# On some $P-Q$ modular equations of degree 45 

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#### Abstract

On page 330 of his second notebook, Srinivasa Ramanujan recorded a $P-Q$ modular equation of degree 45 , proof of which has been given by Bruce C. Berndt via theory of modular forms. We in this paper, give a simple proof of the same using the identities of Ramanujan and also establish few new $P-Q$ modular equations of degree 45. Further using these, we establish certain new modular equations of signature 3.


Keywords and Phrases: Theta functions, modular equations.

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## 1. Introduction

Throughout this paper, we assume that $|q|<1$ and define

$$
(a ; q)_{\infty}:=\prod_{n=0}^{\infty}\left(1-a q^{n}\right), \quad a, q \in \mathbf{C}
$$

The Ramanujan's general theta function [18, p. 197], [4, p. 34] is defined by

$$
f(a, b):=\sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}=(-a ; a b)_{\infty}(-b ; a b)_{\infty}(a b ; a b)_{\infty}, \quad|a b|<1
$$

Following Ramanujan, we define

$$
f(-q):=f\left(-q,-q^{2}\right)=(q ; q)_{\infty}=: q^{-1 / 24} \eta(z), q=e^{2 \pi i z}, \operatorname{Im}(z)>0
$$

where $\eta(z)$ denotes the Dedekind eta-function. Also for convenience, we denote

$$
f\left(-q^{n}\right)=\left(q^{n} ; q^{n}\right)_{\infty}=f_{n}
$$

The ordinary hypergeometric series is defined by

$$
{ }_{2} F_{1}\left[\begin{array}{c}
a, b \\
c
\end{array} ; z\right]:=\sum_{k=0}^{\infty} \frac{(a)_{k}(b)_{k}}{k!(c)_{k}} z^{k}, \quad|z|<1
$$

where $(a)_{0}:=1$ and $(a)_{n}:=a(a+1)(a+2) \cdots(a+n-1), \quad n \in \mathbf{Z}^{+}$.

Suppose that

$$
\frac{{ }_{2} F_{1}\left[\frac{1}{r}, \frac{r-1}{r} ; 1-\beta\right]}{1}{ }_{2} F_{1}\left[\frac{1}{r}, \frac{r-1}{r} ; \beta\right] \quad{ }_{2} F_{1}\left[\begin{array}{c}
\left.\frac{1}{r}, \frac{r-1}{r} ; 1-\alpha\right]  \tag{1.1}\\
1
\end{array}\right.
$$

holds for some $n \in \mathbf{Z}^{+}$, then the relation induced by between $\alpha$ and $\beta$ by (1.1) is called a modular equation of degree $n$ in signature $r$. The case $r=2$ is called classical case. Ramanujan has recorded many modular equation in his Notebooks $[18,19]$ both in classical and alternate theories ( $r=3,4$ and 6 ), proof of all modular equations recorded by Ramanujan can be seen in $[4,5,6]$.

In the unorganized pages of his second notebook [18], Ramanujan recorded 23 identities, so called $P-Q$ eta function identities or $P-Q$ modular equations. These are identities involving quotients of eta function, which are designated by $P$ or $Q$ by Ramanujan. Proofs of these $P-Q$ identities employing various modular equations of Ramanujan or via the theory of modular forms have been given by Berndt and L.-C. Zhang [9]. Similar 14 identities involving ratio of Dedikind's eta function found on page 55 of Ramanujan's Lost notebook [19] have been proved by Berndt [7] employing the above methods. These $P-Q$ modular equations play a very important role in evaluations of class invariants, continued fractions and ratios of theta functions, for more details, see $[1,2,3,11,13,15,24,25,26,27,30]$.

On page 330 of his notebooks [18], Ramanujan has recorded a $P-Q$ modular equation of degree 45 , proof of which has been given by Berndt [ 5 , Entry 39, p. 192] by employing theory of modular forms. In Section 2 of this paper, we give a simple proof of Entry 39 [5] and also establish certain new identities involving ratios of theta functions in degree 45.

Ramanujan [18] has recorded many Russell [20, 21], Schläfli [22]and Weber [29] type modular equations, proofs of these can be seen in [6]. For a wonderful introduction to Ramanujan's work on modular equations, refer [17]. In his notebooks [18], Ramanujan has recorded several modular equations in the theory of signature 3 , details of these can be seen in $[6$, 8]. For recent works on the same, one may refer $[12,10,14,28,16,23]$. Motivated by these, in Section 3 of this paper, employing the $P-Q$ identities obtained in Section 2, we deduce certain new mixed modular equations in the theory of signature 3 which are analogous to Ramanujan-Schläfli and Ramanujan-Weber type modular equations in classical theory.

## 2. $P-Q$ Modular Equations

In this Section, we establish certain $P-Q$ modular equations. We first recall the following Lemmas:

Lemma 2.1. [19, p. 366] [4, Entry 1(iv), p. 345] If

$$
\begin{equation*}
A:=\frac{f_{1}}{q^{1 / 12} f_{3}} \quad \text { and } \quad A_{3}:=\frac{f_{3}}{q^{1 / 4} f_{9}} \tag{2.1}
\end{equation*}
$$

then

$$
\left(A A_{3}\right)^{3}+\frac{27}{\left(A A_{3}\right)^{3}}=\left(\frac{A_{3}}{A}\right)^{6}-9 .
$$

Lemma 2.2. [18, p. 324][5, p. 221] If

$$
\begin{equation*}
B:=\frac{f_{1}}{q^{1 / 12} f_{3}} \quad \text { and } \quad B_{5}:=\frac{f_{5}}{q^{5 / 12} f_{15}} \tag{2.2}
\end{equation*}
$$

then

$$
\left(B B_{5}\right)^{2}+\frac{9}{\left(B B_{5}\right)^{2}}+5=\left(\frac{B_{5}}{B}\right)^{3}-\left(\frac{B}{B_{5}}\right)^{3} .
$$

Lemma 2.3. [18, p. 325][5, p. 223] If

$$
\begin{equation*}
C:=\frac{f_{1}}{q^{1 / 6} f_{5}} \quad \text { and } \quad C_{3}:=\frac{f_{3}}{q^{1 / 2} f_{15}} \tag{2.3}
\end{equation*}
$$

then

$$
\left(C C_{3}\right)^{3}+\frac{125}{\left(C C_{3}\right)^{3}}=\left(\frac{C_{3}}{C}\right)^{6}-9\left(\frac{C_{3}}{C}\right)^{3}-9\left(\frac{C}{C_{3}}\right)^{3}-\left(\frac{C}{C_{3}}\right)^{6} .
$$

Theorem 2.1. [18, p. 330][5, p. 192] If

$$
P:=\frac{f_{1} f_{5}}{q^{1 / 2} f_{3} f_{15}} \quad \text { and } \quad Q:=\frac{f_{3} f_{15}}{q^{3 / 2} f_{9} f_{45}}
$$

then

$$
\left(\frac{Q}{P}\right)^{2}-3\left(\frac{Q}{P}\right)=P Q+\frac{9}{P Q}+3
$$

Proof. Let $B$ and $B_{5}$ be as defined in Lemma 2.2, we set $B_{3}:=\frac{f_{3}}{q^{1 / 4} f_{9}}$ and $B_{15}=\frac{f_{15}}{q^{5 / 4} f_{45}}$, then

$$
\begin{equation*}
P=B B_{5} \quad \text { and } \quad Q=B_{3} B_{15} . \tag{2.4}
\end{equation*}
$$

Now from Lemma 2.2 and (2.4), we have

$$
\frac{B_{5}^{6}}{P^{3}}-\frac{P^{3}}{B_{5}^{6}}=P^{2}+\frac{9}{P^{2}}+5 .
$$

Solving this for $\frac{P^{3}}{B_{5}^{6}}$, we find that

$$
\begin{equation*}
\frac{P^{3}}{B_{5}^{6}}=\frac{-K_{1} \pm \sqrt{K_{1}^{2}+4}}{2} \tag{2.5}
\end{equation*}
$$

where $K_{1}=P^{2}+\frac{9}{P^{2}}+5$. The identity (2.5) implies

$$
\begin{equation*}
\frac{B_{5}^{6}}{P^{3}}=\frac{K_{1} \pm \sqrt{K_{1}^{2}+4}}{2} \tag{2.6}
\end{equation*}
$$

Similarly, we deduce

$$
\begin{equation*}
\frac{Q^{3}}{B_{15}^{6}}=\frac{-K_{2} \pm \sqrt{K_{2}^{2}+4}}{2} \tag{2.7}
\end{equation*}
$$

where $K_{2}=Q^{2}+\frac{9}{Q^{2}}+5$. The identity (2.7) gives

$$
\begin{equation*}
\frac{B_{15}^{6}}{Q^{3}}=\frac{K_{2} \pm \sqrt{K_{1}^{2}+4}}{2} \tag{2.8}
\end{equation*}
$$

Multiplying (2.5) and (2.7), we obtain

$$
\frac{(P Q)^{3}}{\left(B_{5} B_{15}\right)^{6}}=\frac{K_{1} K_{2} \mp K_{1} \sqrt{K_{2}^{2}+4} \mp K_{2} \sqrt{K_{1}^{2}+4}+\sqrt{\left(K_{1}^{2}+4\right)\left(K_{2}^{2}+4\right)}}{4}
$$

Similarly multiplying (2.6) and (2.8), we have

$$
\frac{\left(B_{5} B_{15}\right)^{6}}{(P Q)^{3}}=\frac{K_{1} K_{2} \pm K_{1} \sqrt{K_{2}^{2}+4} \pm K_{2} \sqrt{K_{1}^{2}+4}+\sqrt{\left(K_{1}^{2}+4\right)\left(K_{2}^{2}+4\right)}}{4}
$$

Now, adding the above two identities, we deduce that

$$
\begin{equation*}
\frac{(P Q)^{3}}{\left(B_{5} B_{15}\right)^{6}}+\frac{\left(B_{5} B_{15}\right)^{6}}{(P Q)^{3}}=\frac{K_{1} K_{2}+\sqrt{\left(K_{1}^{2}+4\right)\left(K_{2}^{2}+4\right)}}{2} \tag{2.9}
\end{equation*}
$$

Similarly from $(2.5),(2.6),(2.7)$ and (2.8), we deduce that

$$
\begin{equation*}
\left(\frac{P}{Q}\right)^{3}\left(\frac{B_{15}}{B_{5}}\right)^{6}+\left(\frac{Q}{P}\right)^{3}\left(\frac{B_{5}}{B_{15}}\right)^{6}=\frac{-K_{1} K_{2}+\sqrt{\left(K_{1}^{2}+4\right)\left(K_{2}^{2}+4\right)}}{2} \tag{2.10}
\end{equation*}
$$

From Lemma 2.1, we have

$$
\left(B B_{3}\right)^{3}+\frac{27}{\left(B B_{3}\right)^{3}}=\left(\frac{B_{3}}{B}\right)^{3}\left\{\left(\frac{B_{3}}{B}\right)^{3}-9\left(\frac{B}{B_{3}}\right)^{3}\right\}
$$

Changing $q \rightarrow q^{5}$ in the above identity, we see that

$$
\left(B_{5} B_{15}\right)^{3}+\frac{27}{\left(B_{5} B_{15}\right)^{3}}=\left(\frac{B_{15}}{B_{5}}\right)^{3}\left\{\left(\frac{B_{15}}{B_{5}}\right)^{3}-9\left(\frac{B_{5}}{B_{15}}\right)^{3}\right\}
$$

Next, multiplying the above two identities and then employing (2.4), we deduce that

$$
\begin{aligned}
& (P Q)^{3}+\frac{3^{6}}{(P Q)^{3}}+27\left\{\frac{(P Q)^{3}}{\left(B_{5} B_{15}\right)^{6}}+\frac{\left(B_{5} B_{15}\right)^{6}}{(P Q)^{3}}\right\} \\
& =\left(\frac{Q}{P}\right)^{3}\left[\left(\frac{Q}{P}\right)^{3}+81\left(\frac{P}{Q}\right)^{3}-9\left\{\left(\frac{P}{Q}\right)^{3}\left(\frac{B_{15}}{B_{5}}\right)^{6}+\left(\frac{Q}{P}\right)^{3}\left(\frac{B_{5}}{B_{15}}\right)^{6}\right\}\right] .
\end{aligned}
$$

Now using (2.9) and (2.10) in the above identity and upon simplification, we obtain

$$
\begin{aligned}
& 2\left[(P Q)^{3}+\frac{3^{6}}{(P Q)^{3}}\right]+27 K_{1} K_{2}-\left(\frac{Q}{P}\right)^{3}\left[2\left\{\left(\frac{Q}{P}\right)^{3}+81\left(\frac{P}{Q}\right)^{3}\right\}+9 K_{1} K_{2}\right] \\
& =-9 \sqrt{\left(K_{1}^{2}+4\right)\left(K_{2}^{2}+4\right)}\left\{\left(\frac{Q}{P}\right)^{3}+3\right\} .
\end{aligned}
$$

Squaring the above on both sides, substituting the values of $K_{1}$ and $K_{2}$ and then factorizing, we arrive at

$$
G(P, Q) H(P, Q)=0
$$

where
$G(P, Q)=Q^{3}-3 P Q^{2}-9 P-3 P^{2} Q-Q^{2} P^{3}$
and

```
H(P,Q) = 17010P\mp@subsup{P}{}{9}\mp@subsup{Q}{}{6}-103518\mp@subsup{P}{}{8}\mp@subsup{Q}{}{5}+98415\mp@subsup{P}{}{8}Q+10935\mp@subsup{P}{}{8}\mp@subsup{Q}{}{3}-161838\mp@subsup{P}{}{4}\mp@subsup{Q}{}{5}+
6561P\mp@subsup{P}{}{4}\mp@subsup{Q}{}{3}-198288\mp@subsup{P}{}{6}\mp@subsup{Q}{}{5}+157464\mp@subsup{P}{}{6}Q+4374Q\mp@subsup{Q}{}{8}\mp@subsup{P}{}{5}-10935\mp@subsup{Q}{}{4}\mp@subsup{P}{}{5}+51759\mp@subsup{Q}{}{6}\mp@subsup{P}{}{5}-
18225Q 8}\mp@subsup{P}{}{3}-59049\mp@subsup{Q}{}{4}\mp@subsup{P}{}{3}-12393Q\mp@subsup{Q}{}{6}\mp@subsup{P}{}{3}+52488\mp@subsup{P}{}{6}\mp@subsup{Q}{}{3}+8748\mp@subsup{P}{}{7}\mp@subsup{Q}{}{4}+59049\mp@subsup{P}{}{5}
Q 15 - 19683P P
108P 11 Q & -68040 P}\mp@subsup{P}{}{8}\mp@subsup{Q}{}{7}-11664\mp@subsup{P}{}{11}\mp@subsup{Q}{}{4}-16038\mp@subsup{P}{}{8}\mp@subsup{Q}{}{9}-405\mp@subsup{P}{}{11}\mp@subsup{Q}{}{6}-5076\mp@subsup{P}{}{4}\mp@subsup{Q}{}{11}
11664P\mp@subsup{P}{}{7}\mp@subsup{Q}{}{8}-39204\mp@subsup{P}{}{4}\mp@subsup{Q}{}{9}+55404P\mp@subsup{P}{}{7}\mp@subsup{Q}{}{6}-4356\mp@subsup{P}{}{6}\mp@subsup{Q}{}{11}+6156\mp@subsup{P}{}{9}\mp@subsup{Q}{}{8}-144342\mp@subsup{P}{}{6}\mp@subsup{Q}{}{7}-
3645P\mp@subsup{P}{}{9}\mp@subsup{Q}{}{4}-37341\mp@subsup{P}{}{6}\mp@subsup{Q}{}{9}-13122P\mp@subsup{P}{}{11}\mp@subsup{Q}{}{2}-1998\mp@subsup{Q}{}{11}\mp@subsup{P}{}{2}-32805P\mp@subsup{P}{}{9}\mp@subsup{Q}{}{2}+648\mp@subsup{P}{}{2}\mp@subsup{Q}{}{9}-
```



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72P 12 Q Q + 135PP'4}\mp@subsup{Q}{}{7}+24\mp@subsup{P}{}{14}\mp@subsup{Q}{}{9}-3\mp@subsup{P}{}{13}\mp@subsup{Q}{}{10}+639\mp@subsup{P}{}{9}\mp@subsup{Q}{}{10}-17\mp@subsup{P}{}{9}\mp@subsup{Q}{}{12}
15P 11 Q 年 -9 P}\mp@subsup{}{}{11}\mp@subsup{Q}{}{12}-22599P\mp@subsup{P}{}{10}\mp@subsup{Q}{}{5}+19683\mp@subsup{P}{}{10}Q+4374\mp@subsup{P}{}{10}\mp@subsup{Q}{}{3}-405P\mp@subsup{P}{}{13}\mp@subsup{Q}{}{6}
```



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486 P}\mp@subsup{}{}{12}\mp@subsup{Q}{}{5}-11502\mp@subsup{P}{}{10}\mp@subsup{Q}{}{7}-225\mp@subsup{P}{}{7}\mp@subsup{Q}{}{12}-46\mp@subsup{P}{}{3}\mp@subsup{Q}{}{14}-222\mp@subsup{P}{}{4}\mp@subsup{Q}{}{13}+8\mp@subsup{P}{}{6}\mp@subsup{Q}{}{13}
9936 P}\mp@subsup{}{3}{4}\mp@subsup{Q}{}{10}-1104\mp@subsup{P}{}{5}\mp@subsup{Q}{}{12}-255\mp@subsup{Q}{}{13}\mp@subsup{P}{}{2}-84P\mp@subsup{Q}{}{14}-729P\mp@subsup{Q}{}{10}+\mp@subsup{P}{}{15}\mp@subsup{Q}{}{10}
243P}\mp@subsup{P}{}{14}\mp@subsup{Q}{}{5}+2187\mp@subsup{P}{}{12}\mp@subsup{Q}{}{3}
```

From the definitions of $P$ and $Q$, we have

$$
P=q^{-1 / 2}\left(1-q-q^{2}+q^{3}-q^{4}-q^{5}+3 q^{6}-2 q^{8}+\cdots\right)
$$

and

$$
Q=q^{-3 / 2}\left(1-q^{3}-q^{6}+q^{9}-q^{12}-q^{15}+3 q^{18}-2 q^{24}+\cdots\right)
$$

Using these in $G(P, Q)$ and $H(P, Q)$, we find that
$G(P, Q)=q^{9 / 2}\left(-12+21 q-15 q^{2}-15 q^{3}+33 q^{4}-15 q^{5}-51 q^{6}+\cdots\right)$
and
$H(P, Q)=q^{-47 / 2}\left(-27+16 q-1161 q^{2}+2025 q^{3}+4455 q^{4}-\cdots\right)$.
Clearly $q^{47 / 2} G(P, Q) \rightarrow 0$ as $q \rightarrow 0$, whereas $q^{47 / 2} H(P, Q) \nrightarrow 0$ as $q \rightarrow 0$. Hence by analytic continuation $G(P, Q)=0$ in $|q|<1$. Thus,

$$
Q^{3}-3 P Q^{2}-9 P-3 P^{2} Q-Q^{2} P^{3}=0
$$

which is equivalent to the required result.
Remark. Comparing the definition [18, p. 330] of $u$ and $v$ and that of $P$ and $Q$ of Theorem 2.1, we see that $P=\frac{v}{u}$ and $Q=u$, using these in Theorem 2.1, we deduce Entry 39 [5, p. 192].

Theorem 2.2. If

$$
R:=q^{1 / 3} \frac{f_{1} f_{15}}{f_{3} f_{5}} \quad \text { and } \quad S:=q \frac{f_{3} f_{45}}{f_{9} f_{15}}
$$

then

$$
\begin{aligned}
& \left(\frac{R}{S}\right)^{6}+\left(\frac{S}{R}\right)^{6}+(R S)^{6}+\frac{1}{(R S)^{6}}-\left[(R S)^{3}+\frac{1}{(R S)^{3}}\right]\left\{\left(\frac{R}{S}\right)^{6}+\left(\frac{S}{R}\right)^{6}+46\right\}+92 \\
& =9\left[(R S)^{3 / 2}-\frac{1}{(R S)^{3 / 2}}\right]\left\{\left(\frac{R}{S}\right)^{9 / 2}+\left(\frac{S}{R}\right)^{9 / 2}+2\left[\left(\frac{R}{S}\right)^{3 / 2}+\left(\frac{S}{R}\right)^{3 / 2}\right]\right\} \\
& -9\left[(R S)^{9 / 2}-\frac{1}{(R S)^{9 / 2}}\right]\left\{\left(\frac{R}{S}\right)^{3 / 2}+\left(\frac{S}{R}\right)^{3 / 2}\right\} .
\end{aligned}
$$

Proof. Following (2.3), we set $C_{9}=\frac{f_{9}}{q^{3 / 2} f_{45}}$, then we note that

$$
\begin{equation*}
R=\frac{C}{C_{3}} \quad \text { and } \quad S=\frac{C_{3}}{C_{9}} \tag{2.11}
\end{equation*}
$$

Now, from Lemma 2.3 and (2.11), we have

$$
\frac{R^{3} C_{3}^{6}}{5^{3 / 2}}+\frac{5^{3 / 2}}{R^{3} C_{3}^{6}}=\frac{1}{5^{3 / 2}}\left[\frac{1}{R^{6}}-R^{6}-\frac{9}{R^{3}}-9 R^{3}\right]
$$

Solving the above for $\frac{R^{3} C_{3}^{6}}{5^{3 / 2}}$, we see that

$$
\begin{equation*}
\frac{R^{3} C_{3}^{6}}{5^{3 / 2}}=\frac{K_{1} \pm \sqrt{K_{1}^{2}-4}}{2} \tag{2.12}
\end{equation*}
$$

where

$$
K_{1}=\frac{1}{5^{3 / 2}}\left[\frac{1}{R^{6}}-R^{6}-\frac{9}{R^{3}}-9 R^{3}\right]
$$

From (2.12), we have

$$
\begin{equation*}
\frac{5^{3 / 2}}{R^{3} C_{3}^{6}}=\frac{K_{1} \mp \sqrt{K_{1}^{2}-4}}{2} \tag{2.13}
\end{equation*}
$$

Changing $q$ to $q^{3}$ in (2.12) and (2.13), we respectively obtain

$$
\begin{equation*}
\frac{S^{3} C_{9}^{6}}{5^{3 / 2}}=\frac{K_{2} \pm \sqrt{K_{2}^{2}-4}}{2} \tag{2.14}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{5^{3 / 2}}{S^{3} C_{9}^{6}}=\frac{K_{2} \mp \sqrt{K_{2}^{2}-4}}{2} \tag{2.15}
\end{equation*}
$$

where

$$
K_{2}=\frac{1}{5^{3 / 2}}\left[\frac{1}{S^{6}}-S^{6}-\frac{9}{S^{3}}-9 S^{3}\right] .
$$

Multiplying (2.12) and (2.15), we deduce that

$$
4(R S)^{3}=K_{1} K_{2} \mp K_{1} \sqrt{K_{2}^{2}-4} \pm K_{2} \sqrt{K_{1}^{2}-4}-\sqrt{\left(K_{1}^{2}-4\right)\left(K_{2}^{2}-4\right)}
$$

Similarly, multiplying (2.13) and (2.14), we see that

$$
\frac{4}{(R S)^{3}}=K_{1} K_{2} \pm K_{1} \sqrt{K_{2}^{2}-4} \mp K_{2} \sqrt{K_{1}^{2}-4}-\sqrt{\left(K_{1}^{2}-4\right)\left(K_{2}^{2}-4\right)}
$$

Now, adding the above two identities, we find that

$$
2\left[(R S)^{3}+\frac{1}{(R S)^{3}}\right]-K_{1} K_{2}=-\sqrt{\left(K_{1}^{2}-4\right)\left(K_{2}^{2}-4\right)}
$$

Squaring the above on both sides, substituting the values of $K_{1}$ and $K_{2}$ in the resulting identity and then factorizing, we arrive at
$(1+R S)^{2}\left(1-R S+R^{2} S^{2}\right)^{2}\left(-R^{3} S^{3}+46 R^{6} S^{6}-92 R^{9} S^{9}-18 S^{6} R^{9}-18 R^{6} S^{9}-\right.$
$R^{15} S^{15}+R^{6} S^{18}+9 R^{6} S^{15}+R^{18} S^{6}+9 R^{15} S^{6}+S^{12}+R^{12}+46 R^{12} S^{12}-$
$9 R^{12} S^{3}+18 R^{12} S^{9}-9 R^{3} S^{12}+18 R^{9} S^{12}+9 R^{6} S^{3}+9 R^{3} S^{6}-9 R^{12} S^{15}-$
$\left.9 R^{15} S^{12}-R^{15} S^{3}-R^{3} S^{15}\right)=0$.
By definitions of $R$ and $S$, it follows that $R S \neq-1$ and $R S+\frac{1}{R S} \neq 1$, hence the third factor must be zero, which is equivalent to the required result.

Theorem 2.3. If

$$
\begin{equation*}
M:=\frac{f_{1} f_{5}}{q^{2} f_{9} f_{45}} \quad \text { and } \quad N:=q^{4 / 3} \frac{f_{1} f_{45}}{f_{5} f_{9}} \tag{2.16}
\end{equation*}
$$

then
$N^{6}+\frac{1}{N^{6}}+\left(M^{4}+\frac{9^{4}}{M^{4}}\right)+10\left(M^{3}+\frac{9^{3}}{M^{3}}\right)+45\left(M^{2}+\frac{9^{2}}{M^{2}}\right)+140\left(M+\frac{9}{M}\right)$
$+414=\left(N^{3}+\frac{1}{N^{3}}\right)\left[2\left(M^{2}+\frac{9^{2}}{M^{2}}\right)+35\left(M+\frac{9}{M}\right)+170\right]$

Proof. Changing $q$ to $q^{5}$ in Lemma 2.1, we have

$$
\begin{equation*}
\left(A_{5} A_{15}\right)^{3}+\frac{27}{\left(A_{5} A_{15}\right)^{3}}+9=\left(\frac{A_{15}}{A_{5}}\right)^{6} \tag{2.17}
\end{equation*}
$$

where

$$
\begin{equation*}
A_{5}=\frac{f_{5}}{q^{5 / 12} f_{15}} \quad \text { and } \quad A_{15}=\frac{f_{15}}{q^{5 / 4} f_{45}} \tag{2.18}
\end{equation*}
$$

Now, rearranging the terms in Lemma 2.1 correspondingly as in (2.17) and then multiplying the resultant identity with (2.17), we obtain

$$
\begin{aligned}
& \left(A A_{3} A_{5} A_{15}\right)^{3}+\frac{27^{2}}{\left(A A_{3} A_{5} A_{15}\right)^{3}}+27\left[\left(\frac{A A_{3}}{A_{5} A_{15}}\right)^{3}+\left(\frac{A_{5} A_{15}}{A A_{3}}\right)^{3}\right] \\
& +9\left[\left(A A_{3}\right)^{3}+\frac{27}{\left(A A_{3}\right)^{3}}\right]+9\left[\left(A_{5} A_{15}\right)^{3}+\frac{27}{\left(A_{5} A_{15}\right)^{3}}\right]+81=\left(\frac{A_{3} A_{15}}{A A_{5}}\right)^{6}
\end{aligned}
$$

(2.19)

Using (2.16), (2.1) and (2.18) in (2.19), we see that (2.19) takes the form $M^{3}+\frac{9^{3}}{M^{3}}+27\left(N^{3}+\frac{1}{N^{3}}\right)+9\left[\left(\frac{f_{1}}{f_{9}}\right)^{3}+3^{3}\left(\frac{f_{9}}{f_{1}}\right)^{3}\right]$
$+9\left[\left(\frac{f_{5}}{f_{45}}\right)^{3}+3^{3}\left(\frac{f_{45}}{f_{5}}\right)^{3}\right]+81=\left(\frac{Q}{P}\right)^{6}$,
where $P$ and $Q$ are as defined in Theorem 2.1. The above identity can be rewritten as
$\left(\frac{Q}{P}\right)^{6}-\left(M^{3}+\frac{9^{3}}{M^{3}}\right)-27\left(N^{3}+\frac{1}{N^{3}}\right)-81$

$$
\begin{equation*}
=9\left(M^{3 / 2}+\frac{3^{3}}{M^{3 / 2}}\right)\left(N^{3 / 2}+\frac{1}{N^{3 / 2}}\right) . \tag{2.20}
\end{equation*}
$$

Next, using (2.16) in Theorem 2.1, we see that

$$
\left(\frac{Q}{P}\right)^{2}-3\left(\frac{Q}{P}\right)=M+\frac{9}{M}+3
$$

Cubing the above identity throughout and after simplification, we have

$$
\left(\frac{Q}{P}\right)^{6}-\left(\frac{Q}{P}\right)^{3}(27+9 S)-S^{3}=0
$$

where $S=M+\frac{9}{M}+3$. Solving the above identity for $\left(\frac{Q}{P}\right)^{3}$ gives

$$
2\left(\frac{Q}{P}\right)^{3}=(27+9 S) \pm \sqrt{(27+9 S)^{2}+4 S^{3}}
$$

Squaring the above identity on both sides yields
(2.21) $2\left(\frac{Q}{P}\right)^{6}=(27+9 S)^{2}+2 S^{3} \pm(27+9 S) \sqrt{(27+9 S)^{2}+4 S^{3}}$.

Now eliminating $\left(\frac{Q}{P}\right)^{6}$ between (2.20) and (2.21), we see that
$\pm(27+9 S) \sqrt{(27+9 S)^{2}+4 S^{3}}$
$=\left[2\left\{\left(M^{3}+\frac{9^{3}}{M^{3}}\right)+27\left(N^{3}+\frac{1}{N^{3}}\right)+81\right\}-(27+9 S)^{2}-2 S^{3}\right]$
$+18\left(M^{3 / 2}+\frac{3^{3}}{M^{3 / 2}}\right)\left(N^{3 / 2}+\frac{1}{N^{3 / 2}}\right)$.
Squaring the above identity and simplifying, we obtain
$(27+9 S)^{2}\left((27+9 S)^{2}+4 S^{3}\right)-\left[a^{2}+18^{2}\left(M^{3}+\frac{3^{6}}{M^{3}}+54\right)\left(N^{3}+\frac{1}{N^{3}}+2\right)\right]$
$=2 a b$,
where

$$
a=2\left\{M^{3}+\frac{3^{6}}{M^{3}}+27\left(N^{3}+\frac{1}{N^{3}}\right)+81\right\}-(27+9 S)^{2}-2 S^{3}
$$

and

$$
b=18\left(M^{3 / 2}+\frac{3^{3}}{M^{3 / 2}}\right)\left(N^{3 / 2}+\frac{1}{N^{3 / 2}}\right)
$$

Again squaring the above identity and after simplification we arrive at

$$
U(M, N) \quad V(M, N)=0
$$

where

$$
\begin{aligned}
& U(M, N)=M^{4}+6561 N^{6}-162 N^{3} M^{2}-315 N^{3} M^{3}+7290 M N^{6}-170 N^{3} M^{4}+ \\
& 1260 M^{3} N^{6}-35 M^{5} N^{3}-2 M^{6} N^{3}+3645 N^{6} M^{2}+414 M^{4} N^{6}+140 M^{5} N^{6}- \\
& 162 N^{9} M^{2}-315 N^{9} M^{3}+45 N^{6} M^{6}+10 N^{6} M^{7}+N^{6} M^{8}+N^{12} M^{4}-170 N^{9} M^{4}- \\
& 35 N^{9} M^{5}-2 N^{9} M^{6}
\end{aligned}
$$

and

$$
\begin{aligned}
& V(M, N)=-13122 N^{3} M^{3}+708588 M^{6}-34992 N^{3} M^{4}-104976 M^{3} N^{6}- \\
& 32805 M^{5} N^{3}-14742 M^{6} N^{3}+236196 N^{6} M^{2}-122472 M^{4} N^{6}-56862 M^{5} N^{6}- \\
& 13122 N^{9} M^{3}-19440 N^{6} M^{6}-6318 N^{6} M^{7}-1512 N^{6} M^{8}-34992 N^{9} M^{4}- \\
& 32805 N^{9} M^{5}-14742 N^{9} M^{6}-3645 N^{9} M^{7}-432 N^{9} M^{8}-18 M^{9} N^{3}+12 M^{11} N^{6}+ \\
& M^{12} N^{6}+81 N^{12} M^{6}-18 N^{9} M^{9}-3645 M^{7} N^{3}-432 M^{8} N^{3}-144 N^{6} M^{9}+ \\
& 36 N^{6} M^{10}+531441 N^{6}+81 M^{6} .
\end{aligned}
$$

Now by the definitions of $M$ and $N$, we have

$$
M=q^{2}\left(1-q-q^{2}+q^{6}+2 q^{7}+q^{9}-3 q^{10}-q^{12}-q^{15}+\cdots\right)
$$

and

$$
N=q^{4 / 3}\left(1-q-q^{2}+2 q^{5}-q^{6}+q^{9}+2 q^{10}-3 q^{11}-2 q^{12}+2 q^{14}+3 q^{15}-\cdots\right)
$$

Using these in the definition of $U(M, N)$ and $V(M, N)$, we see that

$$
U(M, N)=120 q^{27}-600 q^{28}+470 q^{29}+3555 q^{30}-13140 q^{31}+21290 q^{32}-\cdots
$$

and

$$
V(M, N)=1862625+q^{-16}-18 q^{-15}+129 q^{-14}-498 q^{-13}+906 q^{-12}+\cdots
$$

Clearly $q^{16} U(M, N) \rightarrow 0$ as $q \rightarrow 0$, where as $q^{16} V(M, N) \nrightarrow 0$ as $q \rightarrow 0$. Hence by analytic continuation, we have

$$
U(M, N)=0,
$$

which is equivalent to the required result.

## Theorem 2.4. If

$$
\begin{equation*}
K:=q^{1 / 6} \frac{f_{1} f_{9}}{f_{3}^{2}} \quad \text { and } \quad L:=q^{5 / 6} \frac{f_{5} f_{45}}{f_{15}^{2}} \tag{2.22}
\end{equation*}
$$

then

$$
\left(\frac{K}{L}\right)^{3}+\left(\frac{L}{K}\right)^{3}+6=6(K L)^{3}+9(K L)^{2}+\frac{1}{(K L)^{2}}+3 K L+\frac{1}{K L} .
$$

Proof. Employing (2.16) and (2.22) in Theorem 2.1, we see that

$$
\begin{equation*}
M+\frac{9}{M}=\frac{1}{(K L)^{2}}-\frac{3}{K L}-3, \tag{2.23}
\end{equation*}
$$

Changing $q$ to $q^{5}$ in Lemma 2.1, we have

$$
\begin{equation*}
\left(A_{5} A_{15}\right)^{3}+\frac{27}{\left(A_{5} A_{15}\right)^{3}}=\left(\frac{A_{15}}{A_{5}}\right)^{6}-9, \tag{2.24}
\end{equation*}
$$

where

$$
A_{5}:=\frac{f_{5}}{q^{5 / 12} f_{15}} \text { and } A_{15}:=\frac{f_{15}}{q^{5 / 4} f_{45}} .
$$

Now, multiplying the identity in Lemma 2.1 and (2.24) and then employing (2.16) and (2.22) in the resulting identity, we obtain
$M^{3}+\frac{9^{3}}{M^{3}}+27\left(N^{3}+\frac{1}{N^{3}}\right)$

$$
\begin{equation*}
=\frac{1}{(K L)^{3}}\left[\frac{1}{(K L)^{3}}+81(K L)^{3}-9\left\{\left(\frac{K}{L}\right)^{3}+\left(\frac{L}{K}\right)^{3}\right\}\right] . \tag{2.25}
\end{equation*}
$$

Next, solving for $N^{3}+\frac{1}{N^{3}}$ in Theorem 2.3, we have

$$
\begin{equation*}
2\left(N^{3}+\frac{1}{N^{3}}\right)=-B \pm \sqrt{B^{2}-4 C} \tag{2.26}
\end{equation*}
$$

where
$B=2\left(M^{2}+\frac{9^{2}}{M^{2}}\right)+35\left(M+\frac{9}{M}\right)+170$
and

$$
C=M^{4}+\frac{9^{4}}{M^{4}}+10\left(M^{3}+\frac{9^{3}}{M^{3}}\right)+45\left(M^{2}+\frac{9^{2}}{M^{2}}\right)+140\left(M+\frac{9}{M}\right)+414 .
$$

Now, setting $T=M+\frac{9}{M}$, we see that

$$
B=2\left(T^{2}-18\right)+35 T+170
$$

and

$$
C=\left(\left(T^{2}-18\right)^{2}-162\right)+10\left(T^{3}-27 T\right)+45\left(T^{2}-18\right)+140 T+412 .
$$

Next, using (2.26) in (2.25), we deduce that
(2.27) $\pm 27 \sqrt{B^{2}-4 C}=2 S-2\left(T^{3}-27 T\right)+27 B$,
where
$S=\frac{1}{(K L)^{3}}\left[\frac{1}{(K L)^{3}}+81(K L)^{3}-9\left\{\left(\frac{K}{L}\right)^{3}+\left(\frac{L}{K}\right)^{3}\right\}\right]$.
Upon squaring (2.27) on both sides and employing (2.23), we arrive at

$$
X(K, L) \quad Y(K, L)=0,
$$

where

$$
X(K, L)=K L+K^{2} L^{2}-6 K^{3} L^{3}-K^{6}+3 K^{4} L^{4}+6 K^{6} L^{6}+9 K^{5} L^{5}-L^{6}
$$

and

$$
Y(K, L)=K L+K^{2} L^{2}-36 K^{3} L^{3}-K^{6}+138 K^{4} L^{4}+141 K^{6} L^{6}-216 K^{5} L^{5}-L^{6} .
$$

Now from the definitions of $K$ and $L$, we have

$$
K=q^{1 / 6}\left(1-q-q^{2}+2 q^{3}-2 q^{4}-q^{5}+5 q^{6}-4 q^{7}-3 q^{8}+9 q^{9}-7 q^{10}-4 q^{11}+\cdots\right)
$$

and

$$
L=q^{5 / 6}\left(1-q^{5}-q^{10}+2 q^{15}+\cdots\right) .
$$

Employing these in $X(K, L)$ and $Y(K, L)$, we note that

$$
X(K, L)=6 q^{2}-18 q^{3}+102 q^{5}-216 q^{6}-54 q^{7}+1020 q^{8}-1620 q^{9}-912 q^{10}+\cdots
$$

and

$$
Y(K, L)=6 q^{2}-48 q^{3}+225 q^{4}-663 q^{5}+984 q^{6}+711 q^{7}-7170 q^{8}+15285 q^{9}-\cdots .
$$

Now, for sufficiently small $q,|q|<1$, we see that $X(K, L) \rightarrow 0$ faster than $Y(K, L)$ and hence

$$
X(K, L)=0
$$

which is equivalent to the required result.

## 3. Mixed Modular Equations of Signature 3

In this Section, by employing the $P-Q$ modular equations deduced in Section 2 , we obtain certain new mixed modular equations in the theory of signature 3.

Lemma 3.1. We have

$$
\begin{equation*}
\frac{f_{1}}{q^{1 / 12} f_{3}}=3^{1 / 4}\left(\frac{(1-\alpha)}{\alpha}\right)^{1 / 12} \tag{3.1}
\end{equation*}
$$

Proof. Employing Corollary 3.4 and Corollary 3.5 of [6, p. 104], the Lemma follows.

Theorem 3.1. If $\alpha, \beta, \gamma$ and $\delta$ have degrees $1,3,5$ and 15 respectively, then
(i) $Z^{2}-3 Z=X+\frac{9}{X}+3$,
(ii) $W^{6}+\frac{1}{W^{6}}+Y^{6}+\frac{1}{Y^{6}}-\left[Y^{3}+\frac{1}{Y^{3}}\right]\left[W^{6}+\frac{1}{W^{6}}+46\right]+92$
$=9\left[Y^{9 / 2}-\frac{1}{Y^{9 / 2}}\right]\left[W^{3 / 2}+\frac{1}{W^{3 / 2}}\right]-9\left[Y^{3 / 2}-\frac{1}{Y^{3 / 2}}\right]$

$$
\times\left[W^{9 / 2}+\frac{1}{W^{9 / 2}}+2\left\{W^{3 / 2}+\frac{1}{W^{3 / 2}}\right\}\right]
$$

(iii) $Y^{6}+\frac{1}{Y^{6}}+X^{4}+\frac{9^{4}}{X^{4}}+10\left[X^{3}+\frac{9^{3}}{X^{3}}\right]+45\left[X^{2}+\frac{9^{2}}{X^{2}}\right]+140\left[X+\frac{9}{X}\right]+414$
$=\left(Y^{3}+\frac{1}{Y^{3}}\right)\left[2\left(X^{2}+\frac{9^{2}}{X^{2}}\right)+35\left(X+\frac{9}{X}\right)+170\right]$
and
(iv) $W^{3}+\frac{1}{W^{3}}+6=\frac{6}{Z^{3}}+\frac{3^{2}}{Z^{2}}+Z^{2}+\frac{3}{Z}+Z$,
where

$$
\begin{align*}
X & =3\left\{\frac{(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)}{\alpha \beta \gamma \delta}\right\}^{1 / 12}  \tag{3.2}\\
Y & =\left\{\frac{\alpha \beta(1-\gamma)(1-\delta)}{\gamma \delta(1-\alpha)(1-\beta)}\right\}^{1 / 12}  \tag{3.3}\\
Z & =\left\{\frac{\alpha \gamma(1-\beta)(1-\delta)}{\beta \delta(1-\alpha)(1-\gamma)}\right\}^{1 / 12} \tag{3.4}
\end{align*}
$$

and

$$
\begin{equation*}
W=\left\{\frac{\alpha \delta(1-\beta)(1-\gamma)}{\beta \gamma(1-\alpha)(1-\delta)}\right\}^{1 / 12} \tag{3.5}
\end{equation*}
$$

Proof. Employing (3.1) in (3.2) and (3.4) and then using them in Theorem 2.1, we obtain (i). Proof of (ii) follows from (3.1), (3.3), (3.5) and Theorem 2.2. Proof of (iii) follows from (3.1), (3.2), (3.3) and Theorem 2.3. Employing (3.1), (3.4), (3.5) in Theorem 2.4, proof of (iv) follows.

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