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Super (a, d)-$G + e$-antimagic total labeling of $G_u[S_n]$

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Abstract

Let $G = (V, E)$ be a simple graph and H be a subgraph of G. Then G admits an H-covering, if every edge in $E(G)$ belongs to at least one subgraph of G that is isomorphic to H. An (a, d)-H-antimagic total labeling of G is a bijection $f : V(G) \cup E(G) \rightarrow \{1, 2, 3, ..., |V(G)| + |E(G)|\}$ such that for all subgraphs H' of G isomorphic to H, the H' weights $w(H') = \sum_{v \in V(H')} f(v) + \sum_{e \in E(H')} f(e)$ constitute an arithmetic progression $\{a, a + d, a + 2d, ..., a + (n - 1)d\}$, where a and d are positive integers and n is the number of subgraphs of G isomorphic to H. The labeling f is called a super (a, d)-H-antimagic total labeling if $f(V(G)) = \{1, 2, 3, ..., |V(G)|\}$. In [9], authors have posed an open problem to characterize the super (a, d)-$G + e$-antimagic total labeling of the graph $G_u[S_n]$, where $n \geq 3$ and $4 \leq d \leq p + q + 2$. In this paper, a partial solution to this problem is obtained.

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Keywords: *H-covering; super (a, d)-H-antimagic; star graphs.*

1. Introduction

Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be simple and finite graphs. Let $|V(G)| = v_G, |E(G)| = e_G, |V(H)| = v_H$ and $|E(H)| = e_H$. Chartrand and Lesniak [3] is referred to graph-theoretic terminology.

An antimagic labeling was first introduced by Hartsfield and Ringel's[7] in 1990. According to Hartsfield and Ringel's[7], a graph G is called antimagic if its edges are labeled with integers $\{1, 2, ..., |E(G)|\}$ such that no two vertices have the same weight, where a weight of a vertex is the sums of the labels of the edges incident to a vertex. Hartsfield and Ringel's [7] conjectured that every connected graph with at least three vertices admits an antimagic labeling. They also made a weak conjecture that every tree with at least three vertices admits an antimagic labeling. These two conjectures were partially shown to be accurate by several authors, but they are still remain unsolved. For a detailed and interesting review of these conjectures, one can see chapter 6 of [5]. This concept motivated several authors to make a study on antimagic labelings and introduced many types of antimagic labelings in [5, 12]. Gutienrez and Lladó [6] developed H cover labeling. For the first time, Inayah et al.[8] in 2009 introduced the (a, d)-H-antimagic cover labeling which was developed from (a, d)-edge-antimagic labeling.

Lih[11] defines a magic labeling of type (1,1,0) of a planar graph and a consecutive magic labeling of type (1,1,0) of a planar graph. In 2012, Ahmed et al.[1] studied the consecutive magic labeling of type (1,1,0) in the name of d-antimagic labeling for a plane graph if for every positive integer s, the set of s-sided face weights is $\{a_s, a_s + d, a_s + 2d, \ldots, a_s + (f_s - 1)d\}$ for some positive integers as a_s and d, where f_s is the number of the s^{th} side face, where a face F weight is the sum of all the vertices labels, edges labels and a face label of F. A d-antimagic labeling is called super if the smallest possible label appears on the vertices. Several authors have studied such labeling for several families of graphs see [5]. Gutienrez and Lladó [6] defined (super) H-magic labeling, which is related to a magic labeling of type (1,1,0).

An edge covering of G is a family of different subgraphs $H_1, H_2, H_3, \ldots, H_k$ such that any edge of $E(G)$ belongs to at least one of the subgraphs $H_j, 1 \leq j \leq k$. If the $H_j{}'$s are isomorphic to a given graph H, then G is said to admit an H-covering. Gutienrez and Lladó [6] defined H-magic labeling, which is a generalization of Kotzig and Rosa's edge magic total labeling see in [10]. A bijection $f : V(G) \cup E(G) \to \{1, 2, 3, ..., v_G + e_G\}$ is called an H-magic labeling of G if there exists a positive integer k such that each subgraph H' of G isomorphic to H satisfies $w(H') = \sum_{v \in V(H')} f(v) + \sum_{e \in E(H')} f(e) = k$. In this case, G is called a H-magic. When $f(V(G)) = \{1, 2, 3, \dots, v_G\}$, then G is known as a H-super magic graph. On the other hand, Inayah et al. [8] introduced (a,d)-H-antimagic total labeling of G which is defined as a bijection $f : V(G) \cup E(G) \to \{1, 2, 3, \dots, v_G + e_G\}$ such that for all subgraphs H' of G isomorphic to H, the set of H'-weights form an arithmetic progression $a, a+d, a+2d, ..., a+(n-1)d$, where a and d are some positive integers, and n is the number of subgraphs isomorphic to H. If $f(V(G)) = \{1, 2, 3, \dots, v_G\}$, then f is a super (a,d)-H-antimagic total labeling and G is super (a,d)-H-antimagic. This labeling is the general case of super (a,d)-edge-antimagic total labelings.

If $H \cong K_2$ then the super (a,d)-H-antimagic labeling is also called super (a,d)-edge-antimagic total labeling which was introduced in [13]. A study on some basic properties of such labelings is done and the proof of the following theorem is presented in [13].

Theorem 1.1. *[8] If G has a super (a,d)-H-antimagic total labeling and t is the number of subgraphs of G isomorphic to H, then G has a super (a',d)-H-antimagic total labeling, where $a' = [(v_G+1)v_H + (2v_G + e_G + 1)e_H] - a - (t-1)d$.*

The (super) (a,d)-H-antimagic labeling related to a super d-antimagic labeling can be found [11, 2, 5].

In 2015, Laurencea and Kathiresan [4] obtained an upper bound of d for any graph G, and they investigated the existence of super (a,d)-P_3-antimagic total labeling of star graph S_n. First, they observed that S_n admits a P_h-covering for $h = 2, 3$ and the star S_n contains $t = \binom{n}{h-1}$ subgraphs $P_h, h = 2, 3$, which are denoted by $P_h^j, 1 \le j \le h$. In 2005, Sugeng et al.[14] investigated the case $h = 2$. In 2015, $h = 3$ case was investigated by Laurencea and Kathiresan [4]. They observed that, if the star $S_n, n \ge 3$ admits the super (a,d)-P_3-antimagic total labeling, then $d \in \{0, 1, 2\}$. Also,they proved the star S_n, $n \ge 3$ has super $(4n+7, 0)$-P_3-antimagic total labeling and S_n admits a super $(a, 2)$-P_3-antimagic total labeling only if $n = 3$.

In [9], they investigated the super (a,d)-H-antimagic total labeling of star related graphs $G_u[S_n]$ is defined as follows. Let G be a (p,q) graph and S_n be a star with n edges. Fix a vertex u of G. Then $G_u[S_n]$ is the graph obtained by identifying the vertex u with the centre of S_n. Let w be any vertex of S_n. Then $G+e, e=uw$, is a subgraph of $G_u[S_n]$. The graph $G_u[S_n]$ contains exactly n subgraphs isomorphic to $G+e$.

Let $G' \cong G_u[S_n]$. Let $v_1, v_2, \ldots, v_p$ and $w_1, w_2, \ldots, w_n$ be the vertices of G and S_n respectively. Let $e_1, e_2, \ldots, e_q$ and $e_{q+1}, e_{q+2}, ..., e_{q+n}$ be the edges of G and S_n respectively. Then $|V(G')| = p+n$ and $|E(G')| = q+n$. They proved the following theorem and posed an open problem.

Theorem 1.2. *[9] If the graph $G_u[S_n], n \geq 2$, admits a super (a,d)-$(G+e)$-antimagic total labeling, then $d \leq p+q+2$.*

Problem 1.3. *[9] For each d, $4 \leq d \leq p+q+2$, either find the super (a,d)-$(G+e)$-antimagic total labeling of the graph $G_u[S_n]$, $n \geq 3$, or prove that this labeling does not exist.*

In this paper, a partial solution to the above problem is presented.

2. Main Results

Let x and y be two positive integers with $x < y$. Throughout the paper, $[x,y]$ denotes the set $\{i \in \mathbf{N} : x \leq i \leq y\}$. Also, $[n]$ denotes the set of all positive integers less than or equal to n.

Theorem 2.1. *If the graph $G_u[S_n], n \geq 3$, admits the super (a,d)-$G+e$-antimagic total labeling, where $d = p+q+2$, then $a = \frac{(p+1)(p+2)}{2} + (p+n)(q+1) + \frac{(q+1)(q+2)}{2}$.*

Proof. Let $G' \cong G_u[S_n]$. Suppose there exists a bijection $f : V(G') \cup E(G') \to \{1,2,3,...,p+q+2n\}$ which is a super $(a,d)-G+e-$ antimagic total labeling of G'. Let $w(H') = \sum_{v \in V(H')} f(v) + \sum_{e \in E(H')} f(e)$ be the weight of the subgraph H' isomorphic to $G+e$ and let $W = \{w(H') : H' \cong G+e\} = \{a, a+d, a+2d, ..., a+(t-1)d\}$ be the set of H' weights and t be the number of subgraphs. Here $t = n$. The minimum possible weight of H' is at least $\frac{(p+1)(p+2)}{2} + (q+1)(p+n) + \frac{(q+1)(q+2)}{2}$ (i.e.,) $a \geq \frac{(p+1)(p+2)}{2} + (q+1)(p+n) + \frac{(q+1)(q+2)}{2}$. The maximum possible weight of H' is not more than $(p+1)(p+n) - \frac{p(p+1)}{2} + (q+1)(p+q+2n) - \frac{q(q+1)}{2}$,i.e., $a+(t-1)d \leq (p+1)(p+n) - \frac{p(p+1)}{2} + (q+1)(p+q+2n) - \frac{q(q+1)}{2}$. Since $d = p+q+2$, it follows that $a \leq \frac{(p+1)(p+2)}{2} + (q+1)(p+n) + \frac{(q+1)(q+2)}{2}$. Hence $a = \frac{(p+1)(p+2)}{2} + (q+1)(p+n) + \frac{(q+1)(q+2)}{2}$. $\square$

Theorem 2.2. *If the graph $G_u[S_n], n \geq 3$, admits the super $(a, p+q+2-i)$-$G+e$-antimagic total labeling, then $L \leq a \leq L+(n-1)i, 1 \leq i \leq p+q-2$, where $L = \frac{(p+1)(p+2)}{2} + (p+n)(q+1) + \frac{(q+1)(q+2)}{2}$.*

Proof. The proof of this result is similar to the above Theorem 2.1. □

Observation 2.3. *If the graph $G_u[S_n]$ admits the super (a,d)-$G+e$-antimagic total labeling, then*

$$nw(G) + \sum_{i=1}^{n} s_i = an + \frac{n(n-1)d}{2},$$

where $w(G) = a - s_1$, a is the weight of the subgraph $H \cong G + \{e = ux_1\}$ and $s_i = f(x_i u) + f(x_i), 1 \leq i \leq n$, $x_i \in V(S_n)$ and $x_i u \in E(S_n)$.

Theorem 2.4. *There is no super (a,d)-$G+e$-antimagic total labeling of the graph $G_u[S_n]$, $n \geq 4$ and even, where $d = p+q+2$.*

Proof. Suppose the graph $G_u[S_n], n \geq 3$ admits the super (a,d)-$G+e$-antimagic total labeling f. By Theorem 2.1, we get $a = \frac{(p+1)(p+2)}{2} + (p+n)(q+1) + \frac{(q+1)(q+2)}{2}$ and hence the subgraph $H_1 \cong G+e$ has weight a and the vertices and edges labels take values from $\{1,2,3,\ldots,p,p+1\}$ and $\{p+n+1, p+n+2, \ldots, p+n+q+1\}$ respectively. Let $e = ux_1$ be an edge with $x_1 \in V(S_n)$. Let $f(ux_1) = y$ and $f(x_1) = x$. Then $x \in \{1,2,3,\ldots,p,p+1\}$ and $y \in \{p+n+1, p+n+2, \ldots, p+n+q+1\}$. Let $s_1 = f(ux_1)+f(x_1) = x+y$. Then $w(G) = a - s_1$, where $p+n+2 \leq s_1 \leq 2p+q+n+2$, that is $[p+n+2 \;\; 2p+q+n+2]$. Let $x' \in \{p+2, p+3, \ldots, p+n-1, p+n\}$ and $y' \in \{p+n+q+2, p+n+q+3, \ldots, p+n+q+n\}$. Then $2p+n+q+4 \leq x'+y' \leq 2p+3n+q$. From Observation 2.3, we get

$$na - (n-1)s_1 + \sum_{i=2}^{n} s_i = an + \frac{n(n-1)(p+q+2)}{2},$$

where $s_i = f(x_i u) + f(x_i), x_i \in V(S_n), x_i u \in E(S_n), 1 \leq i \leq n$ and hence

$$x + y = s_1 = \frac{1}{n-1}\sum_{i=2}^{n} s_i - \frac{n(p+q+2)}{2}. \tag{2.1}$$

If $S = \{s_2, s_3, \ldots, s_n\} \in \{2p+q+n+4, 2p+q+n+5, \ldots, 2p+q+3n\}$ with $s_2 \neq s_3 \neq \ldots \neq s_n$, then substituting the first $(n-1)$ values in the equation

(2.1) we get the minimum possible s_1 is $s_1 \geq 2p+q+n+3-\frac{n}{2}(p+q+1)$ and substituting the last $(n-1)$ values from S in the equation (2.1), the maximum possible s_1 is found to be, $s_1 \leq 2p+q+3n+1-\frac{n}{2}(p+q+3)$. Therefore, $2p+q+n+3-\frac{n}{2}(p+q+1) \leq s_1 \leq 2p+q+3n+1-\frac{n}{2}(p+q+3)$. If $n=2k$, then $p(2-k)+q(1-k)+k+3 \leq s_1 \leq p(2-k)+q(1-k)+3k+1$. If $k \geq 2$ then $s_1 \notin [p+2k+2 \;\; 2p+q+2k+2]$, which is a contradiction. □

Theorem 2.5. *There is no super (a,d)-$G+e$-antimagic total labeling of the graph $G_u[S_3]$, where $d = p+q+2$.*

Proof. Suppose $G' \cong G_u[S_3]$ admits the super (a,d)-$G+e$-antimagic total labeling. Then there exists a bijective function $f : V(G') \cup E(G') \to \{1,2,3,\ldots,|V(G')+E(G')|\}$ with the subgraphs $G+e$ weights form an arithmetic progression $\{a, a+d, a+2d\}$. Let $\{x_1,x_2,x_3\}$ and $\{e_{q+1} = ux_1, e_{q+2} = ux_2, e_{q+3} = ux_3\}$ be the vertex and edge set of S_3. From Theorem 2.1, we get $a = \frac{(p+1)(p+2)}{2} + (p+n)(q+1) + \frac{(q+1)(q+2)}{2}$ and

$$f(ux_1) + f(x_1) = s_1 = \frac{1}{2}(s_2+s_3) - \frac{3(p+q+2)}{2}.$$

If $s_2, s_3 \in \{2p+q+7, 2p+q+8, 2p+q+9\}$ with $s_2 \neq s_3$ then the possible 2-tuples (s_2,s_3) are $(2p+q+7, 2p+q+8), (2p+q+7, 2p+q+9)$ and $(2p+q+8, 2p+q+9)$. If $(s_2,s_3) = (2p+q+7, 2p+q+8)$, then $s_1 = p+4+\frac{1}{2}(p-q+1)$. The vertices and edges of the subgraphs H_2 and H_3 receive the labels as $\{p+2, p+3\}$ and $\{p+q+5, p+q+6\}$ and hence the possible sum of these labels are $\{2p+q+7, 2p+q+8, 2p+q+9\}$. Suppose $w(H_3) = a+2d = a+2(p+q+2)$. From Observation 2.3, $w(H_3) = w(G)+t$, where $t \in \{2p+q+7, 2p+q+8, 2p+q+9\}$, which implies, $a - s_1 + t = a + 2p + 2q + 4$. If $t = 2p+q+7$ or $2p+q+8$ then substituting the values s_1 and t in $2p+2q+4+s_1-t = 0$, it follows, $p < 0$, which is a contradiction. If $t = 2p+q+9$ then substituting the values s_1 and t in $2p+2q+4+s_1-t=0$, it follows, $p = q+\frac{19}{3}$ is not an integer, which is a contradiction. If $(s_2,s_3) = (2p+q+7, 2p+q+9)$, or $(2p+q+8, 2p+q+9)$ then we get $s_1 < p+5$, which is a contradiction. □

Theorem 2.6. *The graph $G_u[S_3]$ admits the super (a,d)-$G+e$-antimagic total labeling, where $4 \leq d \leq \lfloor \frac{p}{2} \rfloor + 3$.*

Proof. Let $V(G_u[S_3]) = \{v_i, w_1, w_2, w_3, 1 \leq i \leq p\}$ and $E(G_u[S_3]) = \{e_i, e_{q+1}, e_{q+2}, e_{q+3}, 1 \leq i \leq q\}$. Then $|V(G_u[S_3])| = p+3$ and $|E(G_u[S_3])| = q+3$. A bijection $f : V \cup E \to \{1,2,\ldots,p+q+6\}$ is defined as follows:

Label the vertices $v_i, 1 \leq i \leq p$ with the elements of A and label the edges $e_i, 1 \leq i \leq q$ with the elements of B in any order, where $A = [p+2] - \{p-(2i-7), p-(i-4)\}, 4 \leq i \leq \lfloor \frac{p}{2} \rfloor + 3$ and $B = \{p+3+i, 1 \leq i \leq q-2\} \cup \{p+q+3, p+q+4\}$. The weight of G is the same for all the weights of the subgraphs $G+e_j, j=1,2,3$. So, it is enough to find the labels of vertices and edges of the star S_3. Now, for each $i, 4 \leq i \leq \lfloor \frac{p}{2} \rfloor + 3$, the labeling f_i^1 is given by

$$\begin{aligned} f_i^1(w_1) &= p-(2i-7) \\ f_i^1(w_2) &= p-(i-4) \\ f_i^1(w_3) &= p+3 \\ f_i^1(e_{q+1}) &= p+q+2 \\ f_i^1(e_{q+2}) &= p+q+5 \\ f_i^1(e_{q+3}) &= p+q+6. \end{aligned}$$

Thus, the induced sums of the labels of vertices and edges of S_3 are $2p+q-2i+9$,$2p+q-i+9$ and $2p+q+9$. Hence, $d=i, 4 \leq i \leq \lfloor \frac{p}{2} \rfloor + 3$. □

Example 2.7. *The graph $G_u[S_n] \cong C_5[S_3]$ admits the super $(93,5)$-C_5+e-antimagic total labeling which is shown in Figure 1.*

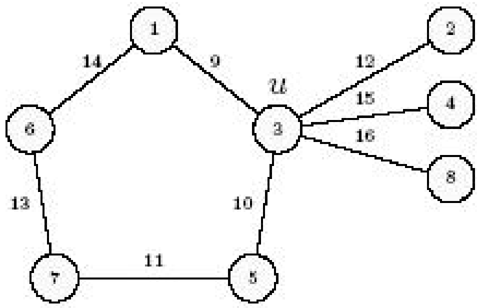

Figure 1: Super $(93,5)$-C_5+e-antimagic total labeling of $C_5[S_3]$.

Theorem 2.8. *Let $G_u[S_n]$ be a graph of order $p+n$ and size $q+n$ with $n \geq 3$, $i \geq 1$ and $p,q \geq 2i(n-1)$. Then the graph $G_u[S_n]$ admits the super (a,d)-$G+e$-antimagic total labeling, where $d=4i$.*

Proof. Let $V(G_u[S_n]) = \{v_i, 1 \leq i \leq p\} \cup \{w_j, 1 \leq j \leq n\}$ and $E(G_u[S_n]) = \{e_i, 1 \leq i \leq q\} \cup \{e_{q+j}, 1 \leq j \leq n\}$. Then $|V(G_u[S_n])| = p+n$ and $|E(G_u[S_n])| = q+n$. A bijection $f: V \cup E \to \{1,2,\ldots,p+q+2n\}$ is defined as follows:

Label the vertices $v_j, 1 \leq j \leq p$ with the elements of A and label the edges $e_j, 1 \leq j \leq q$ with the elements of B in any order, where $A = [p+n] - \{p+n-2i(n-r)i, 1 \leq r \leq n\}$ and $B = [p+q+2n] - \{p+q+2n-2i(n-r), 1 \leq r \leq n\}$. The weight of G is the same for all the weights of the subgraphs $G+e_j, j = 1,2,3,\ldots,n$. So, it is enough to find the labels of vertices and edges of the star S_n. Now, for each i, the labeling f_i^2 is defined by $f_i^2(w_r) = p+n-2i(n-r), 1 \leq r \leq n$ and $f_i^2(e_{q+r}) = p+q+2n-2i(n-r), 1 \leq r \leq n$.

Thus, the induced sums of the labels of vertices and edges of S_n are $2p+q+3n-4i(n-r), 1 \leq r \leq n$ form an arithmetic progression with $d = 4i$. $\square$

Example 2.9. *The graph $G_u[S_n] \cong W_5[S_4]$ admits the super $(210,4)$-W_5+ e-antimagic total labeling and is given in Figure 2.*

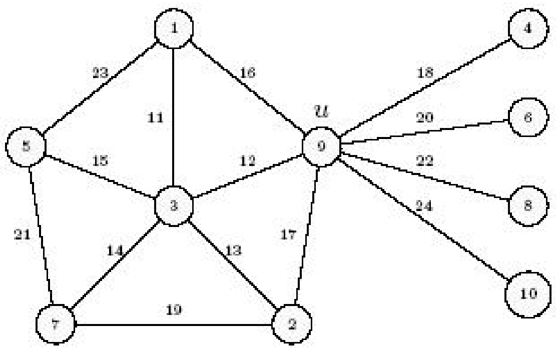

Figure 2: Super $(210,4)$-W_5+e-antimagic total labeling of $W_5[S_4]$.

Theorem 2.10. *Let $G_u[S_n]$ be a graph of order $p+n$ and size $q+n$ with $n \geq 3$, $i \geq 1$ and $p,q \geq 2i(n-1)$. Then the graph $G_u[S_n]$ admits the super (a,d)-$G+e$-antimagic total labeling, where $d = 4i-2$.*

Proof. Let $V(G_u[S_n]) = \{v_i, 1 \leq i \leq p\} \cup \{w_j, 1 \leq j \leq n\}$ and $E(G_u[S_n]) = \{e_i, 1 \leq i \leq q\} \cup \{e_{q+j}, 1 \leq j \leq n\}$. Then $|V(G_u[S_n])| = p+n$ and $|E(G_u[S_n])| = q+n$. A bijection $f : V \cup E \to \{1,2,\ldots,p+q+2n\}$ is defined as follows:

Label the vertices $v_j, 1 \leq j \leq p$ with the elements of A and label the edges $e_j, 1 \leq j \leq q$ with the elements of B in any order, where $A = [p+n] - \{p+n-(2i-1)(n-r), 1 \leq r \leq n\}$ and $B = [p+q+2n] - \{p+q+2n-(2i-1)(n-r), 1 \leq r \leq n\}$. The weight of G is the same for all the weights of the subgraphs $G+e_j, j=1,2,3,\ldots,n$. So, it is enough to find the labels of vertices and edges of the star S_n. Now, for each i, the labeling f_i^3 is defined by $f_i^3(w_r)=p+n-(2i-1)(n-r), 1\leq r\leq n$ and $f_i^3(e_{q+r})=p+q+2n-(2i-1)(n-r), 1\leq r\leq n$.

Thus, the induced sums of the labels of vertices and edges of S_n are $2p+q+3n-(4i-2)(n-r), 1\leq r\leq n$ form an arithmetic progression with $d=4i-2$. □

Example 2.11. *The graph $G_u[S_n]\cong L_8[S_5]$ admits the super $(936,6)$-L_8+e-antimagic total labeling and is given in Figure 3.*

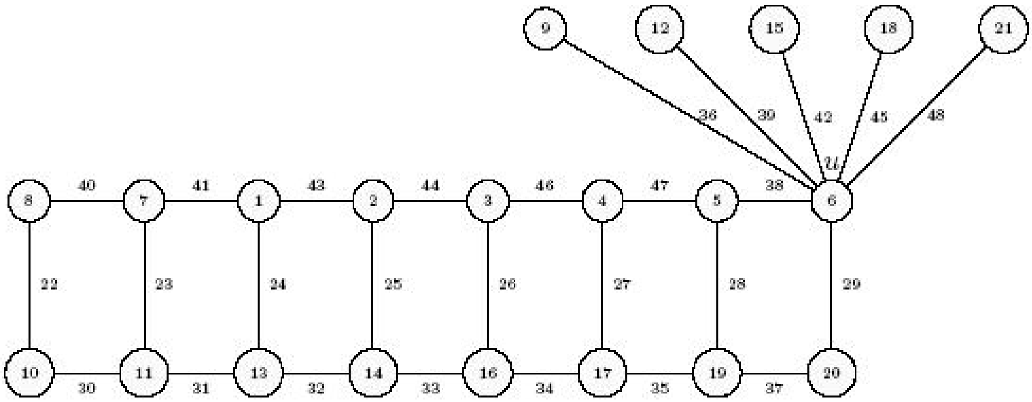

Figure 3: Super $(936,6)$-L_8+e-antimagic total labeling of $L_8[S_5]$.

Theorem 2.12. *Let $G_u[S_n]$ be a graph of order $p+n$ and size $q+n$ with $n\geq 3$, $i\geq 2$ and $p,q\geq 2i(n-1)$. Then the graph $G_u[S_n]$ admits the super (a,d)-$G+e$-antimagic total labeling, where $d=4i-3$.*

Proof. Let $V(G_u[S_n])=\{v_i, 1\leq i\leq p\}\cup\{w_j, 1\leq j\leq n\}$ and $E(G_u[S_n])=\{e_i, 1\leq i\leq q\}\cup\{e_{q+j}, 1\leq j\leq n\}$. Then $|V(G_u[S_n])|=p+n$

and $|E(G_u[S_n])| = q + n$. A bijection $f : V \cup E \rightarrow \{1, 2, \ldots, p + q + 2n\}$ is defined as follows:

Label the vertices $v_j, 1 \leq j \leq p$ with the elements of A and label the edges $e_j, 1 \leq j \leq q$ with the elements of B in any order, where $A = [p+n]-\{p+n-(2i-1)(n-r), 1 \leq r \leq n, i \geq 2\}$ and $B = [p+q+2n]-\{p+q+2n-(2i-2)(n-r), 1 \leq r \leq n, i \geq 2\}$. The weight of G is the same for all the weights of the subgraphs $G+e_j, j = 1, 2, 3, \ldots, n$. So, it is enough to find the labels of vertices and edges of the star S_n. Now, for each i, the labeling f_i^4 is defined by $f_i^4(w_r) = p+n-(2i-1)(n-r), 1 \leq r \leq n, i \geq 2$ and $f_i^4(e_{q+r}) = p+q+2n-(2i-2)(n-r), 1 \leq r \leq n, i \geq 2$. Thus, the induced sums of the labels of vertices and edges of S_n are $2p+q+3n-(4i-3)(n-r), 1 \leq r \leq n$ form an arithmetic progression with $d = 4i - 3$. □

Example 2.13. *The graph $G_u[S_n] \cong D_{4,6}[S_5]$ admits the super $(448, 5)$-$D_{4,6} + e$-antimagic total labeling and is given in Figure 4.*

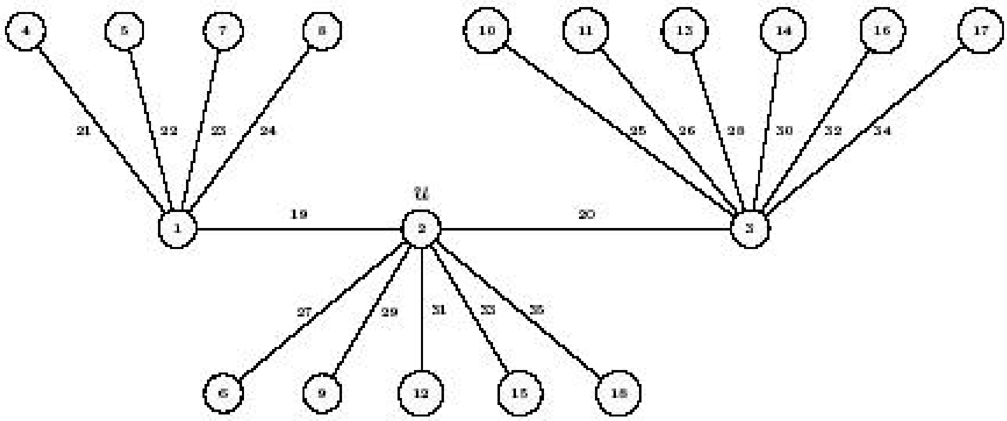

Figure 4: Super $(448, 5)$-$D_{4,6} + e$-antimagic total labeling of $D_{4,6}[S_5]$.

Theorem 2.14. *Let $G_u[S_n]$ be a graph of order $p + n$ and size $q + n$ with $n \geq 3$, $i \geq 2$ and $p, q \geq 3i(n-1)$. Then the graph $G_u[S_n]$ admits the super (a, d)-$G + e$-antimagic total labeling, where $d = 4i - 1$.*

Proof. Let $V(G_u[S_n]) = \{v_i, 1 \le i \le p\} \cup \{w_j, 1 \le j \le n\}$ and $E(G_u[S_n]) = \{e_i, 1 \le i \le q\} \cup \{e_{q+j}, 1 \le j \le n\}$. Then $|V(G_u[S_n])| = p+n$ and $|E(G_u[S_n])| = q+n$. A bijection $f : V \cup E \to \{1, 2, \ldots, p+q+2n\}$ is defined as follows:

Label the vertices $v_j, 1 \le j \le p$ with the elements of A and label the edges $e_j, 1 \le j \le q$ with the elements of B in any order, where $A = [p+n] - \{p+n-3i(n-r), 1 \le r \le n, i \ge 2\}$ and $B = [p+q+2n] - \{p+q+n-(i-1)(n-r), 1 \le r \le n, i \ge 2\}$. The weight of G is the same for all the weights of the subgraphs $G+e_j, j = 1, 2, 3, \ldots, n$. So, it is enough to find the labels of vertices and edges of the star S_n. Now, for each i, the labeling f_i^5 is defined by $f_i^5(w_r) = p+n-3i(n-r), 1 \le r \le n, i \ge 2$ and $f_i^5(e_{q+r}) = p+q+2n-(i-1)(n-r), 1 \le r \le n, i \ge 2$. Thus, the induced sums of the labels of vertices and edges of S_n are $2p+q+3n-(4i-1)(n-r), 1 \le r \le n, i \ge 2$ form an arithmetic progression with $d = 4i-1$. □

Example 2.15. *The graph $G_u[S_n] \cong T[S_4]$ admits the super $(1648, 11)$-$T+e$-antimagic total labeling and is given in Figure 5.*

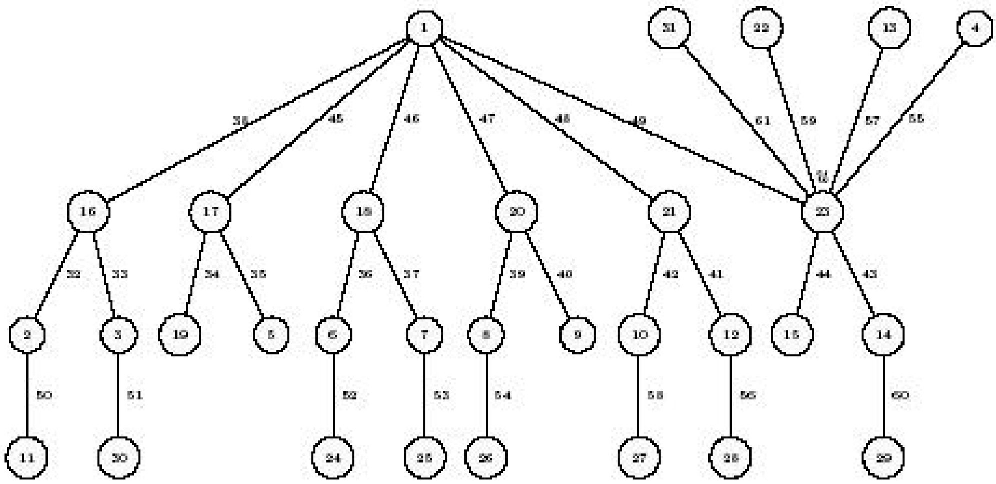

Figure 5: Super $(1648, 11)$-$T+e$-antimagic total labeling of $T[S_4]$.

3. Conclusion and Scope

In this paper, a partial solution is obtained to an open problem posed by Kathiresan and David Laurencea [9]. First, the graph $G_u[S_n], n$ is even, has no super $(a, p+q+2)$-$G+e$-antimagic total labeling is proved. Then the existence of super (a,d)-$G+e$-antimagic total labeling of the graph $G_u[S_n]$ for the given d values is established. The rest of the solution to the problem remains still open.

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References

[1] A. Ahmad, M. Bača, M. Lascsáková and A. Semaničová-Feňovčíková, “Super magic and antimagic labelings of disjoint union of plane graphs”, *Science international*, vol. 24, no. 1, pp. 21-25, 2012. [On line]. Available: https://bit.ly/3tNMxRE

[2] M. Bača, M. Numan and A. Semaničová-Feňovčíková, “Super d-antimagic labelings of disjoint union of generalized prisms”, *Utilitas mathematica*, vol. 103, pp. 299-310, 2017.

[3] G. Chartrand and L. Lesniak, *Graphs & Digraphs*, 4th ed. Boca Raton: CRC Press, 2005.

[4] S. David Laurence and K. M. Kathiresan, “On super (A,d)-ph-antimagic total labeling of stars”, *AKCE International journal of graphs and combinatorics*, vol. 12, no. 1, pp. 54–58, 2015. doi: 10.1016/j.akcej.2015.06.008

[5] J. A. Gallian, “A dynamic survey of graph labelling”, *The electronic journal of combinatorics*, #DS6, 2020.

[6] A. Gutiérrez and A. Lladó, “Magic coverings”, *Journal of combinatorial mathematics and combinatorial computing*, vol. 55, pp. 43-56, 2005.

[7] N. Hartsfield and G. Ringel, *Pearls in graph theory*, Boston: Academic Press, 1994.

[8] N. Inayah, A. N. M. Salman, R. Simanjuntak, “On (a, d)-H-antimagic coverings of graphs”, *Journal of combinatorial mathematics and combinatorial computing*, vol. 71, pp. 273-281, 2009.

[9] K. Kathiresan and S. D. Laurence, “On super (a, d)-H-antimagic total covering of star related graphs”, *Discussiones mathematicae graph theory*, vol. 35, no. 4, p. 755, 2015. doi: 10.7151/dmgt.1832

[10] A. Kotzig and A. Rosa, “Magic valuations of finite graphs”, *Canadian mathematical bulletin*, vol. 13, no. 4, pp. 451–461, 1970. doi: 10.4153/cmb-1970-084-1

[11] K.-W Lih, “On magic and consecutive labelings of plane graphs”, *Utilitas mathematica*, vol. 24, pp. 165-197, 1983.

[12] S. K. Patel and J. Vasava, “Some results on (a, d)-distance antimagic labeling”, *Proyecciones (Antofagasta)*, vol. 39, no. 2, pp. 361–381, 2020. doi: 10.22199/issn.0717-6279-2020-02-0022

[13] R. Simanjuntak, M. Miller and F. Bertault, “Two new (a, d)-antimagic graph labelings”, in *Proceedings of the Eleventh Australasian Workshop on Combinatorial Algorithms (AWOCA 2000): July 29th - August 1st, 2000, Hunter Valley New South Wales*, J. Ryab, Ed., 2000, pp. 179-189.

[14] K. A. Sugeng, M. Miller, Slamin, and M. Bača, “(a,d)-edge-antimagic total labelings of Caterpillars”, *Lecture Notes in Computer Science*, vol. 3330, pp. 169-180, 2005.

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