cases}$$

$$\sigma(\lambda) = \begin{cases} \text{if } \lambda = \overline{0} \text{ or } \overline{1} \\ \frac{1}{2} \\ \text{if } \lambda = \overline{0.5} \\ \frac{1}{3} \\ \text{if } \lambda = \overline{0.3} \\ 0 \\ \text{otherwise.} \end{cases} \qquad \sigma^\star(\lambda) = \begin{cases} \text{if } \lambda = \overline{0} \text{ or } \overline{1} \\ \frac{1}{2} \\ \text{if } \lambda = \overline{0.5} \\ \frac{2}{3} \\ \text{if } \lambda = \overline{0.3} \\ 1 \\ \text{otherwise.} \end{cases}$$

Then F is double fuzzy upper weakly α-continuous but none of double fuzzy upper almost α-continuous and double fuzzy upper α-continuous.

Theorem 3.8. *For a fuzzy multifunction $F : (X, \tau, \tau^\star) \to (Y, \sigma, \sigma^\star)$, then the following are equivalent:*

1. *F is double fuzzy lower almost α-continuous.*
2. *$\alpha C_{\tau,\tau^\star}(F^u(\mu), r, s) \leq F^u(C_{\sigma,\sigma^\star}(\mu, r, s))$ for any μ is (r, s)-fuzzy β-open.*
3. *$\alpha C_{\tau,\tau^\star}(F^u(\mu), r, s) \leq F^u(C_{\sigma,\sigma^\star}(\mu, r, s))$ for any μ is (r, s)-fuzzy semiopen.*
4. *$F^l(\mu) \leq \alpha I_{\tau,\tau^\star}(F^l(I_{\sigma,\sigma^\star}(C_{\sigma,\sigma^\star}(\mu, r, s), r, s)), r, s)$ for any μ is (r, s)-fuzzy preopen.*

Proof. (1) $\Rightarrow$ (2): Let $\mu \in I^Y$ and μ be (r, s)-fuzzy β-open. Since $C_{\sigma,\sigma^\star}(\mu, r, s)$ is (r, s)-fuzzy regular closed, by Theorem 3.3 (3) we have $F^u(C_{\sigma,\sigma^\star}(\mu, r, s))$ is (r, s)-fuzzy α-closed and $F^u(\mu) \leq F^u(C_{\sigma,\sigma^\star}(\mu, r, s))$. Then, we have $\alpha C_{\tau,\tau^\star}(F^u(\mu), r, s) \leq F^u(C_{\sigma,\sigma^\star}(\mu, r, s))$.
(2) $\Rightarrow$ (3): This is obvious from every (r, s)-fuzzy semiopen set is (r, s)-fuzzy β-open set.

(3) $\Rightarrow$(1): Let $\mu \in I^Y$ and μ be (r,s)-fuzzy regular closed. Then μ is (r,s)-fuzzy semiopen and $\alpha C_{\tau,\tau^\star}(F^u(\mu),r,s) \leq F^u(\mu)$. Therefore, $F^u(\mu)$ is (r,s)-fuzzy α-closed. Then, F is double fuzzy lower almost α-continuous.
(1) $\Rightarrow$ (4): Let $\mu \in I^Y$ be (r,s)-fuzzy preopen. Since $I_{\sigma,\sigma^\star}(C_{\sigma,\sigma^\star}(\mu,r,s),r,s)$ is $r$fuzzy regular open, by Theorem 3.3 (2), $F^l(I_{\sigma,\sigma^\star}(C_{\sigma,\sigma^\star}(\mu,r,s),r,s))$ is (r,s)-fuzzy α-open. Hence $F^l(\mu) \leq F^l(I_{\sigma,\sigma^\star}(C_{\sigma,\sigma^\star}(\mu,r,s),r,s))$
$= \alpha I_{\tau,\tau^\star}(F^l(I_{\sigma,\sigma^\star}(C_{\sigma,\sigma^\star}(\mu,r,s),r,s)),r,s)$.
(4) $\Rightarrow$ (1): Let $\mu \in I^Y$ and μ be (r,s)-fuzzy regular open. Since μ is (r,s)-fuzzy preopen, $F^l(\mu) \leq \alpha I_{\tau,\tau^\star}(F^l(I_{\sigma,\sigma^\star}(C_{\sigma,\sigma^\star}(\mu,r,s),r,s)),r,s) = \alpha I_{\tau,\tau^\star}(F^l(\mu),r,s)$. Then, $F^l(\mu)$ is (r,s)-fuzzy α-open. It follows from Theorem 3.3 (2) that F is double fuzzy lower almost α-continuous. □

In view of Theorem 3.8, we formulate the following result without proof.

Theorem 3.9. *For a fuzzy multifunction $F : (X,\tau,\tau^\star) \to (Y,\sigma,\sigma^\star)$, then the following are equivalent:*

1. *F is double fuzzy upper almost α-continuous.*
2. *$\alpha C_{\tau,\tau^\star}(F^l(\mu),r,s) \leq F^l(C_{\sigma,\sigma^\star}(\mu,r,s))$ for any μ is (r,s)-fuzzy β-open.*
3. *$\alpha C_{\tau,\tau^\star}(F^l(\mu),r,s) \leq F^l(C_{\sigma,\sigma^\star}(\mu,r,s))$ for any μ is (r,s)-fuzzy semiopen.*
4. *$F^u(\mu) \leq \alpha I_{\tau,\tau^\star}(F^u(I_{\sigma,\sigma^\star}(C_{\sigma,\sigma^\star}(\mu,r,s),r,s)),r,s)$ for any μ is (r,s)-fuzzy preopen.*

Theorem 3.10. *If $F : (X,\tau,\tau^\star) \to (Y,\sigma,\sigma^\star)$ is double fuzzy lower α-continuous multifunction and $H : (Y,\sigma) \to (Z,\eta)$ is a double lower almost α-continuous function, the $H \circ F$ is a double fuzzy lower almost α-continuous multifunction.*

Proof. Let F be double fuzzy lower α-continuous, H be double fuzzy almost α-continuous and $\gamma \in I^Z$, $\eta(\gamma) \geq r$. Then we have, $(H \circ F)^l(\gamma) = F^l(H^l(\gamma))$ is (r,s)-fuzzy α-open with $\sigma(H^l(\gamma)) \geq \eta(\gamma) \geq r$ and γ is (r,s)-fuzzy regular open. Thus $H \circ F$ is double fuzzy lower almost α-continuous. □

In view of Theorem 3.10, we formulate the following result without proof.

Theorem 3.11. *If $F : (X,\tau,\tau^\star) \to (Y,\sigma,\sigma^\star)$ is double fuzzy upper α-continuous multifunction, $H : (Y,\sigma) \to (Z,\eta)$ is a double fuzzy upper α-continuous function and F and H are normalized, the $H \circ F$ is a double fuzzy upper almost α-continuous multifunction.*

Theorem 3.12. *Let $\{F_i : i \in \Gamma\}$ be a family of double fuzzy lower almost α-continuous. Then $\bigcup_{i\in\Gamma} F_i$ is double fuzzy lower almost α-continuous.*

Proof. Let $\mu \in I^Y$, then $(\bigcup_{i\in\Gamma} F_i)^l(\mu) = \bigvee_{i\in\Gamma} F_i^l(\mu)$. Since $\{F_i : i \in \Gamma\}$ is a family of double fuzzy lower almost α-continuous multifunctions, $F_i^l(\mu)$ is (r,s)-fuzzy α-open for any μ is (r,s)-fuzzy regular open. Then $(\bigcup_{i\in\Gamma} F_i)^l(\mu) = \bigvee_{i\in\Gamma} F_i^l(\mu)$ is (r,s)-fuzzy α-open for any μ is (r,s)-fuzzy regular open. Hence $\bigcup_{i\in\Gamma} F_i$ is fuzzy lower almost α-continuous. □

Definition 3.13. *Let $F : (X,\tau) \to (Y,\sigma)$ be a fuzzy multifunction. Then F is called:*

1. *double fuzzy upper weakly α-continuous at a fuzzy point $x_t \in dom(F)$ if $x_t \in F^u(\mu)$ for each $\mu \in I^Y$, $\sigma(\mu) \geq r$ and $\sigma^\star(\mu) \leq s$, there exists an (r,s)-fuzzy α-open set $\lambda \in I^X$ and $x_t \in \lambda$ such that $\lambda \wedge dom(F) \leq F^u(C_{\tau,\tau^\star}(\mu,r,s))$.*
2. *double fuzzy lower weakly α-continuous at a fuzzy point $x_t \in dom(F)$ if $x_t \in F^u(\mu)$ for each $\mu \in I^Y$, $\sigma(\mu) \geq r$ and $\sigma^\star(\mu) \leq s$, there exists an (r,s)-fuzzy α-open set $\lambda \in I^X$ and $x_t \in \lambda$ such that $\lambda \leq F^l(C_{\tau,\tau^\star}(\mu,r,s))$.*
3. *double fuzzy upper (lower) weakly α-continuous if it is double fuzzy upper (lower) weakly α-continuous at every $x_t \in dom(F)$.*

Proposition 3.14. *If F is normalized, then F is double fuzzy upper weakly α-continuous at a fuzzy point $x_t \in dom(F)$ if, and only if $x_t \in F^u(\mu)$ for each $\mu \in I^Y$, $\sigma(\mu) \geq r$ and $\sigma^\star(\mu) \leq s$, there exists $\lambda \in I^X$, $\tau(\lambda) \geq r$, $\tau^\star(\lambda) \leq s$ and $x_t \in \lambda$ such that $\lambda \leq F^u(C_{\tau,\tau^\star}(\mu,r,s))$.*

Theorem 3.15. *For a fuzzy multifunction $F : (X,\tau,\tau^\star) \to (Y,\sigma,\sigma^\star)$, then the following are equivalent:*

1. *F is double fuzzy lower weakly α-continuous.*
2. *$F^l(\mu) \leq I_{\tau,\tau^\star}(C_{\tau,\tau^\star}(I_{\tau,\tau^\star}(F^l(C_{\sigma,\sigma^\star}(\mu,r,s)),r,s),r,s),r,s)$ if $\sigma(\mu) \geq r$ and $\sigma^\star(\mu) \leq s$.*
3. *$C_{\tau,\tau^\star}(I_{\tau,\tau^\star}(C_{\tau,\tau^\star}(F^u(I_{\sigma,\sigma^\star}(\mu,r,s)),r,s),r,s),r,s) \leq F^u(\mu)$ if $\sigma(\overline{1}-\mu) \geq r$ and $\sigma^\star(\overline{1}-\mu) \leq s$.*

4. $\alpha C_{\tau,\tau^\star}(F^u(I_{\sigma,\sigma^\star}(\mu,r,s)),r,s) \leq F^u(\mu)$ *if* $\sigma(\overline{1}-\mu) \geq r$ *and* $\sigma^\star(\overline{1}-\mu) \leq s$.

5. $\alpha C_{\tau,\tau^\star}(F^u(I_{\sigma,\sigma^\star}(C_{\sigma,\sigma^\star}(\mu,r,s),r,s)),r,s) \leq F^u(C_{\sigma,\sigma^\star}(\mu,r,s))$.

6. $F^l(I_{\sigma,\sigma^\star}(\mu,r,s)) \leq \alpha I_{\tau,\tau^\star}(F^l(C_{\sigma,\sigma^\star}(I_{\sigma,\sigma^\star}(\mu,r,s),r,s)),r,s)$.

7. $F^l(\mu) \leq \alpha I_{\tau,\tau^\star}(F^l(C_{\sigma,\sigma^\star}(\mu,r,s)),r,s)$ *if* $\sigma(\mu) \geq r$ *and* $\sigma^\star(\mu) \leq s$.

8. $\alpha C_{\tau,\tau^\star}(F^u(I_{\sigma,\sigma^\star}(\mu,r,s)),r,s) \leq F^u(\mu)$ *for any* μ *is* (r,s)*-fuzzy regular closed.*

9. $\alpha C_{\tau,\tau^\star}(F^u(\mu),r,s) \leq F^u(C_{\sigma,\sigma^\star}(\mu,r,s))$ *if* $\sigma(\mu) \geq r$ *and* $\sigma^\star(\mu) \leq s$.

Proof. (1) $\Rightarrow$ (2): Let $x_t \in dom(F)$, $\mu \in I^Y$, $\sigma(\mu) \geq r$ and $\sigma^\star(\mu) \leq s$ and $x_t \in F^l(\mu)$. Then there exist (r,s)-fuzzy α-open set $\lambda \in I^X$ and $x_t \in \lambda$ such that $\lambda \leq F^l(C_{\sigma,\sigma^\star}(\mu,r,s))$. Since λ is (r,s)-fuzzy α-open, we have $x_t \in \lambda \leq I_{\tau,\tau^\star}(C_{\tau,\tau^\star}(I_{\tau,\tau^\star}(F^l(C_{\sigma,\sigma^\star}(\mu,r,s)),r,s),r,s),r,s)$ and hence $F^l(\mu) \leq I(C_{\tau,\tau^\star}(I(F^l(C_{\sigma,\sigma^\star}(\mu,r,s)),r,s),r,s),r,s)$.
(2) $\Rightarrow$ (3): Let $\sigma(\overline{1}-\mu) \geq r$ and $\sigma^\star(\overline{1}-\mu) \leq s$. By (2), $F^l(\overline{1}-\mu) \leq I(C_{\tau,\tau^\star}(I_{\tau,\tau^\star}(F^l(C_{\sigma,\sigma^\star}(\overline{1}-\mu,r,s)),r,s),r,s),r,s)$. Then we have
$C_{\tau,\tau^\star}(I(C_{\tau,\tau^\star}(F^u(I_{\sigma,\sigma^\star}(\mu,r,s)),r,s),r,s),r,s) \leq F^u(\mu)$.
(3) $\Rightarrow$ (4): Let $\sigma(\overline{1}-\mu) \geq r$ and $\sigma^\star(\overline{1}-\mu) \leq s$. By (3), we have $C_{\tau,\tau^\star}(I(C_{\tau,\tau^\star}(F^u(I_{\sigma,\sigma^\star}(\mu,r,s)),r,s),r,s),r,s) \leq F^u(\mu)$. It follows that
$\alpha C_{\tau,\tau^\star}(F^u(I_{\sigma,\sigma^\star}(\mu,r,s)),r,s) \leq F^u(\mu)$.
(4) $\Rightarrow$ (5): Let $\mu \in I^Y$. Then $\sigma(\overline{1}-[C_{\sigma,\sigma^\star}(\mu,r,s)]) \geq r$ by (4), we have $\alpha C_{\tau,\tau^\star}(F^u(I_{\sigma,\sigma^\star}(C_{\sigma,\sigma^\star}(\mu,r,s),r,s)),r,s) \leq F^u(C_{\sigma,\sigma^\star}(\mu,r,s))$.
(5) $\Rightarrow$ (6): Let $\mu \in I^Y$. Then $\overline{1}-[F^l(I_{\sigma,\sigma^\star}(\mu,r,s))]$
$= F^u(C_{\sigma,\sigma^\star}(\overline{1}-\mu,r,s)) \geq \alpha C_{\tau,\tau^\star}(F^u(I_{\sigma,\sigma^\star}(C_{\sigma,\sigma^\star}(\overline{1}-\mu,r,s),r,s)),r,s)$
$= \alpha C_{\tau,\tau^\star}(\overline{1}-[F^l(C_{\sigma,\sigma^\star}(I_{\sigma,\sigma^\star}(\mu,r,s),r,s))],r,s)$
$= \overline{1}-[\alpha I_{\tau,\tau^\star}(F^l(C_{\sigma,\sigma^\star}(I_{\sigma,\sigma^\star}(\mu,r,s),r,s)),r,s)]$. Hence $F^l(I_{\sigma,\sigma^\star}(\mu,r,s)) \leq \alpha I_{\tau,\tau^\star}(F^l(C_{\sigma,\sigma^\star}(I_{\sigma,\sigma^\star}(\mu,r,s),r,s)),r,s)$.
(6) $\Rightarrow$(7): The proof is obvious.
(7) $\Rightarrow$ (1): Let $x_t \in dom(F)$, $\mu \in I^Y$ and $\sigma(\mu) \geq r$ and $\sigma^\star(\mu) \leq s$ with $x_t \in F^l(\mu)$, we have by (7), $x_t \in F^l(\mu) \leq \alpha I_{\tau,\tau^\star}(F^l(C_{\sigma,\sigma^\star}(\mu,r,s)),r,s) = \lambda$ (say). Thus, there exist (r,s)-fuzzy α-open set $\lambda \in I^X$ and $x_t \in \lambda$ such that $\lambda \leq F^l(C_{\sigma,\sigma^\star}(\mu,r,s))$. Then, F is fuzzy lower weakly α-continuous.
(4) $\Rightarrow$(8): The proof is obvious.
(8) $\Rightarrow$ (9): Let $\mu \in I^Y$ and $\sigma(\mu) \geq r$ and $\sigma^\star(\mu) \leq s$ then $C_{\sigma,\sigma^\star}(\mu,r,s)$ is (r,s)-fuzzy regular closed set and by (8), we have $\alpha C_{\tau,\tau^\star}(F^u(\mu),r,s) \leq \alpha C_{\tau,\tau^\star}(F^u(I_{\sigma,\sigma^\star}(C_{\sigma,\sigma^\star}(\mu,r,s),r,s)),r,s) \leq F^u(C_{\sigma,\sigma^\star}(\mu,r,s))$.

(9) $\Rightarrow$ (7): Let $\mu \in I^Y$ and $\sigma(\mu) \geq r$ and $\sigma^\star(\mu) \leq s$. Then $\overline{1} - [\alpha I_{\tau,\tau^\star}(F^l(C_{\sigma,\sigma^\star}(\mu, r, s)), r, s)] = \alpha C_{\tau,\tau^\star}(\overline{1} - [F^l(C_{\sigma,\sigma^\star}(\mu, r, s))], r, s) = \alpha C_{\tau,\tau^\star}(F^u(\overline{1} - [C_{\sigma,\sigma^\star}(\mu, r, s)]), r, s) \leq F^u(C_{\sigma,\sigma^\star}(\overline{1} - [C_\sigma(\mu, r, s)], r, s)) = \overline{1} - [F^l(I_{\sigma,\sigma^\star}(C_{\sigma,\sigma^\star}(\mu, r, s), r, s))]$. Then, $F^l(\mu) \leq F^l(I_{\sigma,\sigma^\star}(C_{\sigma,\sigma^\star}(\mu, r, s), r, s)) \leq \alpha I_{\tau,\tau^\star}(F^l(C_{\sigma,\sigma^\star}(\mu, r, s)), r, s)$. □

Proposition 3.16. *1. Every double fuzzy upper almost α-continuous multifunction is double fuzzy upper weakly α-continuous.*

2. Every double fuzzy lower almost α-continuous multifunction is double fuzzy lower weakly α-continuous.

The following example shows that the converses of the Proposition 3.16 are not true.

Example 3.17. *Let* $X = \{x_1, x_2\}$, $Y = \{y_1, y_2, y_3\}$ *and* $F : (X, \tau, \tau^\star) \to (Y, \sigma, \sigma^\star)$ *be a fuzzy multifunction defined by* $G_F(x_1, y_1) = 0.1$, $G_F(x_1, y_2) = G_F(x_2, y_1) = \overline{1}$, $G_F(x_1, y_3) = G_F(x_2, y_2) = \overline{0}$, *and* $G_F(x_2, y_3) = 0.3$. *Define* $\mu(y_1) = 0.3$, $\mu(y_2) = \overline{0}$, $\mu(y_3) = 0.5$. *Define fuzzy topologies* $\tau : I^X \to I$ *and* $\sigma : I^Y \to I$ *as follows:*

$$\tau(\lambda) = \begin{cases} 1 & \text{if } \lambda = \overline{0} \text{ or } \overline{1} \\ \frac{1}{2} & \text{if } \lambda \in \overline{0.6} \\ 0 & \text{otherwise} \end{cases} \qquad \tau^\star(\lambda) = \begin{cases} 0 & \text{if } \lambda = \overline{0} \text{ or } \overline{1} \\ \frac{1}{2} & \text{if } \lambda \in \overline{0.6} \\ 1 & \text{otherwise} \end{cases}$$

$$\sigma(\lambda) = \begin{cases} 1 & \text{if } \lambda = \overline{0} \text{ or } \overline{1} \\ \frac{1}{2} & \text{if } \lambda = \mu \\ 0 & \text{otherwise.} \end{cases} \qquad \sigma^\star(\lambda) = \begin{cases} 0 & \text{if } \lambda = \overline{0} \text{ or } \overline{1} \\ \frac{1}{2} & \text{if } \lambda = \mu \\ 1 & \text{otherwise.} \end{cases}$$

Then F is double fuzzy upper weakly α-continuous but none of double fuzzy upper almost α-continuous and double fuzzy upper α-continuous.

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