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An extension of biconservative timelike hypersurfaces in Einstein space

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Abstract

A well-known conjecture of Bang-Yen Chen says that the only biharmonic Euclidean submanifolds are minimal ones, which affirmed by himself for surfaces in 3*-dimensional Euclidean space,* E^3*. We consider an extended version of Chen conjecture (namely,* L_k*-conjecture) on Lorentzian hypersurfaces of the pseudo-Euclidean space* E_1^4 *(i.e. the Einstein space). The biconservative submanifolds in the Euclidean spaces are submanifolds with conservative stress-energy with respect to the bienergy functional. In this paper, we consider an extended condition (namely,* L_k*-biconservativity) on non-degenerate timelike hypersurfaces of the Einstein space* E_1^4*. A Lorentzian hypersurface* $x : M_1^3 \to E_1^4$ *is called* L_k*-biconservative if the tangent part of* $L_k^2 x$ *vanishes identically. We show that* L_k*-biconservativity of a timelike hypersurface of* E_1^4 *(with constant* k*th mean curvature and some additional conditions) implies that its* $(k+1)$ *th mean curvature is constant.*

Keywords: *Timelike hypersurface, Biconservative,* L_k*-biconservative.*

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1. Introduction

The main geometric motivation of the subject of biconservative hypersurfaces is a well-known conjecture of Bang-Yen Chen (in 1987) which states that each biharmonic surface in Euclidean 3-spaces E^3 is harmonic. In 1992, Dimitrić proved that any biharmonic hypersurface in E^m with at most two distinct principal curvatures is minimal ([9]). Let $\phi : M^n \to E^{n+1}$ denotes an isometric immersion of a hypersurface M^n into the $(n+1)$-dimensional Euclidean space with the Laplace operator Δ, the shape operator S associated to a unit normal vector field $\mathbf{n}$ and the ordinary mean curvature H on M^n. The hypersurface M^n is said to be harmonic if ϕ satisfies condition $\Delta\phi = 0$. It is said to be biharmonic if ϕ satisfies condition $\Delta^2\phi = 0$. Also, M^n is said to be biconservative if the tangential part of $\Delta^2\phi$ vanishes identically. A famous law due to Beltrami says that $\Delta\phi = -nH\mathbf{n}$, so the condition $\Delta\phi = 0$ is equivalent to $H \equiv 0$ and the condition $\Delta^2\phi = 0$ is equivalent to $\Delta(H\mathbf{n}) = 0$. In 1995, Hasanis and Vlachos proved an extension of Chen's result to the hypersurfaces in Euclidean 4-space ([10]). Also, in 2013, Akutagawa and Maeta ([1]) have generalized Chen's conjecture on submanifolds in Euclidean n-space. As an extended case, a hypersurface $x : M_p^3 \to E_s^4$, whose mean curvature vector field is an eigenvector of the Laplace operator Δ, has been studied, for instance, in [7, 8] for the Euclidean case (where, $p = s = 0$), and for the Lorentz case in [3, 4] (when $s = 1$ and $p = 0, 1$). On the other hand, Chen himself had found a good relation between the finite type hypersurfaces and biharmonic ones. The theory of finite type hypersurfaces is a well-known subject interested by Chen (for instance, in [5, 6]) and also L. J. Alias, S.M.B. Kashani and others. In [11], Kashani has studied the notion of L_1-finite type Euclidean hypersurfaces as an extension of finite type ones. One can see main results in Chapter 11 of Chen's book ([5]).

The map L_1 is an extension of the Laplace operator $L_0 = \Delta$, which stands for the linearized operator of the first variation of the 2th mean curvature of the hypersurface (see, for instance, [2, 12, 16, 17, 19]). This operator is defined by $L_1(f) = tr(P_1 \circ \nabla^2 f)$ for any $f \in C^\infty(M)$, where $P_1 = nHI - S$ denotes the first Newton transformation associated to the second fundamental from of the hypersurface and $\nabla^2 f$ is the hessian of f. It is interesting to generalize the definition of biharmonic hypersurface by replacing Δ by L_1. Recently, in [14], we have studied the L_1-biharmonic spacelike hypersurfaces in 4-dimentional Minkowski space E_1^4. In this paper, we study the L_k-biconservative Lorentzian hypersurfaces in the Einstein space E_1^4. We show that, every L_k-biconservative Lorentzian hypersurface $x : M_1^3 \to E_1^4$, with constant kth mean curvature and some additional conditions on principal curvatures, has constant $(k+1)$th mean curvature.

2. Preliminaries

In this section, we restate some preliminaries from [2, 12, 13] and [15]-[18]. The 4-dimensional Minkowski space, denoted by E_1^4, is the real vector space R^4 equipped with the scalar product $< x, y >:= -x_1y_1 + \Sigma_{i=2}^4 x_iy_i$, for every $x, y \in R^4$. Any nondegenerate hypersurface M_p^3 of E_1^4, can be endowed with a Riemannian or Lorentzian induced metric of index $p = 0$ or $p = 1$, respectively. Our study will be on a Lorentzian hypersurface of E_1^4, denoted by an isometric immersion $x : M_1^3 \to E_1^4$. The symbols $\tilde{\nabla}$ and $\bar{\nabla}$ stand for the Levi-Civita connection on M_1^3 and E_1^4, respectively. For every tangent vector fields X and Y on M, the Gauss formula is given by $\bar{\nabla}_X Y = \tilde{\nabla}_X Y + < SX, Y > \mathbf{n}$, for every $X, Y \in \chi(M)$, where, $\mathbf{n}$ is a (locally) unit normal vector field on M and S is the shape operator of M relative to $\mathbf{n}$. Every non null vector $X \in E_1^4$ is called time-like, light-like or space-like if $< X, X >$ is negative, zero or positive, respectively.

Definition 2.1. For a Lorentzian vector space V_1^3, a basis $\mathcal{B} := \{e_1, e_2, e_3\}$ is said to be *orthonormal* if it satisfies $< e_i, e_j >= \epsilon_i \delta_i^j$ for $i, j = 1, 2, 3$, where $\epsilon_1 = -1$ and $\epsilon_i = 1$ for $i = 2, 3$. As usual, δ_i^j stands for the Kronecker function. $\mathcal{B}$ is called $pseudo - orthonormal$ if it satisfies $< e_1, e_1 >=< e_2, e_2 >= 0$, $< e_1, e_2 >= -1$ and $< e_i, e_j >= \delta_i^j$, for $i = 1, 2, 3$ and $j = 3$.

As well-known, the shape operator of the Lorentzian hypersurface M_1^3, as a self-adjoint linear map on the tangent space of M_1^3, can be put into one of four possible canonical matrix forms, usually denoted by I, II, III and IV. Where, in cases I and IV, with respect to an orthonormal basis of the tangent space of M_1^3, the matrix representation of the induced metric on M_1^3 is

$$G_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and the shape operator S of M_1^3 can be put into matrix forms

$$B_1 = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \qquad \text{and} \qquad B_4 = \begin{pmatrix} \kappa & \lambda & 0 \\ -\lambda & \kappa & 0 \\ 0 & 0 & \eta \end{pmatrix}, \quad (\lambda \neq 0)$$

respectively. For cases II and III, using a pseudo-orthonormal basis of the tangent space of M_1^3, the induced metric on M_1^3 has matrix form

$$G_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and the shape operator S of M_1^3 can be put into matrix forms

$$B_2 = \begin{pmatrix} \kappa & 0 & 0 \\ 1 & \kappa & 0 \\ 0 & 0 & \lambda \end{pmatrix} \qquad \text{and} \qquad B_3 = \begin{pmatrix} \kappa & 0 & 0 \\ 0 & \kappa & 1 \\ -1 & 0 & \kappa \end{pmatrix},$$

respectively. In case IV, the matrix B_4 has two conjugate complex eigenvalues $\kappa \pm i\lambda$, but in other cases the eigenvalues of the shape operator are real numbers.

Remark 2.2. In two cases II and III, one can substitute the pseudo-orthonormal basis $\mathcal{B} := \{e_1, e_2, e_3\}$ by orthonormal basis $\tilde{\mathcal{B}} := \{\tilde{e_1}, \tilde{e_2}, e_3\}$ where $\tilde{e_1} := \frac{1}{2}(e_1 + e_2)$ and $\tilde{e_2} := \frac{1}{2}(e_1 - e_2)$. Therefore, we obtain new matrix representations $\tilde{B_2}$ and $\tilde{B_3}$ (instead of B_2 and B_3, respectively) as

$$\tilde{B_2} = \begin{pmatrix} \kappa + \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \kappa - \frac{1}{2} & 0 \\ 0 & 0 & \lambda \end{pmatrix} \qquad \text{and} \qquad \tilde{B_3} = \begin{pmatrix} \kappa & 0 & \frac{\sqrt{2}}{2} \\ 0 & \kappa & -\sqrt{2}/2 \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & \kappa \end{pmatrix}$$

After this changes, to unify the notations we denote the orthonormal basis by $\mathcal{B}$ in all cases.

Notation: According to four possible matrix representations of the shape operator of M_1^3, we define its principal curvatures, denoted by unified notations κ_i for $i = 1, 2, 3$, as follow.

In case I, we put $\kappa_i := \lambda_i$, for $i = 1, 2, 3$, where λ_i's are the eigenvalues of B_1.

In cases II, where the matrix representation of S is $\tilde{B}_2$, we take $\kappa_i := \kappa$ for $i = 1, 2$, and $\kappa_3 := \lambda$.

In case III, where the shape operator has matrix representation $\tilde{B}_3$, we take $\kappa_i := \kappa$ for $i = 1, 2, 3$.

Finally, in the case IV, where the shape operator has matrix representation $\tilde{B}_4$, we put $\kappa_1 = \kappa + i\lambda$, $\kappa_2 = \kappa - i\lambda$, and $\kappa_3 := \eta$.

The characteristic polynomial of S on M_1^3 is of the form

$$Q(t) = \prod_{i=1}^{3}(t - \kappa_i) = \sum_{j=0}^{3}(-1)^j s_j t^{3-j},$$

where, $s_0 := 1$, $s_1 = \sum_{j=1}^{3} \kappa_j$, $s_2 := \sum_{1 \leq i_1 < i_2 \leq 3} \kappa_{i_1}\kappa_{i_2}$ and $s_3 := \kappa_1\kappa_2\kappa_3$.

For $k = 1, 2, 3$, the *kth mean curvature* H_k of M is defined by $H_k = \frac{1}{\binom{3}{k}} s_k$. When H_k is identically null, M_1^n is said to be $(k-1)$-*minimal.*

Definition 2.3. *(i) A timelike hypersurface $x : M_1^3 \to E_1^4$, with diagonalizable shape operator, is said to be isoparametric if all of it's principal curvatures are constant.*
(ii) A timelike hypersurface $x : M_1^3 \to E_1^4$, with non-diagonalizable shape operator, is said to be isoparametric if the minimal polynomial of the shape operator is constant.

Remark 2.4. Here we remember Theorem 4.10 from [13], which assures us that there is no isoparametric timelike hypersurface of E_1^4 with complex principal curvatures.

The Newton transformations on the hypersurface, $P_k : \chi(M) \to \chi(M)$, is defined by

$$P_0 = I, \quad P_k = s_k I - S \circ P_{k-1}, \qquad (k = 1, 2, 3), \tag{2.1}$$

where, I is the identity map. The explicit formula $P_k = \sum_{i=0}^{k}(-1)^i s_{k-i} S^i$ (where $S^0 = I$) gives that, P_k is self-adjoint and it commutes with S (see [2, 16]).

Now, we define a notation as

$$(2.2)\ \mu_{j;k} = \sum_{l=0}^{k}(-1)^l\binom{n}{k-l}H_{k-l}\kappa_j^l. \qquad (1 \leq j \leq 3,\ 1 \leq k < 3)$$

Corresponding to the four possible forms $\tilde{B}_i$ (for $1 \leq i \leq 4$) of S, the Newton transformation P_j has different representations. In the case I, where $S_p = \tilde{B_1}$, we have $P_j(p) = diag[\mu_{1;j}(p), \mu_{2;j}(p), \mu_{3;j}(p)]$, for $j = 1, 2$.

When $S = B_2$ (in the case II), we have

$$P_1 = \begin{pmatrix} \kappa+\lambda-\frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \kappa+\lambda+\frac{1}{2} & 0 \\ 0 & 0 & 2\kappa \end{pmatrix}, \qquad P_2 = \begin{pmatrix} (\kappa-\frac{1}{2})\lambda & -\frac{1}{2}\lambda & 0 \\ \frac{1}{2}\lambda & (\kappa+\frac{1}{2})\lambda & 0 \\ 0 & 0 & \kappa^2 \end{pmatrix}.$$

In the case III, we have $S_p = B_3$, and

$$P_1 = \begin{pmatrix} 2\kappa & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 2\kappa & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 2\kappa \end{pmatrix}, \qquad P_2 = \begin{pmatrix} \kappa^2-\frac{1}{2} & -\frac{1}{2} & -\frac{\sqrt{2}}{2}\kappa \\ \frac{1}{2} & \kappa^2+\frac{1}{2} & \frac{\sqrt{2}}{2}\kappa \\ \frac{\sqrt{2}}{2}\kappa & \frac{\sqrt{2}}{2}\kappa & \kappa^2 \end{pmatrix}.$$

In the case IV, $S = B_4$,

$$P_1 = \begin{pmatrix} \kappa+\eta & -\lambda & 0 \\ \lambda & \kappa+\eta & 0 \\ 0 & 0 & 2\kappa \end{pmatrix}, \qquad P_2 = \begin{pmatrix} \kappa\eta & -\lambda\eta & 0 \\ \lambda\eta & \kappa\eta & 0 \\ 0 & 0 & \kappa^2+\lambda^2 \end{pmatrix}.$$

Fortunately, in all cases we have the following important identities, similar to those in [2, 16].

$$(2.3)\quad tr(P_1) = 6H_1,\ tr(P_2) = 3H_2,\ tr(P_1 \circ S) = 6H_2,\ tr(P_2 \circ S) = 3H_3,$$

$$(2.4)\ trS^2 = 9H_1^2 - 6H_2,\ tr(P_1 \circ S^2) = 9H_1H_2 - 3H_3,\ tr(P_2 \circ S^2) = 3H_1H_3.$$

The *linearized operator* arisen from the first variation of the $(j+1)$th mean curvature of M denoted by $L_j : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ is defined by the formula $L_j(f) := tr(P_j \circ \nabla^2 f)$, where, $< \nabla^2 f(X), Y >=< \nabla_X \nabla f, Y >$ for every $X, Y \in \chi(M)$. Associated to the orthonormal frame $\{e_1, e_2, e_3\}$ of tangent space on a local coordinate system in the hypersurface $x : M_1^3 \to E_1^4$, $L_j(f)$ has an explicit expression as

$$L_j(f) = \sum_{i=1}^{3} \epsilon_i \mu_{i,j}(e_i e_i f - \nabla_{e_i} e_i f). \tag{2.5}$$

For a Lorentzian hypersurface $x : M_1^3 \to E_1^4$, with a chosen (local) unit normal vector field $\mathbf{n}$, for an arbitrary vector $a \in E_1^4$ we use the decomposition $a = a^T + a^N$ where $a^T \in TM$ is the tangential component of a, $a^N \perp TM$, and we have the following formulae from [2, 16].

(2.6) $\nabla < x, a >= a^T,\ \nabla < \mathbf{n}, a >= -Sa^T, L_1 x = 6H_2\mathbf{n}, L_2 x = 3H_3\mathbf{n}$

$$\begin{aligned} L_1\mathbf{n} &= -3\nabla H_2 - 3[3H_1H_2 - H_3]\mathbf{n}, \\ L_2\mathbf{n} &= -\nabla H_3 - [3H_1H_3]\mathbf{n}, \end{aligned} \tag{2.7}$$

$$\begin{aligned} L_1^2 x &= 6[2P_2\nabla H_2 - 9H_2\nabla H_2] + 6[L_1H_2 - 9H_1H_2^2 + 3H_2H_3]\mathbf{n}, \\ L_2^2 x &= -9H_3\nabla H_3 + 3(L_2H_3 - 3H_1H_3^2)\mathbf{n}. \end{aligned} \tag{2.8}$$

Assume that a hypersurface $x : M_1^3 \to E_1^4$ satisfies the condition $L_k^2 x = 0$ for an integer $k \in \{0, 1, 2\}$, then it is said to be L_k*-biharmonic*. In the case $k = 0$, we have $L_0 = \Delta$ and L_0-biharmoniciy is the same ordinary harmonicity which has been studied in [3, 4]. By equalities (2.8), a hypersurface $x : M_1^3 \to E_1^4$ is L_1-biharmonic if and only if it satisfies two following conditions:

$$\begin{array}{l} (i)\ L_1H_2 = 3(3H_1H_2^2 - H_2H_3) = H_2 tr(S^2 \circ P_1), \\ (ii)\ P_2\nabla H_2 = \frac{9}{2}H_2\nabla H_2. \end{array} \tag{2.9}$$

A timelike hypersurface $x : M_1^3 \to E_1^4$ is said to be L_1*-biconservative*, if its 2nd mean curvature satisfies the condition (2.9)(ii).

Also, $x : M_1^3 \to E_1^4$ is L_2-biharmonic if and only if it satisfies two following conditions:

$$(i)\ L_2H_3 = 3H_1H_3^2 = H_3 tr(S^2 \circ P_2), \quad (ii)\ H_3\nabla H_3 = 0. \tag{2.10}$$

A timelike hypersurface $x : M_1^3 \to E_1^4$ is said to be L_2*-biconservative*, if its 3rd mean curvature satisfies the condition (2.10)(ii).

The structure equations of E_1^4 are given by

$$d\omega_i = \sum_{j=1}^{4} \omega_{ij} \wedge \omega_j, \qquad \omega_{ij} + \omega_{ji} = 0, \tag{2.11}$$

$$d\omega_{ij} = \sum_{l=1}^{4} \omega_{il} \wedge \omega_{lj}. \tag{2.12}$$

With restriction to M, we have $\omega_4 = 0$ and then,

$$0 = d\omega_4 = \sum_{i=1}^{3} \omega_{4,i} \wedge \omega_i. \tag{2.13}$$

By Cartan's lemma, there exist functions h_{ij} such that

$$\omega_{4,i} = \sum_{j=1}^{3} h_{ij}\omega_j, \qquad h_{ij} = h_{ji}. \tag{2.14}$$

This gives the second fundamental form of M, as $B = \sum_{i,j} h_{ij}\omega_i\omega_j e_4$. The mean curvature H is given by $H = \frac{1}{3}\sum_{i=1}^{3} h_{ii}$. From (2.11) -(2.14) we obtain the structure equations of M as follow.

$$d\omega_i = \sum_{j=1}^{3} \omega_{ij} \wedge \omega_j, \omega_{ij} + \omega_{ji} = 0, \tag{2.15}$$

$$d\omega_{ij} = \sum_{k=1}^{3} \omega_{ik} \wedge \omega_{kj} - \frac{1}{2}\sum_{k,l=1}^{3} R_{ijkl}\omega_k \wedge \omega_l, \tag{2.16}$$

for $i, j = 1, 2, 3$, and the Gauss equations

$$R_{ijkl} = (h_{ik}h_{jl} - h_{il}h_{jk}), \tag{2.17}$$

where R_{ijkl} denotes the components of the Riemannian curvature tensor of M.

Let h_{ijk} denote the covariant derivative of h_{ij}.

We have

$$dh_{ij} = \sum_{k=1}^{3} h_{ijk}\omega_k + \sum_{k=1}^{3} h_{kj}\omega_{ik} + \sum_{k=1}^{3} h_{ik}\omega_{jk}. \tag{2.18}$$

Thus, by exterior differentiation of (2.14), we obtain the Codazzi equation

$$h_{ijk} = h_{ikj}. \tag{2.19}$$

Now we recall the definition of an L_k-finite type hypersurface from [11], which is the basic notion of the paper.

Definition 2.5. *An isometrically immersed hypersurface $x : M_1^3 \to E_1^4$ is said to be of L_k-finite type if x has a finite decomposition $x = \sum_{i=0}^{m} x_i$, for some positive integer m, satisfying the condition $L_k x_i = \beta_i x_i$, where, $\beta_i \in R$ and $x_i : M^3 \to E_1^4$ is smooth maps, for $i = 1, 2, \cdots, m$, and x_0 is constant point. If all β_i's are mutually different, M^n is said to be of L_k-m-type. An L_k-m-type hypersurface is said to be null if for at least one i $(1 \le i \le m)$ we have $\beta_i = 0$.*

Now, we see two examples of L_k-biconservative timelike hypersurfaces in E_1^4, for $k = 0, 1, 2$.

Example 2.6. Assume that $\mathcal{D}_1(r)$ be the product $S_1^2(r) \times R \subset E_1^4$ where $r > 0$. It has another representation as

$$\mathcal{D}_1(r) = \{(y_1, ..., y_4) \in E_1^4 | -y_1^2 + y_2^2 + y_3^2 = r^2\},$$

having the spacelike vector field $\mathbf{n}(y) = -\frac{1}{r}(y_1, y_2, y_3, 0)$ as the Gauss map. Clearly, it has two distinct principal curvatures $\kappa_1 = \kappa_2 = \frac{1}{r}$, $\kappa_3 = 0$, and the constant higher order mean curvatures $H_1 = \frac{2}{3}r^{-1}$, $H_2 = \frac{1}{3}r^{-2}$ and $H_3 = 0$.

Example 2.7. Assume that $\mathcal{D}_2(r)$ be the product $S_1^1(r) \times E^2 \subset E_1^4$ where $r > 0$. It has another representation as

$$\mathcal{D}_2(r) = \{(y_1, ..., y_4) \in E_1^4 | -y_1^2 + y_2^2 = r^2\},$$

having the spacelike vector field $\mathbf{n}(y) = -\frac{1}{r}(y_1, y_2, 0, 0, 0)$ as the Gauss map. Clearly, it has two distinct principal curvatures $\kappa_1 = \frac{1}{r}$, $\kappa_2 = \kappa_3 = 0$, and the constant higher order mean curvatures $H_1 = \frac{1}{4r}$, and $H_2 = H_3 = 0$.

Example 2.8. Let $\mathcal{D}_3(r)$ be the product $E_1^2 \times S^1(r) \subset E_1^4$ where $r > 0$. It can be represented as

$$\mathcal{D}_3(r) = \{(y_1, ..., y_4) \in E_1^4 | y_4^2 + y_5^2 = r^2\},$$

with the Gauss map $\mathbf{n}(y) = -\frac{1}{r}(0, 0, 0, y_4, y_5)$. it has two distinct principal curvatures $\kappa_1 = \kappa_2 = 0$, $\kappa_3 = \frac{1}{r}$, and the constant higher order mean curvatures $H_1 = \frac{1}{4r}$, and $H_k = 0$ for $k = 2, 3$.

Example 2.9. Consider the pseudo-sphere

$$S_1^3(r) = \{(y_1, ..., y_4) \in E_1^4 | - y_1^2 + y_2^2 + y_3^2 + y_4^2 = r^2\},$$

(for $r > 0$) with the Gauss map $\mathbf{n}(y) = -\frac{1}{r}(y_1, y_2, y_3, y_4)$. It has three principal curvatures $\kappa_1 = \kappa_2 = \kappa_3 = \frac{1}{r}$ and constant higher order mean curvatures $H_k = \frac{1}{r^k}$, for $k = 1, 2, 3$.

3. Results

In this section, we give four theorems on the L_k-biconservative connected orientable timelike hypersurface in E_1^4 with constant ordinary mean curvature. Theorem 3.1 is appropriated to the case that the shape operator on hypersurface is diagonalizable. Theorems 3.2, 3.3 and 3.4 are related to the cases that the shape operator on hypersurface is of type II, III and IV, respectively.

Theorem 3.1. *Let $x : M_1^3 \to E_1^4$ be a L_k-biconservative timelike hypersurface in the Minkowski 4-space, having diagonalizable shape operator (i.e of type I) with constant kth mean curvature and exactly two distinct principal curvatures (for k a nonnegative integer number less than 3). Then, it has constant $(k+1)$th mean curvature.*

Proof. By assumption, M_1^3 has two distinct principal curvatures λ and μ of multiplicities 2 and 1, respectively. Defining the open subset U of M_1^3 as $U := \{p \in M_1^3 : \nabla H_{k+1}^2(p) \neq 0\}$, we prove that U is empty. Assuming $U \neq \emptyset$, we consider $\{e_1, e_2, e_3\}$ as a local orthonormal frame of principal directions of S on $\mathcal{U}$ such that $Se_i = \lambda_i e_i$ for $i = 1, 2, 3$. By assumption, we have

$$\lambda_1 = \lambda_2 = \lambda, \qquad \lambda_3 = \mu.$$

Therefore, we obtain

$$\mu_{1,2} = \mu_{2,2} = \lambda\mu, \quad \mu_{3,2} = \lambda^2, \quad 3H_2 = \lambda^2 + 2\lambda\mu. \tag{3.1}$$

In the case $k = 1$, by condition (2.9)(ii), we have

$$P_2(\nabla H_2) = \frac{9}{2} H_2 \nabla H_2. \tag{3.2}$$

Then, using the polar decomposition

$$(3.3) \qquad \nabla H_2 = \sum_{i=1}^{3} \epsilon_i < \nabla H_2, e_i > e_i,$$

we see that (3.2) is equivalent to

$$\epsilon_i < \nabla H_2, e_i > (\mu_{i,2} - \frac{9}{2}H_2) = 0$$

on $\mathcal{U}$ for $i = 1, 2, 3$. Hence, for every i such that $< \nabla H_2, e_i > \neq 0$ on $\mathcal{U}$ we get

$$(3.4) \qquad \mu_{i,2} = \frac{9}{2}H_2.$$

By definition, we have $\nabla H_2 \neq 0$ on U, which gives one or both of the following states.

State 1. $< \nabla H_2, e_i > \neq 0$, for $i = 1$ or $i = 2$. By equalities (3.1) and (3.4), we obtain

$$\lambda\mu = \frac{9}{2}(\frac{2}{3}\lambda\mu + \frac{1}{3}\lambda^2),$$

which gives

$$(3.5) \qquad \lambda(6H - \frac{5}{2}\lambda) = 0.$$

If $\lambda = 0$ then $H_2 = 0$. Otherwise, we get $\lambda = \frac{12}{5}H$, $\mu = -\frac{9}{5}H$ and $H_2 = -\frac{72}{25}H^2$.

State 2. $< \nabla H_2, e_3 > \neq 0$. By equalities (3.1) and (3.4), we obtain

$$\lambda^2 = \frac{9}{2}(\frac{2}{3}\lambda\mu + \frac{1}{3}\lambda^2),$$

which gives

$$(3.6) \qquad \lambda(9H - \frac{11}{2}\lambda) = 0.$$

If $\lambda = 0$ then $H_2 = 0$. Otherwise, we have $\lambda = \frac{18}{11}H$, $\mu = -\frac{3}{11}H$ and $H_2 = \frac{216}{121}H^2$.

Therefore, H_2 is constant.

In the case $k = 0$, the main claim is proven in [3, 4]. In the case $k = 2$, from condition (2.10)(ii) we get $e_i(H_3^2) = 0$ for $i = 1, 2, 3$, which means that there is nothing to prove. $\square$

Theorem 3.2. *Let k be a nonnegative integer number less than 3 and $x : M_1^3 \to E_1^4$ be an L_k-biconservative connected orientable timelike hypersurface with shape operator of type II which has exactly two distinct principal curvatures and constant kth mean curvature, then it's $(k+1)$th mean curvature is constant.*

Proof. Suppose that, H_2 is non-constant. Considering the open subset $\mathcal{U} = \{p \in M : \nabla H_2^2(p) \neq 0\}$, we try to show $\mathcal{U} = \emptyset$. By the assumption, with respect to a suitable (local) orthonormal tangent frame $\{e_1, e_2, e_3\}$ on M, the shape operator S has the matrix form $\tilde{B}_2$, such that $Se_1 = (\kappa + \frac{1}{2})e_1 - \frac{1}{2}e_2$, $Se_2 = \frac{1}{2}e_1 + (\kappa - \frac{1}{2})e_2$, $Se_3 = \lambda e_3$ and then, we have $P_2e_1 = (\kappa - \frac{1}{2})\lambda e_1 + \frac{1}{2}\lambda e_2$, $P_2e_2 = -\frac{1}{2}\lambda e_1 + (\kappa + \frac{1}{2})\lambda e_2$ and $P_2e_3 = \kappa^2 e_3$.

When $k = 0$, the result is derived from [3, 4]. In the case $k = 1$, by condition (2.9)(ii), using the polar decomposition $\nabla H_2 = \sum_{i=1}^{3} \epsilon_i e_i(H_2)e_i$, we get

$$
(3.7)\qquad
\begin{array}{l}
(i)\ \ \epsilon_1 e_1(H_2)[(\kappa - \frac{1}{2})\lambda - \frac{9}{2}H_2] = \epsilon_2 e_2(H_2)\frac{\lambda}{2} \\
(ii)\ \ \epsilon_2 e_2(H_2)[(\kappa + \frac{1}{2})\lambda - \frac{9}{2}H_2] = -\epsilon_1 e_1(H_2)\frac{\lambda}{2}, \\
(iii)\ \ \epsilon_3 e_3(H_2)(\kappa^2 - \frac{9}{2}H_2) = 0.
\end{array}
$$

Now, we prove some simple claims.

Claim 1: $e_1(H_2) = e_2(H_2) = e_3(H_2) = 0$.
If $e_1(H_2) \neq 0$, then by dividing both sides of equalities (3.7)(i, ii) by $\epsilon_1 e_1(H_2)$ we get

$$
(3.8)\qquad
\begin{array}{l}
(i)\ \ (\kappa - \frac{1}{2})\lambda - \frac{9}{2}H_2 = \frac{\epsilon_2 e_2(H_2)}{\epsilon_1 e_1(H_2)}\frac{\lambda}{2}, \\
(ii)\ \ \frac{\epsilon_2 e_2(H_2)}{\epsilon_1 e_1(H_2)}[(\kappa + \frac{1}{2})\lambda - \frac{9}{2}H_2] = -\frac{\lambda}{2},
\end{array}
$$

which, by substituting (i) in (ii), gives $\frac{\lambda}{2}(1+u)^2 = 0$, where $u := \frac{\epsilon_2 e_2(H_2)}{\epsilon_1 e_1(H_2)}$. Then $\lambda = 0$ or $u = -1$. If $\lambda = 0$, then we get $H_2 = 0$ from (3.8)(i). Also, by assumption $\lambda \neq 0$ we get $u = -1$ which gives $\kappa\lambda = \frac{9}{2}H_2$, then $\kappa(3\kappa + 4\lambda) = 0$ and finally $\kappa = -\frac{4}{3}\lambda$ (since $\kappa = 0$ gives $H_2 = 0$ again). Hence, we have $H_2 = \frac{2}{9}\kappa\lambda = -\frac{8}{27}\lambda^2$ and $H_1 = -\frac{5}{9}\lambda$, and since H_1 is assumed to be constant, then $\lambda = -\frac{9}{5}H_1$ and $H_2 = -\frac{24}{25}H_1^2$ have to be constant and we have $e_1(H_2) = 0$, which is a contradiction. Therefore, the first claim is proved.

The second part of Claim 1 is $e_2(H_2) = 0$. It can be proven by a similar manner. If $e_2(H_2) \neq 0$, then by dividing both sides of equalities (3.7)(i, ii) by $\epsilon_2 e_2(H_2)$ we get

$$(3.9)\qquad \begin{aligned} &(i)\quad \tfrac{\epsilon_1 e_1(H_2)}{\epsilon_2 e_2(H_2)}[(\kappa-\tfrac{1}{2})\lambda-\tfrac{9}{2}H_2]=\tfrac{\lambda}{2}\\ &(ii)\quad (\kappa+\tfrac{1}{2})\lambda-\tfrac{9}{2}H_2=-\tfrac{\epsilon_1 e_1(H_2)}{\epsilon_2 e_2(H_2)}\tfrac{\lambda}{2}, \end{aligned}$$

which, by substituting (ii) in (i), gives $\frac{\lambda}{2}(1+v)^2=0$, where $v:=\frac{\epsilon_1 e_1(H_2)}{\epsilon_2 e_2(H_2)}$. Then $\lambda=0$ or $v=-1$. If $\lambda=0$, then we get $H_2=0$ from (3.9)(ii). Also, by assumption $\lambda\neq 0$ we get $v=-1$ which gives $\kappa\lambda=\frac{9}{2}H_2$, then $\kappa(3\kappa+4\lambda)=0$ and finally $\kappa=-\frac{4}{3}\lambda$ (since $\kappa=0$ gives $H_2=0$ again). Hence, we have $H_2=\frac{2}{9}\kappa\lambda=-\frac{8}{27}\lambda^2$ and $H_1=-\frac{5}{9}\lambda$, and since H_1 is assumed to be constant, then $\lambda=-\frac{9}{5}H_1$ and $H_2=-\frac{24}{25}H_1^2$ have to be constant and we have $e_2(H_2)=0$, which is a contradiction. Therefore, the second part of Claim 1 is proved.

Now, if $e_3(H_2)\neq 0$, then by (3.7)(iii) we get $\kappa^2=\frac{9}{2}H_2$, then $\kappa(\kappa+6\lambda)=0$, which gives $\kappa=0$ or $\kappa=-6\lambda$. If $\kappa=0$, then $H_2=0$, and if $\kappa=-6\lambda$ then since $H_1=-\frac{11}{3}\lambda$ is assumed to be constant, we get that H_2 is constant and then $e_3(H_2)=0$. Which is a contradiction, so we have $e_3(H_2)=0$.

In the case $k=2$, from condition (2.10)(ii) we get $e_i(H_3^2)=0$ for $i=1,2,3$, which means that there is nothing to prove. □

Theorem 3.3. *Let k be a nonnegative integer number less than 3 and $x: M_1^3\to E_1^4$ be an L_k-biconservative connected orientable timelike hypersurface with shape operator of type III which has constant kth mean curvature, then it's $(k+1)$th mean curvature is constant.*

Proof. When $k=0$, the result is derived from [3, 4]. In the case $k=1$, suppose that, H_2 is non-constant. Considering the open subset $\mathcal{U}=\{p\in M:\nabla H_2^2(p)\neq 0\}$, we try to show $\mathcal{U}=\emptyset$. By the assumption, with respect to a suitable (local) orthonormal tangent frame $\{e_1,e_2,e_3\}$ on M, the shape operator S has the matrix form $\tilde{B}_3$, such that $Se_1=\kappa e_1+\frac{\sqrt{2}}{2}e_3$, $Se_2=\kappa e_2-\frac{\sqrt{2}}{2}e_3$, $Se_3=-\frac{\sqrt{2}}{2}e_1-\frac{\sqrt{2}}{2}e_2+\kappa e_3$ and then, we have $P_2e_1=(\kappa^2-\frac{1}{2})e_1-\frac{1}{2}e_2-\frac{\sqrt{2}}{2}\kappa e_3$, $P_2e_2=\frac{1}{2}e_1+(\kappa^2+\frac{1}{2})e_2+\frac{\sqrt{2}}{2}\kappa e_3$ and $P_2e_3=\frac{\sqrt{2}}{2}\kappa e_1+\frac{\sqrt{2}}{2}\kappa e_2+\kappa^2 e_3$.

Using the polar decomposition $\nabla H_2=\sum_{i=1}^{3}\epsilon_i e_i(H_2)e_i$, from condition (2.9)(ii) we get

$$(3.10)\qquad \begin{aligned} &(i)\quad \epsilon_1 e_1(H_2)[(\kappa^2-\tfrac{1}{2})-\tfrac{9}{2}H_2]+\tfrac{1}{2}\epsilon_2 e_2(H_2)+\tfrac{\sqrt{2}}{2}\epsilon_3 e_3(H_2)\kappa=0,\\ &(ii)\quad \tfrac{-1}{2}\epsilon_1 e_1(H_2)+\epsilon_2 e_2(H_2)[(\kappa^2+\tfrac{1}{2})-\tfrac{9}{2}H_2]+\tfrac{\sqrt{2}}{2}\epsilon_3 e_3(H_2)\kappa=0,\\ &(iii)\quad \epsilon_1 e_1(H_2)\tfrac{-\sqrt{2}}{2}\kappa+\epsilon_2 e_2(H_2)\tfrac{\sqrt{2}}{2}\kappa+\epsilon_3 e_3(H_2)(\kappa^2-\tfrac{9}{2}H_2)=0, \end{aligned}$$

Now, we prove some simple claims.

Claim: $e_1(H_2) = e_2(H_2) = e_3(H_2) = 0$.
If $e_1(H_2) \neq 0$, then by dividing both sides of equalities (3.10)(i, ii, iii) by $\epsilon_1 e_1(H_2)$, and using the identity $H_2 = \kappa^2$ in type III, we get

$$
\text{(3.11)} \qquad \begin{array}{ll}
(i) & -\frac{1}{2} - \frac{7}{2}\kappa^2 + \frac{1}{2}u_1 + \frac{\sqrt{2}}{2}u_2\kappa = 0, \\
(ii) & \frac{-1}{2} + u_1(\frac{1}{2} - \frac{7}{2}\kappa^2) + \frac{\sqrt{2}}{2}u_2\kappa = 0, \\
(iii) & \frac{-\sqrt{2}}{2}\kappa + \frac{\sqrt{2}}{2}u_1\kappa - \frac{7}{2}\kappa^2)u_2 = 0,
\end{array}
$$

where $u_1 := \frac{\epsilon_2 e_2(H_2)}{\epsilon_1 e_1(H_2)}$ and $u_2 := \frac{\epsilon_3 e_3(H_2)}{\epsilon_1 e_1(H_2)}$, which, by comparing (i) and (ii), gives $\kappa^2(u_1 - 1) = 0$. If $\kappa = 0$, then $H_2 = 0$. Assuming $\kappa \neq 0$, we get $u_1 = 1$, which, using (3.11)(iii), gives $u_2 = 0$. Substituting $u_1 = 1$ and $u_2 = 0$ in (3.11)(i), we obtain again $\kappa = 0$, which is a contradiction. Hence $e_1(H_2) \equiv 0$.

Therefore, using the result $e_1(H_2) \equiv 0$, the system of equations (3.10) gives

$$
\text{(3.12)} \qquad \begin{array}{ll}
(i) & \frac{1}{2}\epsilon_2 e_2(H_2) + \frac{\sqrt{2}}{2}\epsilon_3 e_3(H_2)\kappa = 0, \\
(ii) & \epsilon_2 e_2(H_2)(\frac{1}{2} - \frac{7}{2}\kappa^2) + \frac{\sqrt{2}}{2}\epsilon_3 e_3(H_2)\kappa = 0, \\
(iii) & \epsilon_2 e_2(H_2)\frac{\sqrt{2}}{2}\kappa - \epsilon_3 e_3(H_2)\frac{7}{2}\kappa^2 = 0.
\end{array}
$$

Comparing parts $(i), (ii)$ and (iii) of (3.12), we get $\kappa e_2(H_2) = 0$ and $\kappa e_3(H_2) = 0$, hence, using (i), gives $e_2(H_2) = 0$. Then, the second claim (i.e. $e_2(H_2) = 0$) is proved.

Now, using the results $e_1(H_2) = e_2(H_2) = 0$, we get $\kappa e_3(H_2) = 0$, which, using $H_2 = \kappa^2$, implies $\kappa e_3(\kappa^2) = 0$ and then $e_3(\kappa^3) = 0$, and finally $e_3(H_2) = 0$.

In the case $k = 2$, from condition (2.10)(ii) we get $e_i(H_3^2) = 0$ for $i = 1, 2, 3$, which means that there is nothing to prove. $\square$

Theorem 3.4. *Let k be a nonnegative integer number less than 3 and $x : M_1^3 \to E_1^4$ be an L_k-biconservative connected orientable timelike hypersurface with shape operator of type IV which has constant kth mean curvature and a constant real principal curvature. Then, its 2nd and 3rd mean curvatures are constant.*

Proof. When $k = 0$, the result is derived from [3, 4]. In the case $k = 1$, suppose that, H_2 is non-constant. Considering the open subset $\mathcal{U} = \{p \in M : \nabla H_2^2(p) \neq 0\}$, we try to show $\mathcal{U} = \emptyset$. By the assumption

M_1^3 has three distinct principal curvature, then, with respect to a suitable (local) orthonormal tangent frame $\{e_1, e_2, e_3\}$ on M, the shape operator S has the matrix form B_4, such that $Se_1 = \kappa e_1 - \lambda e_2$, $Se_2 = \lambda e_1 + \kappa e_2$, $Se_3 = \eta e_3$ and then, we have $P_2 e_1 = \kappa\eta e_1 + \lambda\eta e_2$, $P_2 e_2 = -\lambda\eta e_1 + \kappa\eta e_2$ and $P_2 e_3 = (\kappa^2 + \lambda^2)e_3$.

Using the polar decomposition $\nabla H_2 = \sum_{i=1}^{3} \epsilon_i e_i(H_2) e_i$, from condition (2.9)(ii) we get

$$\text{(3.13)} \qquad \begin{aligned} &(i)\ \ \epsilon_1 e_1(H_2)(\kappa\eta - \tfrac{9}{2}H_2) = \epsilon_2 e_2(H_2)\lambda\eta, \\ &(ii)\ \ \epsilon_2 e_2(H_2)(\kappa\eta - \tfrac{9}{2}H_2) = -\epsilon_1 e_1(H_2)\lambda\eta, \\ &(iii)\ \ \epsilon_3 e_3(H_2)(\kappa^2 + \lambda^2 - \tfrac{9}{2}H_2) = 0. \end{aligned}$$

Now, we prove three simple claims.

Claim 1: $e_1(H_2) = e_2(H_2) = 0$.
If $e_1(H_2) \neq 0$, then by dividing both sides of equalities (3.13)(i, ii) by $\epsilon_1 e_1(H_2)$ we get

$$\text{(3.14)} \qquad \begin{aligned} &(i)\ \ \kappa\eta - \tfrac{9}{2}H_2 = \tfrac{\epsilon_2 e_2(H_2)}{\epsilon_1 e_1(H_2)}\lambda\eta, \\ &(ii)\ \ \tfrac{\epsilon_2 e_2(H_2)}{\epsilon_1 e_1(H_2)}(\kappa\eta - \tfrac{9}{2}H_2) = -\lambda\eta, \end{aligned}$$

which, by substituting (i) in (ii), gives $\lambda\eta(1 + (\frac{\epsilon_2 e_2(H_2)}{\epsilon_1 e_1(H_2)})^2) = 0$, then $\lambda\eta = 0$. Since by assumption $\lambda \neq 0$, we get $\eta = 0$. So, by (3.14(i)), we have $H_2 = 0$.

Similarly, if $e_2(H_2) \neq 0$, then by dividing both sides of (3.13(i, ii)) by $\epsilon_2 e_2(H_2)$ we get

$$\text{(3.15)} \qquad \begin{aligned} &(i)\ \ \tfrac{\epsilon_1 e_1(H_2)}{\epsilon_2 e_2(H_2)}(\kappa\eta - \tfrac{9}{2}H_2) = \lambda\eta, \\ &(ii)\ \ \kappa\eta - \tfrac{9}{2}H_2 = -\tfrac{\epsilon_1 e_1(H_2)}{\epsilon_2 e_2(H_2)}\lambda\eta, \end{aligned}$$

which, by substituting (i) in (ii), gives $\lambda\eta(1 + (\frac{\epsilon_1 e_1(H_2)}{\epsilon_2 e_2(H_2)})^2) = 0$, then $\lambda\eta = 0$. Since by assumption $\lambda \neq 0$, we get $\eta = 0$. So, by (3.15(ii)), we have $H_2 = 0$.

Claim 2: $e_3(H_2) = 0$.
If $e_3(H_2) \neq 0$, then from equality (3.13(iii)) we have $\kappa^2 + \lambda^2 = \frac{9}{2}H_2$, which gives $\kappa^2 + \lambda^2 = -6\kappa\eta$, where $\eta = 3H_1 - 2\kappa$ and η and H_1 are assumed to be constant on U. So, κ is also constant on U, and then, we obtain $H_2 = \frac{-4}{3}\kappa\eta = \frac{8}{3}\kappa^2 - 4H_1\kappa$ and $H_3 = -6\kappa\eta^2 = -6\kappa(3H_1 - 2\kappa)^2$. are constant on U.

In the case $k = 2$, from condition (2.10)(ii) we get $e_i(H_3^2) = 0$ for $i = 1, 2, 3$, which means that there is nothing to prove. $\square$

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