Proyecciones Journal of Mathematics
Vol. 41, N^o 6, pp. 1439-1456, December 2022.
Universidad Católica del Norte
Antofagasta - Chile

10.22199/issn.0717-6279-5011

Codiskcyclic sets of operators on complex topological vector spaces

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Received : July 2021. Accepted : January 2022

Abstract

Let X be a complex topological vector space and $L(X)$ the set of all continuous linear operators on X. In this paper, we extend the notion of the codiskcyclicity of a single operator $T \in L(X)$ to a set of operators $\Gamma \subset L(X)$. We prove some results for codiskcyclic sets of operators and we establish a codiskcyclicity criterion. As an application, we study the codiskcyclicity of C_0-semigroups of operators.

Subjclass[2010]: *47A16.*

Keywords: *Hypercyclity, supercyclicity, diskcyclicity, codiskcyclicity, C_0-semigroup.*

1. Introduction and Preliminary

Let X be a complex topological vector space and $L(X)$ the set of all continuous linear operators on X. By an operator, we always mean a continuous linear operator.

The most studied notion in linear dynamics is that of hypercyclicity: an operator $T \in L(X)$ is called *hypercyclic* if there exists $x \in X$ such that the orbit of x under T satisfies

$$\overline{Orb(T,x)} = \overline{\{T^n x : n \in \mathbf{N}\}} = X.$$

Such a vector x is called hypercyclic for T. The set of all hypercyclic vectors for T is denoted by $HC(T)$.

Another important notion in linear dynamics is that of supercyclicity which was introduced in [11]. We say that T is *supercyclic* if there exists $x \in X$ whose projective orbit satisfies

$$\overline{\mathbf{C}.Orb(T,x)} = \overline{\{\alpha T^n x : \alpha \in \mathbf{C},\, n \in \mathbf{N}\}} = X.$$

The vector x is called a supercyclic vector for T. We denote by $SC(T)$ the set of all supercyclic vectors. For more information about hypercyclic and supercyclic operators, see [7, 10, 9, 16, 17].

Another notion in linear dynamics which was been studied by many authors is that of codiskcyclicity: an operator T is called *codiskcyclic* if ther exists $x \in X$ whose codisk satisfies orbit

$$\overline{\mathbf{U}.Orb(T,x)} = \overline{\{\alpha T^n x : \alpha \in \mathbf{U},\, n \geq 0\}} = X,$$

when $\mathbf{U} := \{\alpha \in \mathbf{C} : |\alpha| \geq 1\}$. In this case, the vector x is called a codiskcyclic vector for T. The set of all codiskcyclic vectors for T is denoted by $\mathbf{U}C(T)$.

In the case of a separable complex Banach space, an operator T is codiskcyclic if and only if it is codisk transitive, that is for each pair (U,V) of nonempty open sets there exist some $\alpha \in \mathbf{U}$ and some $n \geq 0$ such that $\alpha T^n(U) \cap V \neq \emptyset$.

For a general overview of the codiskcyclicity, see [12, 13, 15, 18].

Recently, some notions of linear dynamical systems were introduced for a set Γ of operators instead of a single operator T, see [1, 2, 3, 4, 5, 6]: A set Γ of operators is called hypercyclic if there exists a vector x in X such that its orbit under Γ satisfies $Orb(\Gamma,x) = \{Tx : T \in \Gamma\}$, is a dense subset of X. If there exits a vector $x \in X$ such that $\mathbf{C}.Orb(\Gamma,x) = \{\alpha Tx : T \in$

Γ, $\alpha \in \mathbf{C}\}$, is a dense subset of X, then Γ is supercyclic. If $x \in X$ is a vector such that $\text{span}\{Orb(\Gamma, x)\} = \text{span}\{Tx : T \in \Gamma\}$ is dense in X, then Γ is cyclic. If there exists a vector $x \in X$ such that its disk orbit under T, $\mathbf{D}.Orb(\Gamma, x) = \{\alpha Tx : T \in \Gamma, \alpha \in \mathbf{D}\}$, is a dense subset X, then Γ is called a diskcyclic, when $\mathbf{D}$ is the unit closed disk. In each case, the vector x is called a hypercyclic, a supercyclic, a cyclic and a diskcyclic vector for Γ, respectively.

In this paper, we continue the study of the dynamics of a set of operator by introducing the concept of codiskcyclicity for a set of operators.

In Section 2, we introduce and study the codiskcyclicity for a set of operators. In particular, we show that the set of codiscyclic vectors of a set Γ is a G_δ type and we prove that codiskcyclicity is preserved under quasi-similarity.

In Section 3, we extend the notion of codisk transitivity of a single operator to a set of operators. We give the relation between this notion and the concept of codiskcyclic and we establish a codiskcyclic criterion.

In Section 4, we study the codiskcyclicity of a C_0-semigroup of operators. We show that the codiskcyclicity and the codisk transitivity are equivalent and we prove that a codiskcyclic C_0-semigroup of operators exists on X if and only if $\dim(X) = 1$ or $\dim(X) = \infty$.

2. Codiskcyclic Sets of Operators

In the following definition, we introduce the notion of the codiskcyclicity of a set of operators instead of a single operator.

Definition 2.1. *We say that a set Γ of operators on X is codiskcyclic if there exists $x \in X$ for which the codisk orbit of x under Γ*

$$\mathbf{U}.Orb(\Gamma, x) := \{\alpha Tx : \alpha \in \mathbf{U}, T \in \Gamma\},$$

is dense in X. The vector x is called a codiskcyclic vector for Γ. The set of all codiskcyclic vectors for Γ is denoted by $\mathbf{U}C(\Gamma)$.

Remark 2.2. *An operator T is codiskcyclic if and only if the set $\Gamma = \{T^n : n \geq 0\}$ is codiskcyclic.*

Example 2.3. *Let f be a nonzero linear form on a locally convex space X and D be a subset of X such that the set $\mathbf{U}D := \{\alpha x : \alpha \in \mathbf{U}, x \in D\}$ is a dense subset of X. For all $x \in X$, let T_x defined by $T_x y = f(y)x$, for* *all $y \in X$. Put $\Gamma_f = \{T_x : x \in D\}$ and let y be a vector of X such that $f(y) \neq 0$. Then $\mathbf{U}.Orb(\Gamma_f, y) = \{\alpha T_x y : x \in D, \alpha \in \mathbf{U}\} = \{\alpha f(y)x : x \in D, \alpha \in \mathbf{U}\} = \mathbf{U}D$. Hence, Γ_f is codiskcyclic.*

A necessary condition for the codiskcyclicity is given by the following proposition.

Proposition 2.4. *Let X be a complex normed space and Γ a subset of $L(X)$. If x is a codiskcyclic vector for Γ, then $\sup\{\|\alpha Tx\| : \alpha \in \mathbf{U}, T \in \Gamma\} = +\infty$.*

Proof. Let $x \in \mathbf{U}C(\Gamma)$. Assume that $\sup\{\|\alpha Tx\| : \alpha \in \mathbf{U}, T \in \Gamma\} = m < +\infty$, and let $y \in X$ such that $\|y\| > m$. Since $x \in \mathbf{U}C(\Gamma)$, there exist $\{\alpha_k\} \subset \mathbf{U}$ and $\{T_k\} \subset \Gamma$ such that $\alpha_k T_k x \longrightarrow y$. Hence, $\|y\| \leq m$, which is a contradiction. $\square$

We denote by $\{\Gamma\}'$ the set of all elements of $L(X)$ which commutes with every element of Γ.

Proposition 2.5. *Let X be a complex topological vector space and Γ a subset of $L(X)$. Assume that Γ is codiskcyclic and let $T \in L(X)$ be with dense range. If $T \in \{\Gamma\}'$, then $Tx \in \mathbf{U}C(\Gamma)$, for all $x \in \mathbf{U}C(\Gamma)$.*

Proof. Let O be a nonempty open subset of X. Since T is continuous and of dense range, $T^{-1}(O)$ is a nonempty open subset of X. Let $x \in \mathbf{U}C(\Gamma)$, then there exist $\alpha \in \mathbf{U}$ and $S \in \Gamma$ such that $\alpha Sx \in T^{-1}(O)$, that is $\alpha T(Sx) \in O$. Since $T \in \{\Gamma\}'$, it follows that $\alpha S(Tx) = \alpha T(Sx) \in O$. Hence, $\mathbf{U}Orb(\Gamma, Tx)$ meets every nonempty open subset of X. From this, $\mathbf{U}Orb(\Gamma, Tx)$ is dense in X. That is, $Tx \in \mathbf{U}C(\Gamma)$. $\square$

The following definition is the notions of quasi-similarity and similarity of sets of operators.

Definition 2.6. *[3] Let X and Y be two complex topological vector spaces, $\Gamma \subset L(X)$, and $\Gamma_1 \subset L(Y)$. We say that Γ is quasi-similar to Γ_1 if there exists a continuous map $\phi : X \longrightarrow Y$ with dense range such that $\forall T \in \Gamma$, $\exists S \in \Gamma_1$ such that $S \circ \phi = \phi \circ T$. If ϕ can be chosen to be a homeomorphism, then Γ and Γ_1 are called similar.*

In the following, we prove that the codiskcyclicity is preserved under quasi-similarity.

Proposition 2.7. *If $\Gamma \subset L(X)$ is quasi-similar to $\Gamma_1 \subset L(Y)$, then Γ is codiskcyclic in X implies that Γ_1 is codiskcyclic in Y. Moreover, $\phi(\mathbf{U}C(\Gamma)) \subset \mathbf{U}C(\Gamma_1)$.*

Proof. Assume that Γ is codiskcyclic in X and let ϕ be the continuous operator as in Definition 2.6. Let O be a nonempty open subset of Y, then $\phi^{-1}(O)$ is a nonempty open subset of X. If $x \in \mathbf{U}C(\Gamma)$, then there exist $\alpha \in \mathbf{U}$ and $T \in \Gamma$ such that $\alpha Tx \in \phi^{-1}(O)$, that is $\alpha\phi(Tx) \in O$. Let $S \in \Gamma_1$ such that $S \circ \phi = \phi \circ T$. Hence, $\alpha S(\phi x) = \alpha\phi(Tx) \in O$. Hence, Γ_1 is codiskcyclic and $\phi x \in \mathbf{U}C(\Gamma_1)$. $\square$

Proposition 2.8. *Let $(c_T)_{T\in\Gamma} \subset \mathbf{R}_+^*$. If $\{c_T T : T \in \Gamma\}$ is codiskcyclic and $(k_T)_{T\in\Gamma}$ is such that $c_T \geq k_T > 0$ for all $T \in \Gamma$, then the set $\{k_T T : T \in \Gamma\}$ is codiskcyclic.*

Proof. Let x be a codiskcyclic vector for $\{c_T T : T \in \Gamma\}$. Since $c_T \geq k_T$ for all $T \in \Gamma$, we have $\mathbf{U}Orb(\{c_T T : T \in \Gamma\}, x) \subset \mathbf{U}Orb(\{k_T T : T \in \Gamma\}, x)$. Since $\mathbf{U}Orb(\{c_T T : T \in \Gamma\}, x)$ is dense in X, it follows that $\mathbf{U}Orb(\{k_T T : T \in \Gamma\}, x)$ is dense in X, this means that $\{k_T T : T \in \Gamma\}$ is codiskcyclic in X. $\square$

Proposition 2.9. *Let $\{X_i\}_{i=1}^n$ be a family of complex topological vector spaces and Γ_i a subset of $L(X_i)$, for $1 \leq i \leq n$. If $\bigoplus_{i=1}^n \Gamma_i$ is a codiskcyclic set in $\bigoplus_{i=1}^n X_i$, then Γ_i is a codiskcyclic set in X_i, for all $1 \leq i \leq n$.*

Proof. If $1 \leq j \leq n$, then $\bigoplus_{i=1}^n \Gamma_i$ is quasi-similar to Γ_j, and the result follow by Proposition 2.7. $\square$

Let X be a complex topological vector space. The following proposition gives a characterization of the set of codiskcyclic vector of set of operators using a countable basis of the topology of X. Note that the set $\mathbf{D}$ is the unit closed disk defined by $\mathbf{D} = \{\alpha \in \mathbf{C} : |\alpha| \leq 1\}$.

Proposition 2.10. *Let X be a second countable complex topological vector space and Γ a subset of $L(X)$. If Γ is codiskcyclic, then*

$$\mathbf{U}C(\Gamma) = \bigcap_{n\geq 1}\left(\bigcup_{\beta\in\mathbf{D}}\bigcup_{T\in\Gamma} T^{-1}(\beta U_n)\right),$$

where $(U_n)_{n\geq 1}$ is a countable basis of the topology of X. As a consequence, $\mathbf{U}C(\Gamma)$ is a G_δ type set.

Proof. Let $x \in X$. Then, $x \in \mathbf{U}C(\Gamma)$ if and only if $\overline{\mathbf{U}Orb(\Gamma, x)} = X$. Equivalently, for all $n \geq 1$, $U_n \cap \mathbf{U}Orb(\Gamma, x) \neq \emptyset$, that is for all $n \geq 1$, there exist $\lambda \in \mathbf{U}$ and $T \in \Gamma$ such that $\lambda Tx \in U_n$. This is equivalent to the fact that for all $n \geq 1$, there exist $\beta \in \mathbf{D}$ and $T \in \Gamma$ such that $x \in T^{-1}(\beta U_n)$. Hence, $x \in \bigcap_{n\geq 1} \bigcup_{\beta \in \mathbf{D}} \bigcup_{T \in \Gamma} T^{-1}(\beta U_n)$. □

3. Density and Codisk Transitivity of Sets of Operators

In the following definition, we introduce the notion of a codisk transitive set of operators which generalize the codisk transitivity of a single operator.

Definition 3.1. *We say that a set Γ of operators on X is codisk transitive if for any pair (U, V) of nonempty open subsets of X, there exist $\alpha \in \mathbf{U}$ and $T \in \Gamma$ such that $T(\alpha U) \cap V \neq \emptyset$.*

Remark 3.2. *An operator $T \in L(X)$ is codisk transitive if and only if $\Gamma = \{T^n : n \geq 0\}$ is codisk transitive.*

Example 3.3. *Assume that X is a locally convex space. Let $x, y \in X$ and let f_y be a linear form on X such that $f_y(y) \neq 0$. Let $T_{f_y,x}$ be an operator defined by $T_{f_y,x}z = f_y(z)x$. Define $\Gamma = \{T_{f_y,x} : x, y \in X$ such that $f_y(y) \neq 0\}$. Let U and V be two nonempty open subsets of X. There exist $x, y \in X$ such that $x \in U$ and $y \in V$. We have $T_{f_y,x}(y) = f_y(y)x$. Since $0 < f_y(y) < 1$, it follows that $x = \frac{1}{f_y(y)}T_{f_y,x}(y)$. Hence $x \in U$ and $x \in \frac{1}{f_y(y)}T_{f_y,x}(V)$, which implies that $U \cap \frac{1}{f_y(y)}T_{f_y,x}(V) \neq \emptyset$. Thus Γ is a codisk transitive.*

In the following proposition, we prove that the codisk transitivity of sets of operators is preserved under quasi-similarity.

Proposition 3.4. *Assume that $\Gamma \subset L(X)$ is quasi-similar to $\Gamma_1 \subset L(Y)$. If Γ is codisk transitive in X, then Γ_1 is codisk transitive in Y.*

Proof. Assume that Γ is codisk transitive and let ϕ be the continuous operator as in Definition 2.6. Let U and V be nonempty open subsets of X. Since ϕ is continuous and of dense range, $\phi^{-1}(U)$ and $\phi^{-1}(V)$ are nonempty and open sets. Since Γ is codisk transitive in X, there exist $y \in \phi^{-1}(U)$ and $\alpha \in \mathbf{U}$, $T \in \Gamma$ with $\alpha Ty \in \phi^{-1}(V)$, which implies that $\phi(y) \in U$ and $\alpha\phi(Ty) \in V$. Let $S \in \Gamma$ such that $S \circ \phi = \phi \circ T$. Then, $\phi(y) \in U$ and $\alpha S\phi(y) \in V$. Thus, $\alpha S(U) \cap V \neq \emptyset$. Hence, Γ_1 is codisk transitive in Y. □

In the following result, we give necessary and sufficient conditions for a set of operators to be codisk transitive.

Theorem 3.5. *Let X be a complex normed space and Γ a subset of $L(X)$. The following assertions are equivalent*:

(i) Γ *is codisk transitive*;

(ii) *For each x, $y \in X$, there exists sequences $\{x_k\}$ in X, $\{\alpha_k\}$ in $\mathbf{U}$ and $\{T_k\}$ in Γ such that $x_k \longrightarrow x$ and $\alpha_k T_k(x_k) \longrightarrow y$*;

(iii) *For each x, $y \in X$ and for W a neighborhood of 0, there exist $z \in X$, $\alpha \in \mathbf{U}$ and $T \in \Gamma$ such that $x - z \in W$ and $\alpha T(z) - y \in W$.*

Proof. $(i) \Rightarrow (ii)$ Let x, $y \in X$. For all $k \geq 1$, let $U_k = B(x, \frac{1}{k})$ and $V_k = B(y, \frac{1}{k})$. Then U_k and V_k are nonempty open subsets of X. Since Γ is codisk transitive, there exist $\alpha_k \in \mathbf{U}$ and $T_k \in \Gamma$ such that $\alpha_k T_k(U_k) \cap V_k \neq \emptyset$. For all $k \geq 1$, let $x_k \in U_k$ such that $\alpha_k T_k(x_k) \in V_k$, then $\|x_k - x\| < \frac{1}{k}$ and $\|\alpha_k T_k(x_k) - y\| < \frac{1}{k}$, this implies that $x_k \longrightarrow x$ and $\alpha_k T_k(x_k) \longrightarrow y$.
$(ii) \Rightarrow (iii)$ Clear.
$(iii) \Rightarrow (i)$ Let U and V be two nonempty open subsets of X. Then there exists x, $y \in X$ such that $x \in U$ and $y \in V$. Since for all $k \geq 1$, $W_k = B(0, \frac{1}{k})$ is a neighborhood of 0, there exist $z_k \in X$, $\alpha_k \in \mathbf{U}$ and $T_k \in \Gamma$ such that $\|x - z_k\| < \frac{1}{k}$ and $\alpha_k T_k(z_k) - y\| < \frac{1}{k}$. This implies that $z_k \longrightarrow x$ and $\alpha_k T_k(z_k) \longrightarrow y$. Since U and V are nonempty open subsets of X, $x \in U$ and $y \in V$, there exists $N \in \mathbf{N}$ such that $z_k \in U$ and $\alpha_k T_k(z_k) \in V$, for all $k \geq N$. □

Let Γ be a subset of $L(X)$. In whats follows, we prove that Γ is codisk transitive if and only if it admits a dense subset of codiskcyclic vectors.

Theorem 3.6. *Let X be a second countable Baire complex topological vector space and Γ a subset of $L(X)$. The following assertions are equivalent*:

(i) $\mathbf{U}C(\Gamma)$ *is dense in X;*

(ii) Γ *is codisk transitive.*

As a consequence, a codisk transitive set is codiskcyclic.

Proof. countable basis of the topology of X.
$(i) \Rightarrow (ii)$: Assume that $\mathbf{U}C(\Gamma)$ is dense in X and let U and V be two nonempty open subsets of X. By Proposition 2.10, we have $\mathbf{U}C(\Gamma) = \bigcap_{n\geq 1} \left(\bigcup_{\beta \in \mathbf{D}} \bigcup_{T \in \Gamma} T^{-1}(\beta U_n) \right)$. Hence, for all $n \geq 1$, $A_n := \bigcup_{\beta \in \mathbf{D}} \bigcup_{T \in \Gamma} T^{-1}(\beta U_n)$

is dense in X. Thus, for all n, $m \geq 1$, we have $A_n \cap U_m \neq \emptyset$ which implies that for all n, $m \geq 1$, there exist $\beta \in \mathbf{U}$ and $T \in \Gamma$ such that $T(\beta U_m) \cap U_n \neq \emptyset$. Hence, Γ is a codisk transitive set.
$(ii) \Rightarrow (i)$: Assume that Γ is codisk transitive. Let n, $m \geq 1$, then there exist $\beta \in \mathbf{U}$ and $T \in \Gamma$ such that $T(\beta U_m) \cap U_n \neq \emptyset$, which implies that $T^{-1}(\frac{1}{\beta}U_n) \cap U_m \neq \emptyset$. Hence, for all $n \geq 1$, we have $\bigcup\limits_{\beta \in \mathbf{D}} \bigcup\limits_{T \in \Gamma} T^{-1}(\beta U_n)$ is dense in X. Since X is a Baire space, it follows that
$\mathbf{U}C(\Gamma) = \bigcap_{n \geq 1} \left(\bigcup_{\beta \in \mathbf{D}} \bigcup_{T \in \Gamma} T^{-1}(\beta U_n) \right)$ is a dense subset of X. □

As a consequence of Theorem 3.6, a codisk transitive set is codiskcyclic. In the following theorem, we prove that the converse holds with an additional assumption.

Theorem 3.7. *Let X be a complex topological vector space and Γ a subset of $L(X)$. Assume that for all T, $S \in \Gamma$ with $T \neq S$, there exists $A \in \Gamma$ such that $T = AS$. The following assertions are equivalent*:

(i) Γ *is codiskcyclic;*

(ii) Γ *is codisk transitive.*

Proof. $(ii) \Rightarrow (i)$ This implication is due to Theorem 3.6.
$(i) \Rightarrow (ii)$ Since Γ is codiskcyclic, there exists $x \in X$ such that $\mathbf{U}Orb(\Gamma, x)$ is a dense subset of X. Let U and V be two nonempty open subsets of X, then there exist α, $\beta \in \mathbf{U}$ with $|\alpha| \geq |\beta|$, and T, $S \in \Gamma$ such that $\alpha Tx \in U$ and $\beta Sx \in V$. There exists $A \in \Gamma$ such that $T = AS$. Hence, $\alpha A(Sx) \in U$ and $\beta A(Sx) \in A(V)$, which implies that $U \cap A(\frac{\alpha}{\beta}V) \neq \emptyset$. Hence, Γ is codisk transitive. □

In the following definition we introduce the notion of strictly codisk transitivity of a set of operators. The case of hypercyclicity (resp, supercyclicity, diskcyclicity) were introduced in [2, 1, 5].

Definition 3.8. *We say that a set Γ of operators on X is strictly codisk transitive if for each pair of nonzero elements x, y in X, there exist some $\alpha \in \mathbf{U}$ and $T \in \Gamma$ such that $\alpha Tx = y$.*

Remark 3.9. *An operator $T \in L(X)$ is strictly codisk transitive if and only if the set $\Gamma = \{T^n : n \geq 0\}$ is strictly codisk transitive.*

Proposition 3.10. *If Γ is strictly codisk transitive, then it is codisk transitive. As a consequence, if Γ is strictly codisk transitive, then it is codiskcyclic.*

Proof. Assume that Γ is strictly codisk transitive. If U and V are two nonempty open subsets of X, then there exist $x, y \in X$ such that $x \in U$ and $y \in V$. Since Γ is strictly codisk transitive, it follows that there exist $\alpha \in \mathbf{U}$ and $T \in \Gamma$ such that $\alpha Tx = y$. Hence, $\alpha Tx \in \alpha T(U)$ and $\alpha Tx \in V$. Thus, $\alpha T(U) \cap V \neq \emptyset$, which implies that Γ is codisk transitive. By Theorem 3.6, we deduce that Γ is codiskcyclic. □

In the following proposition, we prove that the strictly codisk transitivity of sets of operators is preserved under similarity.

Proposition 3.11. *If $\Gamma \subset L(X)$ and $\Gamma_1 \subset L(Y)$ are similar, then Γ is strictly codisk transitive in X if and only if Γ_1 is strictly codisk transitive in Y.*

Proof. Assume Γ and Γ_1 are similar and let ϕ be the homeomorphism as in Definition 2.6. Assume that Γ is strictly codisk transitive in X. Let x, $y \in Y$. There exist a, $b \in X$ such that $\phi(a) = x$ and $\phi(b) = y$. Since Γ is strictly codisk transitive in X, there exist $\alpha \in \mathbf{U}$ and $T \in \Gamma$ such that $\alpha Ta = b$, this implies that $\alpha\phi \circ T(a) = \phi(b)$. Let $S \in \Gamma_1$ such that $S \circ \phi = \phi \circ T$. Hence, $\alpha Sx = y$. Hence Γ_1 is strictly codisk transitive in Y. □

Recall that the strong operator topology (SOT for short) on $L(X)$ is the topology with respect to which any $T \in L(X)$ has a neighborhood basis consisting of sets of the form

$$\Omega = \{S \in L(X) : Se_i - Te_i \in U,\, i = 1, 2, \ldots, k\},$$

where $k \in \mathbf{N}$, $e_1, e_2, \ldots e_k \in X$ are linearly independent and U is a neighborhood of zero in X, see [8].

Let x be an element of a complex topological vector space X. Note that $\mathbf{U}_x$ is the subset of X defined by $\mathbf{U}.\{x\} := \mathbf{U}_x = \{\alpha x : \alpha \in \mathbf{U}\}$.

In the following theorem, the proof is also true for norm-density if X is assumed to be a normed linear space.

Theorem 3.12. *For each pair of nonzero vectors x, $y \in X$ with $y \notin \mathbf{U}_x$, there exists a SOT-dense set $\Gamma_{xy} \subset L(X)$ which is not strictly codisk transitive. Furthermore, $\Gamma \subset L(X)$ is a dense nonstrictly codisk transitive set if and only if Γ is a dense subset of Γ_{xy} for some x, $y \in X$.*

Proof. Fix nonzero vectors x, $y \in X$ such that $y \notin \mathbf{U}_x$ and let Γ_{xy} the set defined by

$$\Gamma_{xy} = \{T \in L(X) : y \notin \mathbf{U}_{Tx}\}.$$

Then Γ_{xy} is not strictly codisk transitive. Let Ω be a nonempty open set in $L(X)$ and $S \in \Omega$. If Sx and y are such that $y \notin \mathbf{U}_{Sx}$, then $S \in \Omega \cap \Gamma_{xy}$. Otherwise, putting $S_n = S + \frac{1}{n}I$, we see that $S_k \in \Omega$ for some k, but $S_k x$ and y are such that $y \notin \mathbf{U}_{S_k x}$. Hence, $\Omega \cap \Gamma_{xy} \neq \emptyset$ and the proof is completed.

We prove the second assertion of the theorem. Suppose that Γ is a dense subset of $L(X)$ that is not strictly codisk transitive. Then there are nonzero vectors x, $y \in X$ such that $y \notin \mathbf{U}_{Tx}$ for all $T \in \Gamma$ and hence $\Gamma \subset \Gamma_{xy}$. To show that Γ is dense in Γ_{xy}, assume that Ω_0 is an open subset of Γ_{xy}. Thus, $\Omega_0 = \Gamma_{xy} \cap \Omega$ for some open set Ω in $L(X)$. Then $\Gamma \cap \Omega_0 = \Gamma \cap \Omega \neq \emptyset$.

For the converse, let Γ be a dense subset of Γ_{xy} for some x, $y \in X$. Then Γ is not strictly codisk transitive. Also, since Γ_{xy} is a dense open subset of $L(X)$, we conclude that Γ is also dense in $L(X)$. Indeed, if Ω is any open set in $L(X)$ then $\Omega \cap \Gamma_{xy} \neq \emptyset$ since Γ_{xy} is dense in $L(X)$. On the other hand, $\Omega \cap \Gamma_{xy}$ is open in Γ_{xy} and so it must intersect Γ since Γ is dense in Γ_{xy}. Thus, $\Omega \cap \Gamma \neq \emptyset$ and so Γ is dense in $L(X)$. □

Corollary 3.13. *Let Γ be a dense subset of $L(X)$. There is a subset Γ_1 of Γ such that $\overline{\Gamma_1} = L(X)$ and Γ_1 is not strictly codisk transitive.*

Proof. Let x, $y \in X$ such that $y \notin \mathbf{U}_x$. By Theorem 3.12, there exists a SOT-dense set $\Gamma_{xy} \subset L(X)$ which is not strictly codisk transitive. Put $\Gamma_1 = \Gamma \cap \Gamma_{xy}$. Then Γ_1 is a nonempty and not strictly codisk transitive since Γ_{xy} is not. Moreover, Γ_1 is dense in $L(X)$ since Γ and Γ_{xy} are dense in $L(X)$. □

In the following definition, we introduce that notion of codiskcyclic transitivity of set of operators. The case of hypercyclicity (resp, supercyclicity, diskcyclicity) were introduced in [2, 1, 5].

Definition 3.14. *We say that a set Γ of operators on X is codiskcyclic transitive if $\mathbf{U}C(\Gamma) = X \setminus \{0\}$.*

Remark 3.15. *An operator $T \in L(X)$ is codiskcyclic transitive if and only if the set $\Gamma = \{T^n : n \geq 0\}$ is codiskcyclic transitive.*

It is clear that a codiskcyclic transitive set is codiskcyclic. Moreover, the next proposition shows that codiskcyclic transitivity of sets of operators implies codisk transitivity.

Proposition 3.16. *If Γ is codiskcyclic transitive, then Γ is codisk transitive.*

Proof. Let U and V be two nonempty open subsets of X. There exists $x \in X \setminus \{0\}$ such that $x \in U$. Since Γ is codiskcyclic transitive, there exists $\alpha \in \mathbf{U}$ and $T \in \Gamma$ such that $\alpha Tx \in V$. This implies that $\alpha T(U) \cap V \neq \emptyset$. Hence, Γ is codisk transitive. $\square$

In the following proposition, we prove that the codiskcyclic transitivity is preserved under similarity.

Proposition 3.17. *Assume that Γ and Γ_1 are similar, then Γ is codiskcyclic transitive on X if and only if Γ_1 is codiskcyclic transitive on Y.*

Proof. Assume Γ and Γ_1 are similar and let ϕ be the homeomorphism as in Definition 2.6. If Γ is a codiskcyclic transitive on X, then by Proposition 2.7, $\phi(\mathbf{U}C(\Gamma)) \subset \mathbf{U}C(\Gamma_1)$. Since ϕ is homeomorphism, the result holds. $\square$

Assume that X is a topological vector space and Γ a subset of $L(X)$. The following result shows that the SOT-closure of Γ is not large enough to have more codiskcyclic vectors than Γ.

Proposition 3.18. *If $\overline{\Gamma}$ stands for the SOT-closure of Γ then $\mathbf{U}C(\Gamma) = \mathbf{U}C(\overline{\Gamma})$.*

Proof. We only need to prove that $\mathbf{U}C(\overline{\Gamma}) \subset \mathbf{U}C(\Gamma)$. Fix $x \in \mathbf{U}C(\overline{\Gamma})$ and let U be an arbitrary open subset of X. Then there is some $\alpha \in \mathbf{U}$ and $T \in \overline{\Gamma}$ such that $\alpha Tx \in U$. The set $\Omega = \{S \in L(X) : \alpha Sx \in U\}$ is a SOT-neighborhood of T and so it must intersect Γ. Therefore, there is some $S \in \Gamma$ such that $\alpha Sx \in U$ and this shows that $x \in \mathbf{U}C(\Gamma)$. $\square$

Corollary 3.19. *Let X be a topological vector space and Γ a subset of $L(X)$. Then Γ is codiskcyclic transitive if and only if $\overline{\Gamma}$ is codiskcyclic transitive.*

Proof. Assume that Γ is codiskcyclic transitive, then $\mathbf{U}C(\overline{\Gamma}) = X \setminus \{0\}$. Since by Proposition 3.18, we have $\mathbf{U}C(\overline{\Gamma}) = \mathbf{U}C(\Gamma)$, it follows that $\mathbf{U}C(\Gamma) = X \setminus \{0\}$. Hence, Γ is codiskcyclic transitive. $\square$

In the next definition, we introduce the notion of codiskcyclic criterion of a set of operators which generalizes the definition of codiskcyclic criterion of a single operator.

Definition 3.20. *We say that a set Γ of operators on X satisfies the criterion of codiskcyclicity if there exist two dense subsets X_0 and Y_0 in X and sequences $\{\alpha_k\}$ of $\mathbf{U}$, $\{T_k\}$ of Γ and a sequence of maps $S_k : Y_0 \longrightarrow X$ such that:*

(i) $\alpha_k T_k x \longrightarrow 0$ *for all* $x \in X_0$;

(ii) $\alpha_k^{-1} S_k x \longrightarrow 0$ *for all* $y \in Y_0$;

(iii) $T_k S_k y \longrightarrow y$ *for all* $y \in Y_0$.

Remark 3.21. *An operator $T \in L(X)$ satisfies the criterion of codiskcyclicity for operators if and only if the set $\Gamma = \{T^n : n \geq 0\}$ satisfies the criterion of codiskcyclicity for sets of operators, see [18].*

Theorem 3.22. *Let X be a second countable Baire complex topological vector space and Γ a subset of $L(X)$. If Γ satisfies the criterion of codiskcyclicity, then $\mathbf{U}C(\Gamma)$ is a dense subset of X. As consequence, Γ is codiskcyclic.*

Proof. Let U and V be two nonempty open subsets of X. Since X_0 and Y_0 are dense in X, there exist x_0 and y_0 in X such that $x_0 \in X_0 \cap U$ and $y_0 \in Y_0 \cap V$. For all $k \geq 1$, let $z_k = x_0 + \alpha_k^{-1} S_k y$. We have $\alpha_k^{-1} S_k y \longrightarrow 0$, which implies that $z_k \longrightarrow x_0$. Since $x_0 \in U$ and U is open, there exists $N_1 \in \mathbf{N}$ such that $z_k \in U$, for all $k \geq N_1$. On the other hand, we have $\alpha_k T_k z_k = \alpha_k T_k x_0 + T_k(S_k y_0) \longrightarrow y_0$. Since $y_0 \in V$ and V is open, there exists $N_2 \in \mathbf{N}$ such that $\alpha_k T_k z_k \in V$, for all $k \geq N_2$. Let $N = \max\{N_1, N_2\}$, then $z_k \in U$ and $\alpha_k T_k z_k \in V$, for all $k \geq N$, that is $\alpha_k T_k(U) \cap V \neq \emptyset$, for all $k \geq N$. Hence, Γ is codisk transitive. By Theorem 3.6 we deduce that $\mathbf{U}C(\Gamma)$ is a dense subset of X. We use again Theorem 3.6 to conclude that Γ is codiskcyclic and this complete the proof. $\square$

4. Codiskcyclic C_0-Semigroups of Operators

In this section we will study the particular case when Γ is a C_0-semigroup of operators.

Recall that a family $(T_t)_{t \in \mathbf{R}_+}$ of operators is called a C_0-semigroup of operators if the following three conditions are satisfied:

(i) $T_0 = I$ the identity operator on X;

(*ii*) $T_{t+s} = T_t T_s$ for all t, $s \in \mathbf{R}_+$;

(*iii*) $\lim_{t\to s} T_t x = T_s x$ for all $x \in X$ and $t \in \mathbf{R}_+$.

For more informations about the theory of C_0-semigroups the reader may refer to [14].

C.

Example 4.1. *Let $X = \mathbf{C}$. For all $t \in \mathbf{R}_+$, let $T_t x = \exp(t)x$, for all $x \in \mathbf{C}$. Then $(T_t)_{t\in\mathbf{R}_+}$ is a C_0-semigroup and we have $\mathbf{U}Orb((T_t)_{t\in\mathbf{R}_+}, 1) = \{\alpha T_t(1) : t \in \mathbf{R}_+, \alpha \in \mathbf{U}\} = \{\alpha y : y \in \mathbf{R}^+, \alpha \in \mathbf{U}\}$. Let $x \in \mathbf{C}\backslash\{0\}$. Then $x = |x|\dfrac{x}{|x|} \in \mathbf{U}Orb((T_t)_{t\in\mathbf{R}_+}, 1)$. Hence, $\overline{\mathbf{U}Orb((T_t)_{t\in\mathbf{R}_+}, 1)} = \mathbf{C}$. Thus, $(T_t)_{t\in\mathbf{R}_+}$ is a codiskcyclic C_0-semigroup of operators and 1 is a codiskcyclic vector for $(T_t)_{t\in\mathbf{R}_+}$.*

Recall from [16, Lemma 5.1], that if X is a complex topological vector space such that $2 \leq \dim(X) < \infty$, then X supports no supercyclic C_0-semigroups of operators.

In the following theorem we will prove that the same result holds in the case of codiskcyclicity on a complex topological vector space.

Theorem 4.2. *Assume that $2 \leq dim(X) < \infty$. Then X supports no codiskcyclic C_0-semigroups.*

Proof. By using [16, Lemma 5.1] and the fact that $\mathbf{U}Orb(\Gamma, x) \subset \mathbf{C}Orb(\Gamma, x)$. $\square$

A necessary and sufficient condition for a C_0-semigroup of operators to be codiskcyclic is given in the next lemma and theorem.

Lemma 4.3. *Let $(T_t)_{t\in\mathbf{R}_+}$ be a codiskcyclic C_0-semigroup of operators on a Banach infinite dimensional space X. If $x \in X$ is a codiskcyclic vector of $(T_t)_{t\in\mathbf{R}_+}$, then the following assertions hold:*

(1) *$T_t x \neq 0$, for all $t \in \mathbf{R}_+$;*

(2) *The set $\{\alpha T_t x : t \geq s, \alpha \in \mathbf{U}\}$ is dense in X, for all $s \in \mathbf{R}_+$.*

Proof. (1) Suppose that $t_0 \in \mathbf{R}_+^*$ is minimal with the property that $T_{t_0}x = 0$. We show first that each $y \in X$ is of the form $y = \alpha T_t x$ for some $t \in [0, t_0]$ and $\alpha \in \mathbf{U}$. Since $x \in \mathbf{U}C(\Gamma)$, there exist a sequence $(t_n)_{n\in\mathbf{N}} \subset [0, t_0]$ and a sequence $(\alpha_n)_{n\in\mathbf{N}} \subset \mathbf{U}$ such that $\alpha_n T_{t_n} x \longrightarrow y$. Without loss of generality we may assume that $(t_n)_{n\in\mathbf{N}}$ converges to some t. By compactness we may assume that $(\alpha_n)_{n\in\mathbf{N}}$ converges to some α and we infer that $y = \alpha T_t x$.

Now take three vectors $y_i = \alpha_i T_{t_i} x \in X$, spanning a two-dimensional subspace, such that each pair y_i, y_j, $i \neq j$, is linearly independent. Assume that $t_1 > t_2 > t_3$. We have then $y_3 = c_1 y_1 + c_2 y_2$. Now we arrive at the contradiction

$$\begin{aligned} 0 &\neq \alpha_3 T_{(t_0+t_3-t_2)}x = T_{(t_0-t_2)}y_3 = c_1 T_{(t_0-t_2)}y_1 + c_2 T_{(t_0-t_2)}y_2 \\ &= c_1\alpha_1 T_{(t_0+t_1-t_2)}x + c_2\alpha_2 T_{t_0}x = 0. \end{aligned}$$

(2) Suppose that there exists $s_0 \in \mathbf{R}_+^*$ such that $\{\alpha T_t x : t \geq s_0, \alpha \in \mathbf{U}\}$ is not dense in X. Hence there exists a bounded open set U such that $U \cap \overline{A} = \emptyset$. Therefore we have $U \subset \overline{\{\alpha T_t x : 0 \leq t \leq s_0, \alpha \in \mathbf{U}\}}$ by using the relation

$$\begin{aligned} X &= \overline{\{\alpha T_t x : t \in \mathbf{R}_+, \alpha \in \mathbf{U}\}} \\ &= \overline{\{\alpha T_t x : t \geq s_0, \alpha \in \mathbf{U}\}} \cup \overline{\{\alpha T_t x : 0 \leq t \leq s_0, \alpha \in \mathbf{U}\}}. \end{aligned}$$

Thus, $\overline{U}$ is compact. Hence X is finite dimensional, which contradicts that X is infinite dimensional. □

Theorem 4.4. *Let $(T_t)_{t\in\mathbf{R}_+}$ be a C_0-semigroup of operators on a separable Banach infinite dimensional space X. Then the following assertions are equivalent:*

(1) *$(T_t)_{t\in\mathbf{R}_+}$ is codiskcyclic;*

(2) *for all y, $z \in X$ and all $\varepsilon > 0$, there exist $v \in X$, $t \in \mathbf{R}_+^*$ and $\alpha \in \mathbf{U}$ such that $\|y - v\| < \varepsilon$ and $\|z - \alpha T_t v\| < \varepsilon$;*

(3) *for all y, $z \in X$, all $\varepsilon > 0$ and for all $l \geq 0$, there exist $v \in X$, $t > l$ and $\alpha \in \mathbf{U}$ such that $\|y - v\| < \varepsilon$ and $\|z - \alpha T_t v\| < \varepsilon$.*

Proof. (1) ⇒ (3): Let $x \in X$ such that $\{\alpha T_t x : t \in \mathbf{R}_+, \alpha \in \mathbf{U}\}$ is dense in X and let $\varepsilon > 0$. For any $y \in X$, there exist $s_1 > 0$ and $\alpha_1 \in \mathbf{U}$ such that $\|y - \alpha_1 T_{s_1} x\| < \varepsilon$. If $l \geq 0$, then by Lemma 4.3, the set $\alpha_1\{\alpha T_t x : t \geq s + l, \alpha \in \mathbf{U}\} := \{\alpha_1 \alpha T_t x : t \geq s + l, \alpha \in \mathbf{U}\}$ is dense in X. For any $z \in X$, there exist $s_2 > l + s_1$ and $\alpha_2 \in \mathbf{U}$ such that $\|z - \alpha_1\alpha_2 T_{s_2} x\| < \varepsilon$. Put $v = \alpha_1 T_{s_1} x$, $t = s_2 - s_1 > l$ and $\alpha = \alpha_2$. Then

we have $\|y - v\| < \varepsilon$ and $\|z - \alpha T_t v\| < \varepsilon$.
(3) $\Rightarrow$ (2): It is obvious.
(2) $\Rightarrow$ (1): Let $\{z_1, z_2, z_3, ...\}$ be a dense sequence in X. We construct sequences $\{y_1, y_2, y_3, ...\} \subset X$, $\{t_1, t_2, t_3, ...\} \subset [0, +\infty)$ and $\{\alpha_1, \alpha_2, \alpha_3, ...\} \subset$ **U** inductively:

- Put $y_1 = z_1$, $t_1 = 0$.
- For $n > 1$, find y_n, t_n and α_n such that

$$\|y_n - y_{n-1}\| \leq \frac{2^{-n}}{\sup\{\|T_{t_j}\| : j < n\}}, \tag{4.1}$$

and

$$\|z_n - \alpha_n T_{t_n} y_n\| \leq 2^{-n}. \tag{4.2}$$

In particular, (4.1) implies that $\|y_n - y_{n-1}\| \leq 2^{-n}$, so that the sequence $(y_n)_{n\geq 1}$ has a limit x. Applying (4.2) and once again (4.1) we infer that

$$\begin{aligned}
\|z_n - \alpha_n T_{t_n} x\| &= \|z_n - \alpha_n T_{t_n} y_n + \alpha_n T_{t_n} y_n - \alpha_n T_{t_n} x\| \\
&\leq \|z_n - \alpha_n T_{t_n} y_n\| + \|\alpha_n T_{t_n}(y_n - x)\| \\
&\leq \|z_n - \alpha_n T_{t_n} y_n\| + \|\alpha_n T_{t_n}\| \textstyle\sum_{i=n+1}^{+\infty} \|y_i - y_{i-1}\| \\
&\leq 2^{-n} + \textstyle\sum_{i=n+1}^{+\infty} 2^{-i} = 2^{-n+1}.
\end{aligned}$$

Given $z \in X$ and $\varepsilon > 0$ there are arbitrarily large n such that $\|z_n - z\| < \frac{\varepsilon}{2}$. Choosing n large enough such that $2^{-n+1} < \frac{\varepsilon}{2}$, we obtain $\|\alpha_n T_{t_n} x - z\| \leq \|z - z_n\| + \|z_n - \alpha_n T_{t_n} x\| < \varepsilon$. Therefore, $\{\alpha T_t x : t \in \mathbf{R}_+, \alpha \in \mathbf{U}\}$ is dense in X. □

As a corollary we obtain a sufficient condition of codiskcyclicity of a C_0-semigroup of operators.

Let X be a separable Banach infinite dimensional space. Denote X_0 the set of all $x \in X$ such that $\lim_{t \longrightarrow \infty} T_t x = 0$, and X_∞ the set of all $x \in X$ such that for each $\varepsilon > 0$ there exist some $w \in X$, $\alpha \in \mathbf{U}$ and some $t > 0$ with $\|w\| < \varepsilon$ and $\|\alpha T_t w - x\| < \varepsilon$.

Theorem 4.5. *Let $(T_t)_{t\in\mathbf{R}_+}$ be a C_0-semigroup of operators on a separable Banach infinite dimensional space X. If both X_∞ and X_0 are dense subsets, then $(T_t)_{t\in\mathbf{R}_+}$ is codiskcyclic.*

Proof. Let $z \in X_\infty$ and $y \in X_0$. Then for each $\varepsilon > 0$ there are arbitrarily large $t > 0$, $\alpha \in \mathbf{U}$ and $w \in X$ such that $\|w\| < \varepsilon$ and $\|\alpha T_t w - x\| < \frac{\varepsilon}{2}$. Since $y \in X_0$, for sufficiently large t we have $\|\alpha T_t y\| < \frac{\varepsilon}{2}$. We put $v = y + w$ and infer $\|z - T_t v\| \leq \|z - T_t w\| + \|\alpha T_t y\| < \varepsilon$,
and

$\|y - v\| = \|w\| < \varepsilon$. By Theorem 4.4, the result holds. □

We use Theorem 3.7 to prove that the codiskcyclicity and codisk transitivity of a C_0-semigroup of operators on a complex topological vector space are equivalent.

Theorem 4.6. *Let $(T_t)_{t \in \mathbf{R}_+}$ be a C_0-semigroup of operators on a complex topological vector space X. Then, the following assertions are equivalent*:

(*i*) *$(T_t)_{t \in \mathbf{R}_+}$ is codiskcyclic;*

(*ii*) *$(T_t)_{t \in \mathbf{R}_+}$ is codisk transitive.*

Proof. Note that if $t_1 > t_2 \geq 0$, then there exists $t = t_1 - t_2$ such that $T_{t_1} = T_t T_{t_2}$. Then use Theorem 3.7. □

Acknowledgment. The authors would like to thank the referee warmly for his suggestions and valuable comments on this paper.

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