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Semi-commutativity of graded rings and graded modules

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Abstract

A ring R is said to be semi-commutative if whenever $a, b \in R$ such that $ab = 0$, then $aRb = 0$. In this article, we introduce the concepts of $g-$semi-commutative rings and $g-N-$semi-commutative rings and we introduce several results concerning these two concepts. Let R be a G-graded ring and $g \in supp(R, G)$. Then R is said to be a $g-$semi-commutative if whenever $a, b \in R$ with $ab = 0$, then $aR_g b = 0$. Also, R is said to be a $g - N-$semi-commutative if for any $a \in R$ and $b \in N(R) \bigcap Ann(a)$, $bR_g \subseteq Ann(a)$. We introduce an example of a G-graded ring R which is $g - N$-semi-commutative for some $g \in supp(R, G)$ but R itself is not semi-commutative. Clearly, if R is a $g-$semi-commutative ring, then R is a $g - N-$semi-commutative ring, however, we introduce an example showing that the converse is not true in general. Several results and examples are investigated. Also, we introduce the concept of $g - NE-$semi-commutative rings and we introduce several results concerning $g - NE-$semi-commutative rings. Let R be a G-graded ring and $g \in supp(R, G)$. Then R is said to be a $g - NE-$ semi-commutative ring if whenever $a \in N(R)$ and $b \in E(R)$ such that $ab = 0$, then $aR_g b = 0$. Clearly, $g-$semi-commutative rings are $g - NE-$semi-commutative, however, we introduce an example ...

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1. Introduction and Preliminaries

Throughout this article, all rings are associative rings. Let G be a group with identity e and R be a ring with unity 1. Then R is a G-graded if $R = \bigoplus_{g\in G} R_g$ where $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$ and this is denoted by (R, G). We denote $supp(R, G) = \{g \in G : R_g \neq 0\}$. The elements of R_g are called homogeneous of degree g where R_g are additive subgroups of R indexed by the elements $g \in G$. If $x \in R$, then x can be written uniquely as $\sum_{g\in G} x_g$, where x_g is the component of x in R_g. Moreover, we denote $h(R) = \bigcup_{g\in G} R_g$. For more details, one can look in [1], [2], [9], [14] and [15].

Proposition 1.1. *([14]) If R is a G-graded ring, then R_e is a subring of R and $1 \in R_e$.*

The symbols $J(R)$, $P(R)$, $N(R)$ and $E(R)$ stand respectively for the Jacobson radical, the prime radical, the set of all nilpotent elements and the set of all idempotent elements of R. Also, for $a \in R$, the set of left annihilators of a is $Ann(a) = \{r \in R : ra = 0\}$.

In [8], a ring R is said to be semicommutative if whenever $a, b \in R$ such that $ab = 0$, then $aRb = 0$. In this article, we introduce the concepts of $g-$semicommutative rings and $g-N-$semicommutative rings and we introduce several results concerning these two concepts. Let R be a G-graded ring and $g \in supp(R, G)$. Then R is said to be a $g-$semicommutative if whenever $a, b \in R$ with $ab = 0$, then $aR_gb = 0$. Also, R is said to be a $g - N-$semicommutative if for any $a \in R$ and $b \in N(R) \bigcap Ann(a)$, $bR_g \subseteq Ann(a)$.

We introduced an example of a G-graded ring R which is $g - N$-semicommutative for some $g \in supp(R, G)$ but R itself is not semicommutative. Clearly, if R is a $g-$ semicommutative ring, then R is a $g - N-$semicommutative ring, however, we introduced an example showing that the converse is not true in general. It was proved that if $a \in R_g$ with $ab = 1$ for some $b \in h(R)$ and R is a $g^{-1} - N-$ semicommutative ring, then $ba = 1$. Several results and examples are investigated.

In this article, we introduce the concept of $g - NE-$semicommutative rings and we introduce several results concerning $g - NE-$semicommutative rings. Let R be a G-graded ring and $g \in supp(R, G)$. Then R is said to be a $g - NE-$ semicommutative ring if whenever $a \in N(R)$ and $b \in E(R)$ such that $ab = 0$, then $aR_gb = 0$.

Clearly, $g-$semicommutative rings are $g-NE-$semicommutative, however, we introduced an example showing that the converse need not be true. It was proved that if R is a $g-NE-$semicommuative ring and $b \in E(R)$ with $R_gbR_g = R_e$, then $b = 1$ and $g^2 = e$. Also, it has been shown that if R is an $e-NE-$semicommuative ring and $x \in R_g$ for some $g \in supp(R, G)$ such that $xy = 1$ for some $y \in h(R)$, then $yx = 1$. Considerable results and examples have been examined.

In [13], a ring R is said to be nil-semicommutative if whenever $a, b \in N(R)$ such that $ab = 0$, then $aRb = 0$. In this article, we introduce the concept of $g-NN-$semicommutative rings; a G-graded ring R is said to be $g-NN-$semicommutative where $g \in supp(R, G)$ if whenever $a, b \in N(R)$ such that $ab = 0$, then $aR_gb = 0$.

Clearly, if R is $g-$semicommutative, then R is $g-NN-$semicommutative, however, we introduced an example showing that the converse is not true in general. Certain results are introduced.

CN rings have been introduced and studied by Drazin in [10]; a ring R is said to be CN ring if $N(R) \subseteq Z(R)$. In this article, we introduce the concept of semi CN rings. A G-graded ring R is said to be semi CN ring if $N(R) \subseteq Z(R_g)$ for some $g \in supp(R, G)$.

Clearly, every CN ring is semi CN ring, however, we introduced an example showing that the converse is not true in general. It was proved that R is semi CN if and only if there exists $g \in supp(R, G)$ such that for all $b \in N(R)$, there exists $n \geq 2$ such that $b - b^n \in Z(R_g)$. Also, if R is semi CN, then there exists $g \in supp(R, G)$ such that $(ab)^n = a^nb^n$ for all $n \geq 2$, for all $a \in N(R)$ and for all $b \in R_g$.

Let R be a G-graded ring. A left R-module M is said to be G-graded if there exist additive subgroups M_g of M such that $M = \bigoplus_{g \in G} M_g$ where $R_gM_h \subseteq M_{gh}$ for all $g, h \in G$. For $m \in M$, $m = \sum_{g \in G} m_g$ where m_g is the component of m in M_g. Also, $supp(M, G) = \{g \in G : M_g \neq 0\}$. Moreover, M_g is R_e-submodule of M for all $g \in G$. For more details, see [14].

Reduced modules have been introduced by Lee and Zhou in [12] and they have been studied in citeBaser Agayev, [4] and [17]. A left R-module M is said to be reduced if whenever $a \in R$ and $m \in M$ such that $a^2m = 0$, then $aRm = 0$. In this article, we introduce the concept of $g-$reduced modules. A G-graded R-module M is said to be $g-$reduced ($g \in supp(M, G)$) if whenever $a \in R_e$ and $m \in M_g$ such that $am = 0$, then $aR_em = 0$. Respective results have been constructed.

Semicommutative modules have been introduced by Buhphang and Rege in [5]. A left R-module M is said to be semicommutative if whenever $a \in R$ and $m \in M$ such that $am = 0$, then $aRm = 0$. In this article, we introduce the concept of $g-$semicommutative modules. A G-graded R-module M is said to be $g-$semicommutative ($g \in supp(M, G)$) if whenever $a \in R_e$ and $m \in M_g$ such that $am = 0$, then $aR_em = 0$.

Clearly, every semicommutative module is $g-$semicommutative for all $g \in supp(M, G)$, however, we introduced an example showing that the converse is not true in general. Numerous results have been examined.

2. $g - N-$semicommutative Rings

In this section, we introduce the concept of $g - N-$semicommutative rings.

Definition 2.1. *Let R be a G-graded ring and $g \in supp(R, G)$. Then R is said to be a $g - N-$semicommutative ring if for any $a \in R$ and $b \in N(R) \bigcap Ann(a)$, $bR_g \subseteq Ann(a)$.*

The following example introduces a G-graded ring R which is $g - N$-semicommutative for some $g \in supp(R, G)$ but R itself is not semicommutative. So, it is important to study $g - N$-semicommutative rings.

Example 2.2. *Let K be a field. Consider $R = \begin{pmatrix} K & K \\ 0 & K \end{pmatrix}$ and $G = Z_4$. Then R is G-graded by $R_0 = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}$, $R_2 = \begin{pmatrix} 0 & K \\ 0 & 0 \end{pmatrix}$ and $R_1 = R_3 = 0$.*
Clearly, $0 \in supp(R, G)$, we prove that R is an $0 - N-$semicommutative ring. Let $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in R$ and $X = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \in Ann(A)$. Then $0 = XA = \begin{pmatrix} xa & xb + yc \\ 0 & zc \end{pmatrix}$ that is $xa = zc = xb + yc = 0$. If $a \neq 0$ and $c \neq 0$, then $x = y = z = 0$, so $Ann(A) = 0$. If $a \neq 0$ and $c = 0$, then $x = 0$, so $Ann(A) = \begin{pmatrix} 0 & K \\ 0 & K \end{pmatrix}$. On the other hand, $N(R) = \begin{pmatrix} 0 & K \\ 0 & 0 \end{pmatrix}$, so $N(R) \bigcap Ann(A) = \begin{pmatrix} 0 & K \\ 0 & 0 \end{pmatrix}$ that is if $B \in N(R) \bigcap Ann(A)$, then $B = \begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix}$ and then $BR_0 = \begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix}\begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} = \begin{pmatrix} 0 & kk_2 \\ 0 & 0 \end{pmatrix} \subseteq$

$Ann(A)$. If $a = 0$ and $c \neq 0$, then $z = 0$, so $Ann(A) = \begin{pmatrix} K & K \\ 0 & 0 \end{pmatrix}$ and then $N(R) \bigcap Ann(A) = \begin{pmatrix} 0 & K \\ 0 & 0 \end{pmatrix}$ that is similar to the previous case. If $a = c = 0$, then $xb = 0$, so we study two cases: $b \neq 0$ and $b = 0$. If $b \neq 0$, then $x = 0$ which is done case. If $b = 0$, then $Ann(A) = R$ (nothing to prove). Finally, R is an $0 - N-$semicommutative ring. On the other hand, R is not semicommutative ring, to see this, choose $B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and $A = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$, then $BA = 0$ but $BRA \neq 0$ since $X = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in R$ such that $BXA \neq 0$.

Definition 2.3. *Let R be a G-graded ring and $g \in supp(R, G)$. Then R is said to be a $g-$semicommutative ring if whenever $a, b \in R$ with $ab = 0$, then $aR_g b = 0$.*

The next Proposition is clear.

Proposition 2.4. *Let R be a G-graded ring and $g \in supp(R, G)$. If R is a $g-$ semicommutative ring, then R is a $g - N-$semicommutative ring.*

The next example shows that the converse of Proposition 2.4 is not true in general and this gives more importance for studying $g-N$-semicommutative rings.

Example 2.5. *Let K be a field. Consider $R = \begin{pmatrix} K & K \\ 0 & K \end{pmatrix}$ and $G = Z_4$. Then R is G-graded by $R_0 = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}$, $R_2 = \begin{pmatrix} 0 & K \\ 0 & 0 \end{pmatrix}$ and $R_1 = R_3 = 0$. In Example 2.2, we proved that R is an $0 - N-$ semicommutative ring. Choose $K = \Re$. If $B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and $A = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}$, then $BA = 0$. On the other hand, $X = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \in R_0$ with $BXA \neq 0$, i.e., $BR_0A \neq 0$. So, R is not an $0-$semicommutaive ring.*

Lemma 2.6. *Let R be a G-graded ring and $g \in supp(R, G)$. If $a \in R_g$ with $ab = 1$ for some $b \in h(R)$, then $b \in R_{g^{-1}}$.*

Proof. Suppose $b \in R_h$. Then $ab \in R_g R_h \subseteq R_{gh}$. But, $ab = 1 \in R_e$, so $0 \neq ab \in R_{gh} \bigcap R_e$ that is $gh = e$ and then $h = g^{-1}$. Hence, $b \in R_{g^{-1}}$. □

Proposition 2.7. *Let R be a G-graded ring, $g \in supp(R, G)$ and $a \in R_g$ with $ab = 1$ for some $b \in h(R)$. If R is a $g^{-1} - N-$ semicommutative ring, then $ba = 1$.*

Proof. By Lemma 2.6, $b \in R_{g^{-1}}$. Assume $x = ba$. Then $x^2 = x$ and $ax = a$. Suppose $y = a - xa$. Then $yx = y$ and $xy = 0$ and then $y^2 = 0$. Hence, $y \in N(R)$. Also, $y(1 - x) = 0$. So, $y \in N(R) \bigcap Ann(1 - x)$. Since R is a $g^{-1} - N-$semicommutative ring, $yR_{g^{-1}}(1 - x) = 0$. In particular, $yb(1 - x) = 0$. Also, $yb = 1 - x$. So, $(1 - x)^2 = 0$. But, $(1 - x)^2 = (1 - x)(1 - x) = 1 - x - x + x^2 = 1 - x$ and hence $1 - x = 0$ that is $x = 1$, i.e., $ba = 1$. □

$a \in R$ is called a regular element if there exists $b \in R$ such that $a = aba$ (see [11]). Also, $a \in R$ is called a strongly regular element if there exists $b \in R$ such that $a = a^2 b$ (see [17]). We introduce the following:

Definition 2.8. *Let R be a G-graded ring and $g \in supp(R, G)$. Then $r \in R$ is said to be a $g-$regular element if there exists $s \in R_g$ such that $r = rsr$.*

Proposition 2.9. *If R is a $g - N-$semicommutative ring, then every $g-$regular element of R is a strongly regular.*

Proof. Let $r \in R$ be a $g-$regular element. Then there exists $s \in R_g$ such that $r = rsr$. Set $x = sr$. Then $r = rx$ and $x^2 = x$. Set $h = r - xr$. Then $hx = h, xh = 0$ and $h^2 = 0$ and then $h \in N(R) \bigcap Ann(h)$. Since R is a $g - N-$semicommutative ring, $hR_g h = 0$. In particular, $hsh = 0, (rs - xrs)(r - xr) = 0, r = rsr = (rs + 1 - xrs)xr = (rs + 1 - xrs)sr^2 \in Rr^2$. Set $y = rs$. Then $r = yr$ and $y^2 = y$. Set $t = r - ry$. Then $t^2 = 0$ and $ty = 0$ and then $t \in N(R) \bigcap Ann(y)$. Since R is a $g - N-$semicommutative ring, $tR_g y = 0$. In particular, $tsy = 0, (rs - rys)y = 0, rsy = rsrs = rysrs, y = y^2 = rsrs = rysrs = r^2 s^2 rs \in r^2 R$. Hence, $r = yr \in r^2 R$. □

Proposition 2.10. *Let R be a G-graded ring and $g \in supp(R, G)$. If R is a $g - N-$semicommutative ring and every element of R is a $g-$regular, then $N(R) = 0$.*

Proof. Let $a \in N(R)$. Then $a^n = 0$. Then $a \in N(R) \bigcap Ann(a^{n-1})$ and since R is a $g-N-$semicommutative ring, $aR_g a^{n-1} = 0$. But, a is a $g-$regular element, i.e., there exists $b \in R_g$ such that $a = aba$ and then $a^{n-1} = a.a^{n-2} = aba.a^{n-2} = aba^{n-1} \in aR_g a^{n-1} = 0$, i.e., $a^{n-1} = 0$. Again, $a \in N(R) \bigcap Ann(a^{n-2})$ and since R is a $g-N-$semicommutative ring, $aR_g a^{n-2} = 0$ and then $a^{n-2} = a.a^{n-3} = aba.a^{n-3} = aba^{n-2} \in aR_g a^{n-2} = 0$, i.e., $a^{n-2} = 0$. Continue on this process to obtain $a = 0$. $\square$

Example 2.11. *Let K be a field. Consider $R = \begin{pmatrix} K & K \\ 0 & K \end{pmatrix}$ and $G = Z_4$. Then R is G-graded by $R_0 = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}$, $R_2 = \begin{pmatrix} 0 & K \\ 0 & 0 \end{pmatrix}$ and $R_1 = R_3 = 0$. In Example 2.2, we proved that R is an $0-N-$semicommutative ring. However, $N(R) = \begin{pmatrix} 0 & K \\ 0 & 0 \end{pmatrix} \neq 0$. So by Proposition 2.10, R has an element which is not $0-$regular. To see this, consider $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in R$. If A is a $0-$regular, then there exists $B \in R_0$ such that $A = BAB$. Now, $B = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$ for some $x, y \in K$ and since $A = BAB$, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x & x+y \\ 0 & y \end{pmatrix}$ and then $x = y = 1$ and $x + y = 1$ which is a contradiction. Hence, A is not $0-$regular.*

3. $g - NE-$Semicommutative Rings

In this section, we introduce the concept of $g - NE-$semicommutative rings.

Definition 3.1. *Let R be a G-graded ring and $g \in supp(R, G)$. Then R is said to be a $g - NE-$semicommutative ring if whenever $a \in N(R)$ and $b \in E(R)$ such that $ab = 0$, then $aR_g b = 0$.*

Clearly, $g-$semicommutative rings are $g-NE-$semicommutative. However, the following example shows that the converse need not be true.

Example 3.2. *Let K be a field. Consider $R = \begin{pmatrix} K & K \\ 0 & K \end{pmatrix}$ and $G = Z_4$. Then R is G-graded by $R_0 = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}$, $R_2 = \begin{pmatrix} 0 & K \\ 0 & 0 \end{pmatrix}$ and $R_1 = R_3 =$*

0. Let $B \in E(R)$. Then $B = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ for some $a, b, c \in K$ and $B^2 = B$ and then $a^2 = a$, $ab + bc = b$ and $c^2 = c$. If $a = c = 0$, then $b = 0$ and then $B = 0$. If $a = c = 1$, then $b = 0$ and then $B = I$. If $a = 0$ and $c = 1$, then b is free and then $B = \begin{pmatrix} 0 & b \\ 0 & 1 \end{pmatrix}$. Finally, if $a = 1$ and $c = 0$, then b is free and then $B = \begin{pmatrix} 1 & b \\ 0 & 0 \end{pmatrix}$. So, $E(R) = \{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & b \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & b \\ 0 & 1 \end{pmatrix} : b \in K\}$. On the other hand, $N(R) = \begin{pmatrix} 0 & K \\ 0 & 0 \end{pmatrix}$. Now, let $A \in N(R)$ and $B \in E(R)$ such that $AB = 0$. Then $A = \begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix}$ for some $k \in K$. If $B = 0$, then $AR_0B = 0$. If $B = \begin{pmatrix} 0 & b \\ 0 & 1 \end{pmatrix}$, then $0 = AB = \begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix}$ and then $k = 0$, so $A = 0$ it follows that $AR_0B = 0$. If $B = \begin{pmatrix} 1 & b \\ 0 & 0 \end{pmatrix}$, then $AR_0B = \begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix}\begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & 0 \end{pmatrix}$ $= \begin{pmatrix} 0 & kK \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Finally, if $B = I$, then $0 = AB = A$ and then $AR_0B = 0$. So, R is $0-NE-$semicommutative. However, if we choose $K = \Re$, $C = \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix} \in R$ and $D = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \in R$, then $CD = 0$ but $CR_0D \neq 0$ since $X = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in R_0$ such that $AXB \neq 0$. Hence, R is not $0-$semicommutative. Also, since $AR_1 = 0$ for all $A \in N(R)$, $AR_1B = 0$ for all $A \in N(R)$ and $B \in E(R)$. So, R is $1-NE-$semicommutative. However, $CR_1D \neq 0$ since $Y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in R_1$ such that $CYD \neq 0$. Hence, R is not $1-$semicommutative.

The following example shows that choosing K to be a field in the above example is necessary.

Example 3.3. *Consider* $R = \begin{pmatrix} Z_4 & Z_4 \\ 0 & Z_4 \end{pmatrix}$ *and* $G = Z_4$*. Then* R *is* G*-graded by* $R_0 = \begin{pmatrix} Z_4 & 0 \\ 0 & Z_4 \end{pmatrix}$, $R_2 = \begin{pmatrix} 0 & Z_4 \\ 0 & 0 \end{pmatrix}$ *and* $R_1 = R_3 = 0$. *Choose* $A = \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}$ *and* $B = \begin{pmatrix} 0 & 3 \\ 0 & 1 \end{pmatrix}$. *Then* $A^2 = 0$, $B^2 = B$ *and* $AB = 0$*, i.e.,* $A \in N(R)$ *and* $B \in E(R)$ *such that* $AB = 0$ *but* $AR_0B \neq 0$ *since* $X = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in R_0$ *such that* $AXB \neq 0$*. Also,* $AR_1B \neq 0$ *since* $Y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in R_1$ *such that* $AYB \neq 0$*. Hence,* R *is neither* $0 - NE-$*semicommutative nor* $1 - NE-$*semicommutative.*

However, we introduce the following example.

Example 3.4. *Let* K *be a field. Consider the ring* $R = \begin{pmatrix} K & K & K \\ 0 & K & K \\ 0 & 0 & K \end{pmatrix}$ *and the group* $G = Z_2$*. Then* R *is* $G-$*graded by* $R_0 = \begin{pmatrix} K & 0 & K \\ 0 & K & 0 \\ 0 & 0 & K \end{pmatrix}$ *and* $R_1 = \begin{pmatrix} 0 & K & 0 \\ 0 & 0 & K \\ 0 & 0 & 0 \end{pmatrix}$*. Choose* $A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ *and* $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix}$*. Then* $A^2 = 0$, $B^2 = B$ *and* $AB = 0$*, i.e.,* $A \in N(R)$ *and* $B \in E(R)$ *such that* $AB = 0$ *but* $AR_0B \neq 0$ *since* $X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in R_0$ *such that* $AXB \neq 0$*. Also,* $AR_1B \neq 0$ *since* $Y = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \in R_1$ *such that* $AYB \neq 0$*. Hence,* R *is neither* $0 - NE-$*semicommutative nor* $1 - NE-$*semicommutative.*

We begin with the following fundamental Lemma.

Lemma 3.5. *Let* R *be a* $g - NE-$*semicommutative ring and* $b \in E(R)$*. Then* $(1 - b)R_gbR_g(1 - b) = 0$*.*

Proof. For any $x \in R_g$, write $a = (1-b)x - (1-b)x(1-b)$. Then $a \in N(R)$ and $1-b \in E(R)$ such that $a(1-b) = 0$. Since R is a $g-NE-$semicommuative ring, $0 = aR_g(1-b) = (1-b)xbR_g(1-b)$. Thus, $(1-b)R_gbR_g(1-b) = \sum_{x\in R_g}(1-b)xbR_g(1-b) = 0$. □

Proposition 3.6. *Let R be a $g-NE-$semicommuative ring and $b \in E(R)$. If $R_gbR_g = R_e$, then $b = 1$ and $g^2 = e$.*

Proof. By Lemma 3.5, $(1-b)R_gbR_g(1-b) = 0$ and so $((1-b)R_gbR_g)^2 = 0$. and then
$1-b = (1-b)^2 = ((1-b).1)^2 \in ((1-b)R_e)^2 = ((1-b)R_gbR_g)^2 = 0$, i.e., $b = 1$. So, $R_e = R_gR_g \subseteq R_{g^2}$ and then $0 \neq R_e = R_e \bigcap R_{g^2}$. This implies that $g^2 = e$. □

In fact, R_e contains all homogeneous idempotent elements.

Proposition 3.7. *Let R be a $G-$ graded ring and b be a homogeneous idempotent element in R. Then $b \in R_e$*

Proof. If $b = 0$, then $b \in R_e$. Suppose $b \neq 0$. Since b is a homogeneous idempotent element, $b \in R_g$ for some $g \in G$ and $b^2 = b$ and then $b = b^2 \in R_gR_g \subseteq R_{g^2}$. So, $0 \neq b \in R_g \bigcap R_{g^2}$ and hence $g = g^2$, i.e., $g = e$. Thus, $b \in R_e$. □

Lemma 3.8. *Let R be an $e-NE-$semicommuative ring and $b \in E(R)$. Then $(1-b)R_eb \subseteq J(R_e)$.*

Proof. By Lemma 3.5, $(1-b)R_ebR_e(1-b) = 0$ and so $((1-b)R_ebR_e)^2 = 0$. This implies that $(1-b)R_eb \subseteq J(R_e)$. □

Lemma 3.9. *[[16]]If M is a graded maximal left ideal of a graded ring R, then M_e is a maximal ideal of R_e.*

Proposition 3.10. *Let R be an $e-NE-$semicommuative ring and b be a homogeneous idempotent element of R. If M is a graded maximal left ideal of R such that $b \notin M_e$, then $(1-b)R_e \subseteq M_e$.*

Proof. By Lemma 3.9, M_e is a maximal ideal of R_e and then $R_eb + M_e = R_e$. By Lemma 3.8, $(1-b)R_eb \subseteq J(R_e) \subseteq M_e$. On the other hand, by Proposition 3.7, $b \in R_e$ and then $(1-b)M_e \subseteq R_eM_e \subseteq M_e$. Hence, $(1-b)R_e = (1-b)R_eb + (1-b)M_e \subseteq M_e$. □

Corollary 3.11. *Let R be an $e-NE-$semicommuative ring, b be a homogeneous idempotent element of R, M be a graded maximal left ideal of R and $a \in R_e$. If $1-ab \in M_e$, then $1-ba \in M_e$.*

Proof. If $b \in M_e$, then $ab \in M_e$ and then $1 = (1-ab)+ab \in M_e$ a contradiction (see Lemma 3.9), so $b \notin M_e$. By Proposition 3.10,
$(1-b)R_e \subseteq M_e$. Since $1-ab = (1-a)+(a-ab)$, $1-a \in M_e$ and $1-ba = (1-a)+((1-b)a)$ implies $1-ba \in M_e$. □

A ring R is said to be directly finite if $xy = 1$ implies $yx = 1$ for all $x, y \in R$. We introduce the following:

Proposition 3.12. *Let R be an $e-NE-$semicommuative ring. If $x \in R_g$ for some $g \in supp(R, G)$ such that $xy = 1$ for some $y \in h(R)$, then $yx = 1$.*

Proof. By Lemma 2.6, $y \in R_{g^{-1}}$. Set $b = yx$, then $b \in R_e$ is a homogeneous idempotent element, $xb = x$ and $by = y$. Since R is an $e-NE-$semicommuative ring, $(1-b)R_eb \subseteq J(R_e)$ by Lemma 3.8. So, we have $(1-b)x = (1-b)xb \in J(R_e)$. Therefore, $1-b = (1-b)xy$. This gives $1 = b = yx$. □

An element $b \in R$ is said to be potent if there exists an integer $n \geq 2$ such that $b^n = b$, see [7]. Clearly, idempotent is potent while $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in M_2(Z)$ is a potent element which is not idempotent. The set of all potent elements of R is denoted by $PE(R)$.

Proposition 3.13. *Let R be a $g-NE-$semicommutative ring. If $a \in N(R)$ and $b \in PE(R)$ such that $ab = 0$, then $aR_gb = 0$.*

Proof. Since $b \in PE(R)$, there exists $n \geq 2$ such that $b^n = b$ and then $b^{n-1} \in E(R)$ such that $ab^{n-1} = 0$ and since R is $g-NE-$semicommutative, $aR_gb^{n-1} = 0$. Thus, $aR_gb = aR_gb^n = aR_gb^{n-1}b = 0$. □

Remark 3.14. *Proposition 3.13 gives us a possibility to introduce another concept, $g-NPE-$semicommutative rings, that is, if R is a G-graded ring and $g \in supp(R, G)$, then R is said to be a $g-NPE-$semicommutative if whenever $a \in N(R)$ and $b \in PE(R)$ such that $ab = 0$, then $aR_gb = 0$. I think it is interesting to study this class, we leave it for the readers or for another work. However, in the next section, we introduce another class of semicommutativity of graded rings, it is called $g-NN-$semicommutative rings.*

4. $g-NN-$Semicommutative and Semi CN rings

In this section, we introduce and study the concepts of $g-NN-$semicommutative rings and semi CN rings.

Definition 4.1. *Let R be a G-graded ring and $g \in supp(R,G)$. Then R is said to be $g-NN-$semicommutative if for every $a, b \in N(R)$ such that $ab = 0$, $aR_gb = 0$.*

Clearly, if R is $g-$semicommutative, then R is $g-NN-$semicommutative. However, the next example shows that the converse is not true in general.

Example 4.2. *Let K be a field. Consider $R = \begin{pmatrix} K & K \\ 0 & K \end{pmatrix}$ and $G = Z_4$. Then R is G-graded by $R_0 = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}$, $R_2 = \begin{pmatrix} 0 & K \\ 0 & 0 \end{pmatrix}$ and $R_1 = R_3 = 0$. Clearly, $N(R) = \begin{pmatrix} 0 & K \\ 0 & 0 \end{pmatrix} = R_2$ and $AB = 0$ for all $A, B \in N(R)$. Let $A = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ for some $a, b \in K$. Then $AR_0B =$ $\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}\begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & aK \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. So, R is $0-NN-$semicommutative. Also,*
$AR_2B = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & K \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.
So, R is $2-NN-$semicommutative. Choose $K = \mathbf{R}$, then $A = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \in R$ such that $AB = 0$. On the other hand,
$C = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \in R_0$ such that $ACB \neq 0$ which implies that R is not $0-$semicommutative.

Proposition 4.3. *For a ring R, consider the ring $\Gamma = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a, b \in R \right\}$. If R is nil-semicommutative and $G = \mathbf{Z}_4$, then Γ is $g-NN-$ semicommutative for some $g \in supp(R,G)$.*

Proof. Consider the following graduation of Γ by G: $\Gamma_0 = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$, $\Gamma_2 = \begin{pmatrix} 0 & R \\ 0 & 0 \end{pmatrix}$ and $R_1 = R_3 = 0$. Clearly, $N(\Gamma) = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a \in N(R), b \in R \right\}$. Let $A, B \in N(\Gamma)$ such that $AB = 0$. Then $A = \begin{pmatrix} x & c \\ 0 & x \end{pmatrix}$ and $B = \begin{pmatrix} y & b \\ 0 & y \end{pmatrix}$ for some $x, y \in N(R)$ and for some $c, b \in R$. Now, $0 = AB = \begin{pmatrix} xy & xb + cy \\ 0 & xy \end{pmatrix}$ which implies that $xy = 0$. Since R is nil-semicommutative, $xRy = 0$ and then $A\Gamma_2 B = \begin{pmatrix} 0 & xRy \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Hence, Γ is $2 - NN-$semicommutative. □

Proposition 4.4. *For a ring R, consider the ring $\Gamma = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a, b \in R \right\}$. If $RN(R) = N(R)R$ $G = \mathbf{Z}_4$, Γ is $g - NN-$semicommutative for all $g \in supp(R, G)$.*

Proof. Consider the graduation of Γ bt G: $\Gamma_0 = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$, $\Gamma_2 = \begin{pmatrix} 0 & R \\ 0 & 0 \end{pmatrix}$ and $R_1 = R_3 = 0$. Clearly, $N(\Gamma) = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a \in N(R), b \in R \right\}$. Let $A, B \in N(\Gamma)$ such that $AB = 0$. Then $A = \begin{pmatrix} x & c \\ 0 & x \end{pmatrix}$ and $B = \begin{pmatrix} y & b \\ 0 & y \end{pmatrix}$ for some $x, y \in N(R)$ and for some $c, b \in R$. Now, $0 = AB = \begin{pmatrix} xy & xb + cy \\ 0 & xy \end{pmatrix}$ which implies that $xy = 0$ and $xb + cy = 0$. Since $N(R)R = RN(R)$, $xRy = (xR)y = (Rx)y = R(xy) = R.0 = 0$ and then $A\Gamma_2 B = \begin{pmatrix} 0 & xRy \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Hence, Γ is $2-NN-$semicommutative. Let $X \in \Gamma_0$. Then $X = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$ for some $\alpha \in R$ and then $AXB = \begin{pmatrix} x\alpha y & x\alpha b + c\alpha y \\ 0 & x\alpha y \end{pmatrix}$. Since $N(R)R = RN(R)$, $x\alpha y = (x\alpha)y = (\alpha x)y =$

$\alpha(xy) = \alpha.0 = 0$ and $x\alpha b + c\alpha y = (x\alpha)b + (c\alpha)y = (\alpha x)b + (\alpha c)y = \alpha(xb + cy) = \alpha.0 = 0$. So, $AXB = 0$ which implies that $A\Gamma_0 B = 0$. Hence, Γ is $0-NN-$semicommutative. Therefore, Γ is $g-NN-$semicommutative for all $g \in supp(R, G)$. □

For the rest of this section, we introduce and study the concept of semi CN rings.

Definition 4.5. *Let R be a G-graded ring. Then R is called semi CN ring if $N(R) \subseteq Z(R_g)$ for some $g \in supp(R, G)$.*

Clearly, every CN ring is semi CN ring. However, the next example shows that the converse is not true in general.

Example 4.6. *Let K be a field. Consider $R = \begin{pmatrix} K & K \\ 0 & K \end{pmatrix}$ and $G = Z_4$. Then R is G-graded by $R_0 = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}$, $R_2 = \begin{pmatrix} 0 & K \\ 0 & 0 \end{pmatrix}$ and $R_1 = R_3 = 0$. Then $N(R) = \begin{pmatrix} 0 & K \\ 0 & 0 \end{pmatrix} = R_2$ and then $N(R)R_2 = (N(R))^2 = R_2 N(R)$ and hence R is semi CN ring. On the other hand, choose, $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in N(R)$ and choose $B = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \in R$, then $AB \neq BA$ and hence $A \notin Z(R)$, so $N(R)Z(R)$. Thus, R is not CN ring.*

Proposition 4.7. *Let R be a G-graded ring. Then R is semi CN if and only if there exists $g \in supp(R, G)$ such that for all $b \in N(R)$, there exists $n \geq 2$ such that $b - b^n \in Z(R_g)$.*

Proof. Suppose that R is semi CN ring. Let $b \in N(R)$. Then there exists $n \geq 2$ such that $b^n = 0$. Since R is semi CN, $bR_g = R_g b$ for some $g \in supp(R, G)$ and then $(b - b^n)R_g = bR_g = R_g b = R_g(b - b^n)$. Hence, $b - b^n \in Z(R_g)$. Conversely, let $b \in N(R)$. Then there exists $m \geq 2$ such that $a^m = 0$. By assumption, there exists $g \in supp(R, G)$ and $n_1 \geq 2$ such that $b - b^{n_1} \in Z(R_g)$. Since $b^{n_1} \in N(R)$, by assumption, there exists $n_2 \geq 2$ such that $b^{n_1} - b^{n_1 n_2} \in Z(R_g)$. Continue on this process, there exists $n_s \geq 2$ such that $(b^{n_1 \dots n_{s-1}} - b^{n_1 \dots n_s}) \in Z(R_g)$ with $n_1 \dots n_s \geq m$. Hence, $b^{n_1 \dots n_s} = 0$ and then $b = b - b^{n_1 \dots n_s} = (b - b^{n_1}) + (b^{n_1} - b^{n_1 n_2}) + \dots + (b^{n_1 \dots n_{s-1}} - b^{n_1 \dots n_s}) \in Z(R_g)$. Hence, R is semi CN ring. □

Proposition 4.8. *Let R be a G-graded ring. If R is semi CN, then there exists $g \in supp(R,G)$ such that $(ab)^n = a^n b^n$ for all $n \geq 2$, for all $a \in N(R)$ and for all $b \in R_g$.*

Proof. Let $a \in N(R)$. Since R is semi CN ring, there exists $g \in supp(R,G)$ such $ab = ba$ for all $b \in R_g$. Suppose that $n \geq 2$. Then

$$(ab)^n = ab.ab....ab = a(ba)(ba)....(ba)b = a(ab)(ab)....(ab)b = \\ a^2(ba)....(ba)b^2 = a^2(ab)....(ab)b^2 = a^3(ba)....(ba)b^3 = ... = a^n b^n.$$

□

5. $g-$semicommutative and $g-$reduced Modules

In this section, we introduce and study the concepts of $g-$semicommutative and $g-$reduced modules.

Definition 5.1. *Let M be a G-graded R-module and $g \in supp(M,G)$. Then M is said to be $g-$semicommutative if whenever $a \in R_e$ and $m \in M_g$ such that $am = 0$, then $aR_e m = 0$.*

Clearly, every semicommutative module is $g-$semicommutative for all $g \in supp(M,G)$. However, the next example shows that the converse is not true in general.

Example 5.2. *Let T be a semicommutative ring. Consider the ring $R = \begin{pmatrix} T & T \\ 0 & T \end{pmatrix}$ and the R-module $M = R$. Consider $G = Z_4$. Then R is G-graded by $R_0 = \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}$, $R_2 = \begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix}$ and $R_1 = R_3 = 0$. So, M is G-graded by $M_g = R_g$ for all $g \in G$. Let $A \in R_0$ and $X \in M_2$ such that $AX = 0$. Then $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ and $X = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$ for some $a, b, x \in T$. Now, $0 = AX = \begin{pmatrix} 0 & ax \\ 0 & 0 \end{pmatrix}$, i.e., $ax = 0$. Since T is semicommutative, $aTx = 0$ and then $AR_0X = \begin{pmatrix} 0 & aTx \\ 0 & 0 \end{pmatrix} = 0$. Hence, M is $2-$semicommutative. Let $Y \in M_0$ such that $AY = 0$. Then $Y = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$ for some $x, y \in T$. Now,*

$0 = AY = \begin{pmatrix} ax & 0 \\ 0 & by \end{pmatrix}$, *i.e.,* $ax = by = 0$. *Since* T *is semicommutative,* $aTx = bTy = 0$ *and then* $AR_0Y = \begin{pmatrix} aTx & 0 \\ 0 & bTy \end{pmatrix} = 0$. *Hence,* M *is* $0-$*semicommutative. So,* M *is* $g-$*semicommutative for all* $g \in supp(M, G)$. *On the other hand,* $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \in R$ *and* $N = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix} \in M$ *such that* $AN = 0$ *while* $ARN \neq 0$ *since* $B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \in R$ *such that* $ABN \neq 0$. *Thus,* M *is not semicommutative.*

Definition 5.3. *Let* M *be a* G*-graded* R*-module and* $g \in supp(M, G)$. *Then* M *is said to be* g*-reduced if whenever* $a \in R_e$ *and* $m \in M_g$ *such that* $a^2m = 0$, *then* $aR_em = 0$.

In [6], an R-module M is said to be rigid if whenever $a \in R$ and $m \in M$ such that $a^2m = 0$, then $am = 0$.

Proposition 5.4. *Let* M *be a* G*-graded* R*-module and* $g \in supp(R, G)$. *If* M *is rigid, then* M *is* $g-$*reduced if and only if* M *is* $g-$*semicommutative.*

Proof. Suppose that M is $g-$reduced. Let $a \in R_e$ and $m \in M_g$ such that $am = 0$. Then $a^2m = a(am) = 0$ and then since M is $g-$reduced, $aR_em = 0$. Hence, M is $g-$semicommutative. Conversely, let $a \in R_e$ and $m \in M_g$ such that $a^2m = 0$. Since M is rigid, $am = 0$ and then since M is $g-$semicommutative, $aR_em = 0$. Hence, M is $g-$reduced. □

Proposition 5.5. *Let* M *be a* G*-graded* R*-module and* $g \in supp(R, G)$. *If* M *is rigid, then* M *is* $g-$*reduced if and only if whenever* $a \in R_e$ *and* $m \in M_g$ *such that* $am = 0$, *we have* $aM_g \bigcap R_em = 0$.

Proof. Suppose that M is $g-$reduced. Let $a \in R_e$ and $m \in M_g$ such that $am = 0$. Then $a^2m = a(am) = 0$ and since M is $g-$reduced, then $aR_em = 0$. Let $x \in aM_g \bigcap R_em$. Then $x = an = rm$ for some $n \in M_g$ and $r \in R_e$ and then $ax = arm \in aR_em = 0$, i.e., $ax = 0$ which implies that $a^2n = 0$. Again, since M is $g-$reduced, $aR_en = 0$ and then $an = a.1.n \in aR_en = 0$, i.e., $an = 0$ and hence $x = 0$. Therefore, $aM_g \bigcap R_em = 0$. Conversely, let $a \in R_e$ and $m \in M_g$ such that $a^2m = 0$. Since M is rigid, and $am = 0$ then by assumption, $aM_g \bigcap R_em = 0$. Now, $aR_em \subseteq R_em$ and $aR_em \subseteq aR_eM_g \subseteq aM_g$. So, $aR_em \subseteq aM_g \bigcap R_em = 0$, i.e., $aR_em = 0$ and hence M is $g-$reduced. □

Conclusion

In this article, we introduced the concepts of $g-$semi-commutative rings and $g-N-$semi-commutative rings and we examined several results concerning these two concepts. We introduced an example of a G-graded ring R which is $g-N$-semi-commutative for some $g \in supp(R, G)$ but R itself is not semi-commutative. If R is a $g-$ semi-commutative ring, then R is a $g-N-$semi-commutative ring, however, we introduced an example showing that the converse is not true in general. Several results and examples are investigated. Also, we introduced the concept of $g-NE-$semi-commutative rings and we examined several results concerning $g-NE-$semi-commutative rings. $g-$semi-commutative rings are $g-NE-$semi-commutative, however, we introduced an example showing that the converse need not be true. Moreover, we introduced the concept of $g-NN-$semi-commutative rings. If R is $g-$semi-commutative, then R is $g-NN-$semi-commutative, however, we introduced an example showing that the converse is not true in general. Certain results have been presented. As a proposal for a future work, we will try to study several concepts in non-commutative rings to see how do they behave in the graded sense throughout the components of a graded non-commutative ring.

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