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# Implications of some types of pairwise closed graphs

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**Abstract**

*The main goal of this paper is to introduce and look into some of the fundamental properties of pairwise strongly closed, pairwise strongly $\alpha$-closed and pairwise quasi $\alpha$-closed graphs. Some characterizations and several properties concerning these graphs are obtained. We also investigate relationships between $(i, j)$- strongly $\alpha$-closed graph $G(f)$ and $(i, j)$- weakly $\alpha$-continuous. We study relationships between $(i, j)$- strongly $\alpha$-closed $(i, j)$-quasi $\alpha$-closed graphs with covering properties. The concepts of pairwise $S^*$-closed and pairwise quasi $H$-closed relatively are stated.*

**Keywords and phrases:** *Pairwise strongly closed graphs; pairwise strongly $\alpha$-closed graphs; pairwise quasi $\alpha$-closed graphs.*

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## 1. Introduction

The concept of bitopological space $(X, \tau_1, \tau_2)$, where $X$ is a non-empty set and $\tau_1, \tau_2$ are different topologies on $X$, was introduced by Kelly [7]. Since the bitopological space is a generalization of topological space, thus it is worth-while to examine that how the being concepts of topological setting can be applied to bitopological category such as sets [16,17,18], covering properties [3,8,9], mappings [4], functions [13,14,15] and multifunctions [2] and others. In 1965, Njastad introduced the concept of $\alpha$-open sets in topological spaces [12]. Similarly, the notion of semi open sets was established by Levine in [11]. The family of all $\alpha$-open sets and of X is denoted by $\alpha O(X)$ and $SO(X)$ respectively. Functions with strongly closed graphs were introduced by T. Noiri [19]. Caldas et al have studied strongly $\alpha$-closed graphs by using $\alpha$-open sets [5]. The idea of $\alpha$-quasi-closed graphs was introduced by Latif and Noiri in [10]. Let $f : X \longrightarrow Y$ be a function of a topological space $X$ into a topological space $Y$. The subset $\{(x, f(x)) : x \in X\}$ of the product space $X \times Y$ is called the graph of f and usually denoted by $G(f)$.

In this work, we define and study some essential concepts of closed graphs in bitopological setting as pairwise strongly closed graph, pairwise strongly $\alpha$-closed graph and pairwise quasi $\alpha$-closed graphs. We also study some of their further principle features. We advance to examine the functions with pairwise closed graphs. In addition, the ideas of pairwise $S^*$-closed and pairwise quasi $H$-closed relatively are considered. Then we recall the following concept.

## 2. Preliminaries and Definitions

In this article, we let $(X, \tau_1, \tau_2)$ and $(Y, \sigma_1, \sigma_2)$ (or briefly $X$, and $Y$) indicate bitopological spaces then for any subset $U \subseteq X$, we will imply the closure of $U$ and the interior of $U$ with respect to topologies $\tau_i$ by $\tau_i - cl(U)$ and $\tau_i - int(U)$ respectively. Also, By $\tau_i$-open cover of $X$, we mean that the cover of $X$ by $\tau_i$-open sets in X. also, $(\tau_1, \tau_2)$-semi-open cover of $X$ means that the cover of $X$ by $(\tau_1, \tau_2)$-semi open sets in $X$, etc. During this work always $i, j = 1, 2$ and $i \neq j$.

**Definition 1.** *A subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is supposed to be:*

1. *$(i, j) - \alpha$-open (see[7]) if $A \subset i - int(j - cl(i - int(A)))$.*

2. *$i$-clopen ([8]) if $A$ is both $i$-closed and $i$-open set in $X$. $A$ is said clopen in $X$ if it is both 1-clopen and 2-clopen in $X$;*

3. *$(i,j)$-semiopen ([1]) if and only if $A \subset i - cl(j - int(A)))$.*

*The family of all $(i,j) - \alpha$-open (resp. $(i,j)$-semi-open) sets of $X$ is denoted by $(i,j) - \alpha O(X)$ (resp. $(i,j) - SO(X)$).*

**Definition 2 (6).** *A bitopological space $(X, \tau_1, \tau_2)$ is said to be pairwise extremally disconnected if and only if the $j$-closure of every $i$-open set is $i$-open and the $i$-closure of every $j$-open set is $j$-open.*

**Definition 3 (10).** *A subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is said to be $(i,j)$-quasi $H$-closed relatively if for any $i$-open cover $\{U_\alpha : \alpha \in \Delta\}$ of $A$, there exists a countable subset $\Delta_0$ of $\Delta$ such that $A \subset \cup\{j{-}cl(U_\alpha)|\alpha \in \Delta_0\}$.*

## 3. Implications between some kinds of closed graphs and Relatives of covering properties in Bitopological spaces

**Definition 4.** *A function $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is said to have $(i,j)$-strongly closed (resp. $(i,j)$-strongly $\alpha$-closed) graph if for each $(x,y) \in (X \times Y) \backslash G(f)$, there exist $\tau_i$-open (resp. $\tau_i - \alpha$-open ) subset $U$ of $X$ and $\sigma_i$-open subset $V$ of $Y$ such that $(x,y) \in U \times V$ and $(U \times \sigma_j - cl(V)) \cap G(f) = \phi$.*

**Definition 5.** *A bitopological space $X$ is called $\tau_i - P$- space if any countable intersection of $\tau_i$-open sets is $\tau_i$-open. $X$ is called $P$- space if $X$ is $\tau_i - P$- space for $i = 1, 2$.*

**Theorem 1.** *If a function $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ has an $(i,j)$-strongly closed graph, then $f^{-1}(C)$ is $\tau_i$-closed in $\tau_i - P$- space $X$ for each subset $C$ which is $(\sigma_i, \sigma_j)$-quasi $H$-closed relatively in $Y$.*

**Proof:** Let $C$ be $(\sigma_i, \sigma_j)$-quasi $H$-closed relatively in $Y$ and $x \notin f^{-1}(C)$. For every $y \in C$, we get $(x,y) \in (X \times Y) \setminus G(f)$, and there exist $\tau_i$-open set $U_y$ having $x$ and an $\sigma_i$-open set $V_y$ having $y$ such that $f(U_y) \cap \sigma_j - cl(U_y) = \phi$, thus $U_y \cap f^{-1}(\sigma_j - cl(V_y)) = \phi$.

Now, the set $\{V_y | y \in C\}$ is $\sigma_i$-open cover of $C$. Since $C$ is $(\sigma_i, \sigma_j)$-quasi $H$-closed relatively, there exists countable subset $C_0$ of $C$ such that $C \subset \cup\{\sigma_j - cl(V_y) | y \in C_0\}$.

Put $W = \cap_{y\in C_0} U_y$. Since $X$ is $\tau_i - P$- space, so $W$ is $\tau_i$-open set containing $x$ and

$f(W) \cap C \subset U_{y\in C_0}[f(W) \cap \sigma - cl(V_y)] \subset U_{y\in C_0}[f(U_y) \cap \sigma - cl(V_y)] = \phi$.

Then, we get $f(W) \cap C = \phi$. Thus $f^{-1}(C)$ is $\tau_i$-closed in $X$.

**Definition 6.** *A bitopological space $X$ is called $\tau_i - \alpha - P$- space if any countable intersection of $\tau_i - \alpha$-open sets is $\tau_i - \alpha$-open. $X$ is said $\alpha - P$-space if $X$ is $\tau_i - \alpha - P$-space for $i = 1, 2$.*

**Theorem 2.** *Let $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ be a function have $(i, j)$-strongly $\alpha$-closed graph $G(f)$, thus $f$ owns the next statement:*
*$(S)$ For any subset $C$ of $Y$ which is $(\sigma_i, \sigma_j)$-quasi $H$-closed relatively, $f^{-1}(C)$ is $\tau_i - \alpha$-closed in $\tau_i - \alpha - P$- space $X$.*

**Proof:** Suppose that $f^{-1}(C)$ is not $\tau_i - \alpha$-closed in $X$. So there exists $x \in \tau_i - \alpha - cl(f^{-1}(C)) \in f^{-1}(C)$. Let $y \in C$. Thus $(x, y) \in (X \times Y) \setminus G(f)$. $(i, j)$-strongly $\alpha$-closed of $G(f)$ gives the existence of $U_y(x) \in \tau_i - \alpha - O(X, x)$ and $V_y \in \sigma_i - O(Y, y)$ such that

$$f(U_y(x)) \cap \sigma_j - cl(V_y) = \phi.(*).$$

Obviously, $\{V_y : y \in C\}$ is $\sigma_i$-open cover of $C$. Due to $C$ is $(\sigma_i, \sigma_j)$-quasi $H$-closed relatively, there exist a countable subcollection $\{V_{y_n} : n \in N\}$ such that $C \subset \cup_{n\in N} \sigma_j - cl(V_{y_n})$. Since $X$ is $\tau_i - \alpha - P$- space, $B = \cap_{n\in N} U_{y_n}(x)$ is $\tau_i - \alpha$-open set in $X$ that satisfies $(*)$. Now,

$$f(B) \cap C \subset f(\cap_{n\in N} U_{y_n}(x)) \cap \cup_{n\in N} \sigma_j - cl(V_{y_n}) \subset \cup_{n\in N}(f(U_{y_n}(x)) \cap \sigma_j - cl(V_{y_n})) = \phi.$$

However, because $x \in \tau_i - \alpha - cl(f^{-1}(C))$ and $B \cap f^{-1}(C) = \phi$, thus $f(B) \cap C = \phi$. This leads to a contradiction. Therefore, the theorem holds.

**Lemma 1.** *Every $i$-clopen subset of a $(i, j)$-quasi $H$-closed space $(X, \tau_1, \tau_2)$ is $(i, j)$-quasi $H$-closed relative to $X$.*

**Proof:** Let $B$ be any $i$-clopen subset of $(i, j)$-quasi $H$-closed space $(X, \tau_1, \tau_2)$. Let $\{U_\gamma : \gamma \in \Delta\}$ be any cover of $B$ by $i$-open sets in $X$. Then the family $E = \{U_\gamma : \gamma \in \Delta\} \cup \{X \setminus B\}$ is a cover of $X$ by $i$-open sets in $X$. Due to $(i, j)$-quasi $H$-closedness of $X$, there exists a countable subcollection

$E = \{U_{\gamma_n} : n \in N\} \cup \{X \setminus B\}$ of $E$ such that $X \subset \cup_{n\in N} j - cl(U_{\gamma_n}) \cup \{X \setminus B\}$. Then, because of $i$-clopenness of $B$, we now infer that the collection $\{j - cl(U_{\gamma_n}) : n \in N\}$ covers $B$. Thus, $B$ is $(i,j)$-quasi $H$-closed relative to $X$.

**Theorem 3.** *If $(Y, \sigma_1, \sigma_2)$ is $(\sigma_i, \sigma_j)$- quasi $H$-closed and $(\sigma_i, \sigma_j)$- extremally disconnected space, then a function $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ with the $(i,j)$-strongly $\alpha$-closed graph $G(f)$ is $(i,j)$- weakly $\alpha$-continuous.*

**Proof:** Let $x \in X$ and $V \in \sigma_i - O(Y, f(x))$. Take any $y \in Y \setminus \sigma_j - cl(V)$. Then $(x, y) \in (X \times Y)\setminus G(f)$. Now the strongly $(i,j)-\alpha$-closedness of $G(f)$ induces the existence of $U_y(x) \in \tau_i - \alpha - O(X, x)$ and $V_y \in \sigma_i - O(Y, y)$ such that

$$f(U_y(x)) \cap \sigma_j - cl(V_y) = \phi ...(*).$$

By extremal disconnectedness of $Y$ inducing the $i$-clopenness of $\sigma_j - cl(V)$, so $Y \setminus \sigma_j - cl(V)$ is also $i$-clopen. Now $\{V_y : y \in Y \setminus \sigma_j - cl(V)\}$ is $\sigma_i$-open cover of $Y \setminus \sigma_j - cl(V)$. By Lemma above, there exists a countable subcollection $\{V_{y_n} : n \in N\}$ such that $Y \setminus \sigma_j - cl(V) \subset \cup_{n\in N}\sigma_j - cl(V_{y_n})$. Let $N = \cap_{n\in N} U_{y_n}(x)$. Since $X$ is $\tau_i-\alpha-P$-space, we get $N \in \tau_i - \alpha_O(X, x)$. Because of $U_{y_n}(x)$ satisfies $(*)$, then we have

$$f(N) \cap (Y \setminus \sigma_j - cl(V)) \subset f(\cap_{n\in N} U_{y_n}(x)) \cap \cup_{n\in N}\sigma_j - cl(V_{y_n}) \subset \cup_{n\in N}(f(U_{y_n}(x)) \cap \sigma_j - cl(V_{y_n}) = \phi$$

by $(*)$.

Thus, $f(N) \subset \sigma_j - cl(V)$ and this implies that f is $(i,j)$- weakly $\alpha$-continuous.

**Definition 7.** *A subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is called $(i,j) - S^*$-closed relative to $X$ if for every $i$-semi-open cover $\{V_\beta : \beta \in \Lambda\}$ of $A$, there exists a countable subset $\Lambda_0$ of $\Lambda$ such that $A \subset \cup\{j - cl(V_\beta) : \beta \in \Lambda_0\}$.*

**Definition 8.** *The graph $G(f)$ of a function $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is said to be $(i,j)$-quasi $\alpha$-closed if for any $(x, y) \in (X\times Y)\setminus G(f)$, there exist $U \in \tau_i - \alpha(X, x)$ and $V \in \sigma_i - SO(Y, y)$ such that $[U \times \sigma_j - cl(V)] \cap G(f) = \phi$.*

**Lemma 2.** *The following properties are equivalent for a graph $G(f)$ of a function $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$:*

1. *The graph $G(f)$ is $(i,j)$-quasi $\alpha$-closed in $X \times Y$;*
2. *For each point $(x,y) \in (X \times Y) \setminus G(f)$, there exist $U \in \tau_i - \alpha(X,x)$ and $V \in (\sigma_i,\sigma_j) - SO(Y,y)$ such that $f(U) \cap \sigma_j - cl(V) = \phi$;*
3. *For each point $(x,y) \in (X \times Y) \setminus G(f)$, there exist $U \in \tau_i - \alpha(X,x)$ and $F \in (\sigma_j,\sigma_i) - RC(Y,y)$ such that $f(U) \cap F = \phi$.*

**Proof:** (1) $\Rightarrow$ (2) let $G(f)$ be $(i,j)$-quasi $\alpha$-closed graph in $X \times Y$ and $(x,y) \notin G(f)$ such that $y \neq f(x)$. So there exist $U \in \tau_i - \alpha(X,x)$ and $V \in \sigma_i - SO(Y,y)$ such that $[U \times \sigma_j - cl(V)] \cap G(f) = \phi$. then $f(U) \cap \sigma_j - cl(V) = \phi$.

(2) $\Rightarrow$ (3) Set $\sigma_j - cl(V) = F \in (j,i) - RC(Y,y)$.

(3) $\Rightarrow$ (1) Put $F = \sigma_j - cl(V)$, where $V = \sigma_i - int(F)$. Since $\sigma_i$-open set is $(\sigma_i,\sigma_j) - SO(Y,y)$, so $[U \times \sigma_j - cl(V)] \cap G(f) = \phi$, which implies that $G(f)$ is $(i,j)$-quasi $\alpha$-closed in $X \times Y$.

**Theorem 4.** *If a function $f : (X,\tau_1,\tau_2) \longrightarrow (Y,\sigma_1,\sigma_2)$ has $(i,j)$-quasi $\alpha$-closed graph, then $f^{-1}(K)$ is $\tau_i - \alpha$-closed in $X$ for any subset $K$ which is $(\sigma_i,\sigma_j) - S^*$-closed relative to $Y$.*

**Proof:** Let $K$ be $(\sigma_i,\sigma_j) - S^*$-closed relative to $Y$ and $x \notin f^{-1}(K)$. For any $y \in K$, we own $(x,y) \in (X \times Y) \setminus G(f)$ and there exist $U_y \in \tau_i - \alpha(X,x)$ and $V_y \in (\sigma_i,\sigma_j) - SO(Y,y)$ such that $f(U_y) \cap \sigma_j - cl(V_y) = \phi$. The collection $\{V_y : y \in K\}$ is $(\sigma_i,\sigma_j)$-semi-open cover of $K$ and there exists a countable number of elements such that $K \subset \cup\{\sigma_j - cl(V_{y_n}) : n \in N\}$.

Set $U = \cap_{n \in N} U_{y_n}$, then $U$ is $\tau_i - \alpha$-open neighborhood of $x$ and $f(U) \cap K = \phi$. Then, we get $U \cap f^{-1}(K) = \phi$. This implies that $f^{-1}(K)$ is $\tau_i - \alpha$-closed in $X$.

## 4. Conclusion

In this present work, we studied the concepts of closed graphs in bitopological setting namely; pairwise strongly closed graph, pairwise strongly $\alpha$-closed graph and pairwise quasi $\alpha$-closed graphs and some further principle features such as pairwise $S^*$-closed and relatively pairwise quasi $H$-closedness.

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