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Spectral analysis for finite rank perturbation of diagonal operator in non-archimedean Banach space of countable type

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Abstract

In this paper we are concerned with the spectral analysis for the classes of finte rank perturbations of diagonal operators in the form $A = D+F$ *where* D *is a diagonal operator and* $F = u_1' \otimes v_1 + ... + u_m' \otimes v_m$ *is an operator of finite rank in the non-archimedean Banach space of countable type. We compute the spectrum of* A *using the theory of Fredholm operators in non archimedean setting and the concept of essential spectrum for linear operators.*

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1. Introduction and Preliminaries

In this paper $\mathbf{K}$ denotes a non trivial field which is complete with respect to a non-archimedean valuation denoted $|.|$ and its residue class fields is formally real.
The spectral analysis for completely continuous linear operators in the non-archimedean setting was studied by Serre [6]. After the work of Diagana and Ramarson was centered to the spectral analysis in the case of "small" perturbation of completely continuous linear operators of the form

$$T = D + K,$$

where D is a diagonal operator and K is completely continuous linear operator in the non-archimedean Hilbert space $\mathbf{E}_w$.
This work will generalized the studies of Diagana and Ramaroson from $\mathbf{E}_w$ to non-archimedean Banach space of countable type.
Define the space c_0 as the collection of all sequences $(\lambda_0, \lambda_1, \lambda_2, ...) \in \mathbf{K}^{\mathbf{N}}$ for which $\lim_i \lambda_i = 0$, i.e

$$c_0 = \{\lambda = (\lambda_i)_{i \in \mathbf{N}} \subset \mathbf{K} : \lim_i \lambda_i = 0\}.$$

It is know that the space c_0 equipped with the norm defined by:

$$for\ each\ \lambda = (\lambda_i)_{i \in \mathbf{N}} \in c_0 \ \|\lambda\|_\infty = \sup_{i \in \mathbf{N}} |\lambda_i|$$

is a non-archimedean Banach space see [2]. The bilinear form $\langle , \rangle : c_0 \times c_0 \to \mathbf{K}$ defined by:

$$for\ each\ x = (x_i), y = (y_i) \in c_0 \ \langle x, y \rangle = \sum_{i=0}^{+\infty} x_i y_i,$$

is an inner product in the non-archimedean sense. Since the residue class field of $\mathbf{K}$ is formally real then $\|x\|_\infty^2 = \langle x, x \rangle$. The non-archimedean Banach space c_0 has a special base denotes by $(e_i)_{i \in \mathbf{N}} = (\delta_{ij})_{i,j \in \mathbf{N}}$ where δ_{ij} is the usual Kronecker symbol.
Moreover we can define the crochet of duality by:

$$\langle , \rangle_d : c_0^{'} \times c_0 \to \mathbf{K}.$$

In this work we study the spectral analysis for an operator with the form

$$T = D + F,$$

where D is a bounded diagonal operator and $F = \sum_{k=1}^{m} u_k^{'} \otimes v_k$ is an operator of finite rank at most m, with $u_k^{'} \in c_0^{'}$ and $v_k \in c_0$ for $k = 1, ..., m$.
Namely, under some suitable assumptions, we will show that the spectrum $\sigma(T)$ of bounded linear operator T is given by

$$\sigma(T) = \sigma_e(D) \cup \sigma_p(T),$$

where $\sigma_e(D)$ is the essential spectrum of D and $\sigma_p(T)$ is the point spectrum of T, that is the set of eigenvalues of T given by $\sigma_p(T) = \{\lambda \in \rho(D) : \det M(\lambda) = 0\}$ with $M(\lambda)$ being the $m \times m$ square matrix $M(\lambda) = (b_{ij})_{i,j=1,..m}$ whose coefficients are given by $b_{ij} = \delta_{ij} + \langle C_\lambda v_i^{'}, v_j \rangle_d$ for $i, j = 1, ..., m$ and $C_\lambda = (D - \lambda I)^{-1}$.

Definition 1.1. *A mapping $A : c_0 \to c_0$ is said to be a bounded linear operator on c_0 whether it is linear and bounded, that is, there exists $C > 0$ such that*

$$for\ all\ u \in c_0\ \|Au\|_\infty \leq C\|u\|_\infty.$$

$\mathcal{B}(c_0)$ denote the collection of all bounded linear operators on c_0, $\mathcal{B}(c_0)$ is a non-archimedean Banach space with the norm $\|.\|_\infty$.
Let $A \in \mathcal{B}(c_0)$, its kernel and range are respectively defined by $N(A) = \{u \in c_0 : Au = 0\}$ and $R(A) = \{Au : u \in c_0\}$.

Proposition 1.2. *Let $A \in \mathcal{B}(c_0)$ then it can be written in a unique fashion as pointwise convergent series*

$$A = \sum_{i,j \in \mathbf{N}} a_{ji} e_i^{'} \otimes e_j, \ \ and \ \ (\forall i \in \mathbf{N}) \lim_j \ a_{ji} \ \|e_j\| = 0,$$

moreover

$$\|A\|_\infty = \sup_{i,j \in \mathbf{N}} \frac{|a_{ji}| \|e_j\|}{\|e_i\|}.$$

Proof. See [1]. □

The adjoint A^* of $A \in B(c_0)$, if it exists, is defined by $< Au, v >=< u, A^*v >$ for all $u, v \in c_0$. In contract with the classical case, the adjoint of an operator may or may not exist. Note that if it exists, the adjoint A^* of an operator A, is unique and has the same norm as A, and hence lies in $B(c_0)$ as well. The properties of the adjoint are easier to express in terms of the canonical orthogonal base of c_0.

1.1. Finite rank operators in c_0

Definition 1.3. *If $A \in \mathcal{B}(c_0)$ is such that $R(A)$ is a finite dimensional subspace of c_0, then A is said to be an operator of finite rank. In this case, the dimension of $R(A)$, that is $\dim R(A)$ is called the rank of A. The collection of all finite rank operators on c_0 will be denoted by $\mathcal{F}(c_0)$.*

It can be easily shown that for each $A \in \mathcal{F}(c_0)$, there exists $u_1^{'}, u_2^{'}, ..., u_m^{'} \in c_0^{'}$ and $v_1, v_2, ..., v_m \in c_0$ such that

$$A = \sum_{k=1}^{m} u_k^{'} \otimes v_k.$$

1.2. Completely continuous operators

Definition 1.4. *A linear operator $K : c_0 \to c_0$ is said to be completely continuous if there exists a sequence $(F_n) \subset \mathcal{F}(c_0)$ such that $\|K - F_n\| \to 0$ as $n \to \infty$. The collection of such linear operators will be denoted $\mathcal{C}(c_0)$.*

Example 1.5. *Classical examples of completely continuous operators include finite rank operators on c_0.*

Example 1.6. *Consider the diagonal operator D defined by, $De_j = \lambda_j e_j$ where $\lambda_j \in \mathbf{K}$ for each $j \in \mathbf{N}$ and suppose that*

$$\lim_{j\to\infty} |\lambda_j| = 0,$$

D is completely continuous. Indeed, consider the sequence of linear operators (D_n) defined on c_0 by $D_n e_j = \lambda_j e_j$ for $j = 0, 1, ..., n$ and $D_n e_j = 0$ for $j \geq n+1$, then $D_n \in \mathcal{F}(c_0)$ and $\lim\limits_{n\to\infty} \|D - D_n\| = \lim\limits_{j\to\infty} \sup\limits_{j\geq n+1} |\lambda_j| = 0$, and hence $D \in \mathcal{C}(c_0)$.

1.3. Fredholm operators

Definition 1.7. *An operator $A \in \mathcal{B}(c_0)$ is said to be a Fredholm operator if it satisfies the following conditions:*

1. *$\eta(A) = \dim N(A)$ is finite,*
2. *$R(A)$ is closed,*
3. *$\delta(A) = \dim(c_0/R(A))$ is finite.*

The collection of all Fredholm linear operators on c_0 will be denoted by $\Phi(c_0)$. If $A \in \Phi(c_0)$, we then define its index by setting $\chi(A) = \eta(A) - \delta(A)$.

Example 1.8. *Any invertible bounded linear operator $A : c_0 \to c_0$ (in particular, the identity operator $I : c_0 \to c_0, I(x) = x$), is a Fredholm operator with index $\chi(A) = 0$ as $\delta(A) = \eta(A) = 0$.*

1.4. Space of countable type

Recall that a topological space is called separable if it has a countable dense subset. Now let E be a normed space over $\mathbf{K}$ such that $E \neq \{0\}$ and suppose that E is separable, then $\mathbf{K}$ must be separable as well. Thus, for normed space the concept of separability is of no use if $\mathbf{K}$ is not separable, however linearizing the notion of separability we obtain a generalization useful for every scalar field $\mathbf{K}$.

Definition 1.9. *A normed space E is of countable type if it contains a countable set whose linear hull is dense in E.*

Example 1.10. *Clearly the span of unit vector $e_1 = (1, 0, ...), e_2 = (0, 1, 0, ...), ...$ is dense in c_0 then c_0 is a Banach space of countable type.*

Proposition 1.11. *Each normed space is linearly heomeorphic to a subspace of c_0. Each infinite-dimensional Banach space of countable type is linearly heomeorphic to c_0.*

Proof. See [2]. □

This result shows that, up to linear homeomorphisms, there exists, for given $\mathbf{K}$, only one infinite-dimensional Banach space of countable type viz c_0.

2. Spectral analysis for the class of operators $T = D + K$

In this section we study the spectral analysis for perturbation of completely continuous linear operators by diagonal operators. Namely, we study the class of operators of the form $T = D + K$, where $D : c_0 \to c_0$ is a diagonal operator defined by $De_j = \lambda_j e_j$ for all $j \in \mathbf{N}$, where $\lambda = (\lambda_j)_{j \in \mathbf{N}}$ is a bounded sequence and $K : c_0 \to c_0$ is completely continuous linear operator.

Definition 2.1. *The resolvent of a bounded linear operator $A : c_0 \to c_0$ is defined by $\rho(A) = \{\lambda \in \mathbf{K} : \lambda I - A$ is a bijection and $(\lambda I - A)^{-1} \in \mathcal{B}(c_0)\}$. The spectrum $\sigma(A)$ of A is then defined by $\sigma(A) = \mathbf{K} \setminus \rho(A)$.*

Definition 2.2. *A scalar $\lambda \in \mathbf{K}$ is called an eigenvalue of $A \in \mathcal{B}(c_0)$, whenever there exists a nonzero $u \in c_0$ (called eigenvector associated to λ) such that $Au = \lambda u$.*

Clearly, eigenvalues of A consist of all $\lambda \in \mathbf{K}$, for which $\lambda I - A$ is not one-to-one; that is $N(\lambda I - A) \neq \{0\}$. The collection of all eigenvalues of A is denoted by $\sigma_p(A)$ (called point spectrum) and is defined by

$$\sigma_p(A) = \{\lambda \in \sigma(A) : N(\lambda I - A) \neq \{0\}\}.$$

Example 2.3. *Consider the diagonal operator $D : c_0 \to c_0$ defined by*

$$\text{for all } \; u = (u_j)_{j\in\mathbf{N}} \in c_0 \; Du = \sum_{j=0}^{\infty} \lambda_j u_j e_j,$$

where $\sup_{j\in\mathbf{N}} |\lambda_j| < +\infty$. Then $\sigma(D) = \overline{\{\lambda_k : k \in \mathbf{N}\}}$ the closure of $\{\lambda_k : k \in \mathbf{N}\}$, i.e

$$\sigma(D) = \{\lambda \in \mathbf{K} : \lim_{j\to\infty} |\lambda - \lambda_j| = 0\}.$$

Definition 2.4. *Define the essential spectrum $\sigma_e(A)$ of a bounded linear operator $A : c_0 \to c_0$ as follows*

$$\sigma_e(A) = \{\lambda \in \mathbf{K} : \lambda I - A \; is \; not \; Fredholm \; operator \; of \; index \; 0\}.$$

Clearly if $\lambda \in \mathbf{K}$ does not belong to neither $\sigma_p(A)$ or $\sigma_e(A)$, then $(\lambda I - A)$ must be injective. $N(\lambda I - A) = \{0\}$ and $R(\lambda I - A)$ is closed with $0 = \dim N(\lambda I - A) = \dim(c_0/R(\lambda I - A))$, consequently $(\lambda I - A)$ must be bijective (injective and surjective) which yield $\lambda \in \rho(A)$.

In view of previous fact, we have $\sigma(A) = \sigma_p(A) \cup \sigma_e(A)$. Define $\Phi_0(c_0) = \{A \in \Phi(c_0) : \chi(A) = 0\}$.

Theorem 2.5. *If $A \in \Phi(c_0)$ and $K \in \mathcal{C}(c_0)$, then $A + K \in \Phi(c_0)$, with $\chi(A + K) = \chi(A)$.*

Proof. See [3]. □

The next theorem is very important in the rest of this paper.

Theorem 2.6. *If $A \in \mathcal{B}(c_0)$, then for all $K \in \mathcal{C}(c_0)$, $\sigma_e(A + K) = \sigma_e(A)$.*

Proof. *If λ does not belong to $\sigma_e(A)$, then $\lambda I - A$ belong to $\Phi(c_0)$ with $\chi(\lambda I - A) = 0$, therefore $\lambda I - A - K$ belong to $\Phi(c_0)$ with $\chi(\lambda I - (A+K)) = 0$ for all $K \in \Phi(c_0)$. Then λ does not belong to $\sigma_e(A + K)$.* □

Corollary 2.7. *For every $K \in \Phi(c_0)$, $\sigma_e(A + K) = \sigma_e(A)$.*

Proposition 2.8. *If $T = D + K$, where $K \in \mathcal{C}(c_0)$, then its spectrum $\sigma(T)$ is giving by $\sigma(T) = \sigma_e(D) \cup \sigma_p(T)$.*

Proof. We have $\sigma(T) = \sigma_e(T) \cup \sigma_p(T)$.
In view of precedent corollary $\sigma_e(T) = \sigma_e(D + K) = \sigma_e(D)$, so it follows that $\sigma(T) = \sigma_e(D) \cup \sigma_p(T)$. □

Lemma 2.9. *If $A \in \Phi_0(c_0)$ and $K \in \mathcal{C}(c_0)$, then the linear operator $A+K$ is invertible if and only if $N(A + K) = \{0\}$.*

Proof. Since $A \in \Phi(c_0)$, with index $\chi(A) = 0$, its follows by using Theorem 2.5, that $A+K$ belong to $\Phi(c_0)$, with index $\chi(A+K) = \chi(A) = 0$. In other word $\eta(A + K) = \delta(A + K)$. Now if $A + K$ is invertible, then $N(A + K) = \{0\}$.
Conversely, if $N(A + K) = \{0\}$ then $\eta(A + K) = \delta(A + K)$, and hence $A + K$ must be surjective, that is $A + K$ is invertible. □

Lemma 2.10. *Consider the finite rank operator*

$$F = \sum_{k=1}^{m} u_k^{'} \otimes v_k,$$

where $u_k^{'} \in c_0^{'}$, $v_k \in c_0$ for $k = 1, ..., m$, then the operator $I - F$ is invertible if and only if $\det P \neq 0$, where P is the $m \times m$ square matrix given by $P = (a_{ij})_{i,j=1,..,m}$, $a_{ij} = \delta_{ij} - \langle u_i^{'}, v_j \rangle_d$.

Proof. Using precedent lemma it follows that the operator $I - F$ is invertible if and only if $N(I-F) = \{0\}$. To complete the proof it is suficient to show that $N(I - F) = \{0\}$ if and only if $\det P \neq 0$.
For that, let $w \in c_0$ such that $(I - F)w = 0$; equivalently

$$w - \sum_{k=1}^{m} \langle u_k^{'}, w \rangle_d v_k = 0. \tag{2.1}$$

Now aplly $\langle .,. \rangle_d$ to the equation (2.1) with respectively $u_1^{'}, ...u_m^{'}$ we obtain the following system of equation

$$P \begin{pmatrix} \langle u_1^{'}, w \rangle_d \\ \vdots \\ \langle u_m^{'}, w \rangle_d \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{2.2}$$

If we suppose that $N(I - F) \neq \{0\}$, then

$$w = \sum_{k=1}^{m} \langle u_k^{'}, w \rangle_d v_k,$$

and hence at least one of the following scalars $\langle u_1^{'}, w \rangle_d, ..., \langle u_m^{'}, w \rangle_d$ is non zero. Consequently the equation (2.2) has at least one non trivial solution which yields $\det P = 0$.
Conversely, if $\det P = 0$, there exist some scalars $\xi_1, ..., \xi_m$ not all zeros, such that with $\xi = (\xi_1, ..., \xi_m)^t$ we have

$$P \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_m \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{2.3}$$

We take $w = \sum\limits_{k=1}^{m} \xi_k v_k$ and we obtain $(I - F)w = 0$.
Now $w \neq 0$, if not $\langle u_k^{'}, w \rangle = \delta_{kj} \xi_j$ for $j = 1, ..., m$ which yields $\xi_j = 0$ for $j = 1, ..., m$ and that contradict the fact that some ξ_j are unique then $N(I - F) \neq 0$. □

Proposition 2.11. *Consider the finite rank operator*

$$F = \sum_{k=1}^{m} u_k^{'} \otimes v_k,$$

where $u_k^{'} \in c_0^{'}, v_k \in c_0$ *for each* $k = 1, 2, ..., m$. *Then the spectrum of* F *is given by*

$$\sigma(F) = \{\lambda \in \mathbf{K} \setminus \{0\} : \det P(\lambda) = 0\} \cup \{0\},$$

where $P(\lambda)$ *is the* $m \times m$ *square matrix given by* $P(\lambda) = (a_{ij}(\lambda))_{i,j}$ *with* $a_{ij}(\lambda)) = \lambda \delta_{ij} - \langle u_i^{'}, v_j \rangle$.

Proof. Consider the operator $\lambda I - F$, clearly $\lambda = 0$ is necessairly in the spectrum of F, because F is not invertible.
Now suppose $\lambda \neq 0$, then $\lambda I - F = \lambda(I - F_\lambda)$, where $F_\lambda = \lambda^{-1}F$ is a finite rank operator, it is clear that $\lambda I - F$ is invertible if and only if $I - F_\lambda$ is invertible, and $(\lambda I - F)^{-1} = \lambda^{-1}(I - F_\lambda)^{-1}$.

Now using Lemma 2.9 it follows that $I - F_\lambda$ is invertible if and only if $N(I - F_\lambda) = \{0\}$. By using the same idea of the precedent lemma it follows that $N(I - F_\lambda) = \{0\}$ if and only if $\det P(\lambda) = \{0\}$. □

3. Spectral analysis for the class of operators $T = D + F$.

In this section we make extensive use of result of previous section to study the spectral analysis of operators of the form $T = D + F$ where $D : c_0 \longrightarrow c_0$ is a diagonal operator and F is an operator of finite rank defined by

$$F = \sum_{k=1}^{m} u_k^{'} \otimes v_k,$$

where $u_k^{'} \in c_0^{'}, v_k \in c_0$ for each $k = 1, 2, ..., m$.

Theorem 3.1. *If $T = D + F$ where D is a bounded diagonal operator on c_0 and $F = \sum\limits_{k=1}^{m} u_k^{'} \otimes v_k$. Then $\lambda \in \sigma_p(T)$ if and only if the following properties holds*

a) $\lambda \notin \sigma_p(D) \, (= \{\lambda_j : j \in \mathbf{N}\})$,

b) $\det M(\lambda) = 0$, *where* $M(\lambda) = (b_{ij})_{i,j=1,2,...,m}$ *with* $b_{ij}(\lambda) = \delta_{ij} + \langle C_\lambda u_i^{'}, v_j \rangle_d$ *for* $i, j = 1, 2, ..., m$ *with* $C_\lambda = (D - \lambda I)^{-1}$.

Proof. Suppose that $\lambda \in \sigma_p(T)$, thus there exists a non null $w \in c_0$ such that $Tw = \lambda w$, equivalently,

$$(\lambda I - D)w = Fw = \sum_{k=1}^{m} \langle u_k^{'}, w \rangle_d v_k. \tag{3.1}$$

Show that $\lambda \notin \sigma_p(D)$.
All the expressions $\langle u_k^{'}, w \rangle_d$ are non zero for $k = 1, 2, ..., m$, if not we will get $(\lambda I - D)w = 0$ with $w \neq 0$. Then $\lambda \in \sigma_p(D)$ and hence there exists $j_0 \in \mathbf{N}$ such that $\lambda = \lambda_{j_0}$, $w = ae_{j_0}$ with $a \in \mathbf{K} \backslash \{0\}$ and

$$0 = \langle u_k^{'}, w \rangle_d = \langle u_k^{'}, ae_{j_0} \rangle_d = a \langle u_k^{'}, e_{j_0} \rangle_d$$

yields $\langle u_k^{'}, e_{j_0}\rangle_d = 0$ for $k = 1, 2, ..., m$, absurd.
Consequently $Fw = (\lambda I - D)w \neq 0$ and hence $\lambda \notin \sigma_p(D)$. Clearly equation (3.1) is equivalent to

$$w + (D - \lambda I)^{-1} \sum_{k=1}^{m} \langle u_k^{'}, w\rangle_d v_k = 0,$$

$$\text{or} \quad w + \sum_{k=1}^{m} \langle u_k^{'}, w\rangle C_\lambda v_k = 0, \tag{3.2}$$

now apply $\langle ., .\rangle_d$ to the equation (3.2) with respectively $u_1^{'}, ..., u_m^{'}$ we obtain the following system of equation

$$M(\lambda) \begin{pmatrix} \langle u_1^{'}, w\rangle_d \\ \vdots \\ \langle u_m^{'}, w\rangle_d \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{3.3}$$

Using the fact that at least one of the following scalars $\langle u_1^{'}, w\rangle_d, ..., \langle u_m^{'}, w\rangle_d$ is non zero it follows that equation (3.3) has at least one non trivial solution which yields $\det M(\lambda) = 0$.

Suppose that $\lambda \notin \sigma_p(D)$ and $\det M(\lambda) = 0$, then there exists some scalars $\xi_1, ..., \xi_m$ not all zero such that with $\xi = (\xi_1, ..., \xi_m)^t$ we have

$$M(\lambda) \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_m \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{3.4}$$

we take $w = -\sum_{k=1}^{m} \xi_k C_\lambda v_k$ and obtain $(T - \lambda I)w = 0$, now $w \neq 0$; to show that it suffice to show that $C_\lambda^{-1} w \neq 0$, by using C_λ yield $w \neq 0$. If $C_\lambda^{-1} w = 0$, it follows that $0 = \langle C_\lambda^{-1} w, v_j\rangle = \delta_{kj}\xi_j$, for $j = 1, ...m$, consequently $\xi_j = 0$, for $j = 1, .., m$, this contradict the fact that some of the ξ_j are non zero. Then $N(T - \lambda I) \neq 0$, that is $\lambda \in \sigma_p(T)$. □

Corollary 3.2. *Let $T = D + F$ and $\lambda \in \rho(D)$. Then*
$\lambda \in \sigma_p(T)$ if and only if $\det M(\lambda) = 0$.

Proof. Since $\lambda \in \rho(D)$ it follows that $\lambda \notin \sigma_p(D)$, then the result follows directly from the previous theorem. □

Corollary 3.3. *If $T = D + F$, where D is a bounded diagonal operator and F is an operator of finite rank defined by $F = \sum_{k=1}^{m} u_k' \otimes v_k$, with $u_k' \in c_0'$ and $v_k \in c_0$ for each $k = 1, ...m$, then the eigenvalues and the spectrum of T are given by*

$$\sigma_p(T) = \{\lambda \in \rho(D) \quad : \det M(\lambda) = 0\}$$

$$\sigma(D) = \{\lambda \in \rho(D) \quad : \det M(\lambda) = 0\} \cup \sigma_e(D).$$

Remark 3.4. *For the characterization of the essential spectrum see [4].*

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