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An overview of cubic intuitionistic $\beta-$subalgebras

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Abstract

The conditions of $\beta-$algebra is enforced into the structure of cubic intuitionistic fuzzy settings. Furthermore, the concept of cubic intuitionistic $\beta-$ subalgebra is expressed and its pertinent properties were explored. Also, discussed about the level set of cubic intuitionistic $\beta-$subalgebras and furnished some fascinating results on the cartesian product of cubic intuitionistic $\beta-$subalgebra. Moreover, the notion of $(\overline{T}, \overline{S}, S, T)$-normed cubic intuitionistic $\beta-$subalgebras have been introduced and relevant results are studied.

Keywords: *Cubic set, Cubic $\beta-$algebra, Cubic $\beta-$subalgebra, Cubic intuitionistic set, Cubic intuitionistic $\beta-$subalgebra.*

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1. Introduction

In 1986, Atanassov[3] presented the notion of intuitionistic fuzzy sets whose elements have degrees of membership and non-membership as an extension of Zadeh's[19] fuzzy sets. The study of fuzzy subgroups with interval valued membership functions has been introduced by Biswas et al.[4] in which the necessary and sufficient condition for an interval valued fuzzy subset to be an interval valued fuzzy subgroup was provided. The thought of $\beta-$algebra was explored by Neggers et al.[15], where two operations were coupled. Aub Ayub Ansari et al.[1] established the concept of fuzzy $\beta-$subalgebras of $\beta-$algebra and discussed some of its analogous outcomes. The notion of interval valued fuzzy $\beta-$subalgebras were developed by Hemavathi et al.[7],[8] and also they have extended the idea of interval valued intuitionistic fuzzy $\beta-$subalgebras with fascinating results. Dutta et al.[6] studied the class of p-summable sequence of interval numbers. The concept of lacunary I-convergent sequences of fuzzy real numbers was introduced by Tirpathy et al. [17, 18]. Further more, some of the algebraic properties such as linearity, symmetric and convergence free have been established. Also the class of fuzzy number sequences bv_p^F has been studied.

The thought of cubic intuitionistic subalgebras and closed cubic intuitionistic ideals of B algebras has been introduced by Tapan Senapati et al[16]. Jun et al.[9] depicted cubic sets, and then this notion is enforced to various algebraic structures. The idea of Cubic subalgebras and ideals have applied into the framework of BCK/BCI algebras by Jun et al.[10],[11]. Besides, they have presented a novel extension of cubic sets and its applications in BCK/BCI algebras and provided various results based on their perception. The notion of Cubic $KU-$subalgebras was provided by Akram et al.[2]. Naveed Yaqoob et al.[14] proposed the thought of Interval valued Intuitionstic $(\overline{S}, \overline{T})-$Fuzzy ideals of Ternary Semigroups.

Young Bae Jun et al.[12] applied Cubic interval valued intuitionistic fuzzy sets into $BCK/BCI-$ algebras. The author discussed the relation between cubic interval valued intuitionistic fuzzy $\beta-$subalgebra and cubic intuitionistic fuzzy $\beta-$ideal and discussed the characterizations between cubic interval valued intuitionistic fuzzy $\beta-$subalgebra and cubic intuitionistic fuzzy $\beta-$ideal. Muralikrishna et al.[13] described Some aspects on cubic fuzzy β-subalgebra of β-algebra. Recently, the concept of binormed intuitionistic fuzzy β-ideals of β-algebras initiated by Borumand Saeid et al.[5] With all these inspiration, this paper provides the study of cubic intuitionistic $\beta-$subalgebras of $\beta-$subalgebras and presents some compelling results. The present work is organized into seven sections: **Section 1** shows the introduction, **section 2** gives some basic definitions and properties of $\beta-$algebra, cubic set, cubic intuitionistic set and so on. **Section 3** describes the concept and operations of cubic intuitionistic $\beta-$subalgebra and their properties. **Section 4**, illustrates the cartesion product on cubic intuitionistic $\beta-$subalgebra. **Section 5** introduces the notion of level set of cubic intuitionistic β-subalgebra and **Section 6** provides the characteristics of $(\overline{T}, \overline{S}, S, T)$-normed cubic intuitionistic $\beta-$subalgebra. **Section 7** presents the conclusion of the work.

2. Preliminaries

This section provides the necessary definitions required for the work.

Definition 2.1. *[4] An interval valued fuzzy set A defined on X is given by* A=$\{(x, [\zeta_A^L(x), \zeta_A^U(x)])\}$ $\forall\ x \in X$ *(briefly denoted by* $A = [\zeta_A^L, \zeta_A^U]$*), where* ζ_A^L *and* ζ_A^U *are two fuzzy sets in X such that* $\zeta_A^L(x) \leq \sigma_A^U(x)$ $\forall\ x \in X$. *Let* $\overline{\zeta}_A(x) = [\zeta_A^L(x), \zeta_A^U(x)]$ $\forall\ x \in X$ *and let* $D[0,1]$ *denotes the family of all closed sub intervals of* $[0,1]$. *If* $\zeta_A^L(x) = \zeta_A^U(x) = c$, *say, where* $0 \leq c \leq 1$, *then* $\overline{\zeta}_A(x) = \overline{c} = [c, c]$ *also for the sake of convenience, to belong to* $D[0,1]$. *Thus* $\overline{\zeta}_A(x) \in D[0,1]$ $\forall\ x \in X$, *and therefore the i_v_ fuzzy set A is given by* $A = \{(x, \overline{\zeta}_A(x))\}$ $\forall\ x \in X$, *where* $\overline{\zeta}_A : X \to D[0,1]$.
Now let us define what is known as $refined$ *mimimum(rmin) of two elements in* $D[0,1]$. *Let us define the symbols* " $\geq$ " , " $\leq$ ", *and* " $=$ " *in case of two elements in* $D[0,1]$. *Consider two elements* $D_1 := [a_1, b_1]$ *and* $D_2 := [a_2, b_2] \in D[0,1]$. *Then* $rmin(D_1, D_2) = [min\{a_1, a_2\}, min\{b_1, b_2\}]$;
$D_1 \geq D_2$ *if and only if* $a_1 \geq a_2,\ b_1 \geq b_2$;
Similarly , $D_1 \leq D_2$ *and* $D_1 = D_2$.

Definition 2.2. *[3] An Intuitionistic fuzzy set (IFS) in a nonempty set X is defined by* $A = \{\langle x, \zeta_A(x), \eta_A(x)\rangle / x \in X\}$ *where* $\zeta_A : X \to [0,1]$ *is a membership function of A and* $\eta_A : X \to [0,1]$ *is a non-membership function of A satisfying* $0 \leq \zeta_A(x) + \eta_A(x) \leq 1$ $\forall\ x \in X$.

Definition 2.3. *[8] An Interval valued intuitionisic fuzzy set A over X is an object having the form* $A = \{\langle x, \overline{\zeta}_A(x), \overline{\eta}_A(x)\rangle / x \in X\}$ *where* $\overline{\zeta}_A : X \to D[0,1]$ *and* $\overline{\eta}_A : X \to D[0,1]$, *where* $D[0,1]$ *is the set of all sub-intervals of* $[0,1]$. *The intervals* $\overline{\zeta}_A(x)$ *and* $\overline{\eta}_A(x)$ *denote the intervals of the grade of* *membership and grade of non-membership of the element x to the set A, where* $\overline{\zeta}_A(x) = [\zeta_A^L(x), \zeta_A^U(x)]$ *and* $\overline{\eta}_A(x) = [\eta_A^L(x), \eta_A^U(x)]$ $\forall\ x \in\ X$*, with the condition* $0 \le \zeta_A^L(x) + \eta_A^L(x) \le 1$ *and* $0 \le \zeta_A^U(x) + \eta_A^U(x) \le 1$*. Also note that* $\overline{\overline{\zeta}}_A(x) = [1 - \zeta_A^U(x), 1 - \zeta_A^L(x)]$ *and* $\overline{\overline{\eta}}_A(x) = [1 - \eta_A^U(x), 1 - \eta_A^L(x)]$*, where* $\overline{A} = \{\langle x, \overline{\overline{\zeta}}_A(x), \overline{\overline{\eta}}_A(x)\rangle / x \in X\}$ *represents the complement of A.*

Definition 2.4. *[8] Let* $A = \{\langle x, \overline{\zeta}_A(x), \overline{\eta}_A(x)\rangle :\ x \in\ X\}$ *be an interval valued intuitionisic fuzzy set in* X*and* f *be a mapping from a set* X *into a set* Y*, then the image of* A *under* f*,* $f(A)$ *is defined as*

$f(A) = \{\langle x, f_{rsup}(\overline{\zeta}_A), f_{rinf}(\overline{\eta}_A)\rangle :\ x \in\ Y\}$*, where*

$$f_{rsup}(\overline{\zeta}_A)(y) = \begin{cases} rsup_{x \in\ f^{-1}(y)}\ \overline{\zeta}_A(x), & \text{if } f^{-1}(y) \neq \emptyset \\ \overline{0}, & \text{otherwise} \end{cases}$$

$$f_{rinf}(\overline{\eta}_A)(y) = \begin{cases} rinf_{x \in\ f^{-1}(y)}\ \overline{\eta}_A(x), & \text{if } f^{-1}(y) \neq \emptyset \\ \overline{1}, & \text{otherwise} \end{cases}$$

Definition 2.5. *[5] A mapping* $T : [0,1] \times [0,1] \to [0,1]$ *is said to be a* $T-$*norm(Triangular norm) if it satisfies the following conditions,*

1. $T(x, 1) = x$ *(boundary condition)*
2. $T(x, y) = T(y, x)$*(commutativity)*
3. $T(T(x, y), z) = T(x, T(y, z))$*(associativity)*
4. $T(x, y) \le T(x, z)\ \ if\ \ y \le z\ \ \forall x, y, z \in [0,1]$*(monotonicity)*

The minimum $T_M(x; y) = min(x; y)$*, the product* $T_P(x; y) = x.y$ *and the Lukasiewicz* $T-$*norm* $T_L(x; y) = max(x + y - 1; 0)\ \forall\ x, y \in [0,1]$ *are some of the* $T-$*norms.*

Definition 2.6. *[14] An interval valued triangular norm denoted by* $\overline{T}-$*norm is a function* $\overline{T} : D[0,1] \times D[0,1] \to D[0,1]$ *if it satisfies the following conditions,*

1. $\overline{T}(\overline{x}, \overline{1}) = \overline{x}$ *(boundary condition)*
2. $\overline{T}(\overline{x}, \overline{y}) = \overline{T}(\overline{y}, \overline{x})$*(commutativity)*
3. $\overline{T}(\overline{T}(\overline{x}, \overline{y}), \overline{z}) = \overline{T}(\overline{x}, \overline{T}(\overline{y}, \overline{z}))$*(associativity)*

4. $\overline{T}(\overline{x},\overline{y}) \leq \overline{T}(\overline{x},\overline{z})$ *if* $\overline{y} \leq \overline{z}$ *(monotonicity)* $\forall\ \overline{x},\overline{y},\overline{z} \in D[0,1]$

The following are some $\overline{T}-$*norms used in general,*

1. *Standard* $\overline{T}-$*norm* $(\overline{T}_M) : \overline{T}(\overline{x},\overline{y}) = rmin(\overline{x},\overline{y})$
2. *Bounded difference* $\overline{T}-$*norm* $(\overline{T}_L) : \overline{T}(\overline{x},\overline{y}) = rmax(\overline{0},\overline{x}+\overline{y}-\overline{1})$
3. *Algebraic product* $\overline{T}-$*norm* $(\overline{T}_P) : \overline{T}(\overline{x},\overline{y}) = \overline{x}\ \overline{y}$
4. *Drastic intersection:*

$$\overline{T}_D : \overline{T}(\overline{x},\overline{y}) = \begin{cases} \overline{x} & when\ \overline{y} = \overline{1} \\ \overline{y} & when\ \overline{x} = \overline{1} \\ \overline{0} & otherwise \end{cases}$$

The minimum $\overline{T}_M(\overline{x};\overline{y}) = rmin(\overline{x};\overline{y})$, *the product* $\overline{T}_P(\overline{x};\overline{y}) = \overline{x}.\overline{y}$ *and the Lukasiewicz* $\overline{T}-$*norm* $\overline{T}_L(\overline{x};\overline{y}) = rmax(\overline{x}+\overline{y}-\overline{1};\overline{0})\ \forall\ \overline{x},\overline{y} \in D[0,1]$ *are some of the* $\overline{T}-$*norms.*

Definition 2.7. *[5] The function* $S : [0,1] \times [0,1] \to [0,1]$ *is called a* $T-$*conorm(Triangular Conorm), if it satisfies the following conditions,*

(i) $S(x,0) = x$
(ii) $S(x,y) = S(y,x)$
(iii) $S(S(x,y),z) = S(x,S(y,z))$
(iv) $S(x,y) \leq S(x,z)$ *if* $y \leq z\ \ \forall x,y,z \in [0,1]$

The maximum $S_M(x;y) = max(x;y)$, *the probabilistic sum* $S_P(x;y) = x + y - x.y$ *and the Lukasiewicz* $T-$*conorm* $S_L(x;y) = min(x+y,1)\forall\ x,y \in [0,1]$ *are some of the* $T-$*conorms.*

Definition 2.8. *[14] An interval valued triangular conorm denoted by* $\overline{T}-$*conorm is a function* $\overline{S} : D[0,1] \times D[0,1] \to D[0,1]$ *if it satisfies the following conditions,*

1. $\overline{S}(\overline{x},\overline{0}) = \overline{x}$ *(boundary condition)*
2. $\overline{S}(\overline{x},\overline{y}) = \overline{S}(\overline{y},\overline{x})$*(commutativity)*
3. $\overline{S}(\overline{S}(\overline{x},\overline{y}),\overline{z}) = \overline{S}(\overline{x},\overline{S}(\overline{y},\overline{z}))$*(associativity)*

4. $\overline{S}(\overline{x},\overline{y}) \leq \overline{S}(\overline{x},\overline{z})$ *if* $\overline{y} \leq \overline{z}$ *(monotonicity)* $\forall\ \overline{x},\overline{y},\overline{z} \in D[0,1]$

The following are some $\overline{T}-$*conorms used in general,*

1. *Standard* $\overline{T}-$*conorm* $(\overline{S}_M) : \overline{S}(\overline{x},\overline{y}) = rmax(\overline{x},\overline{y})$
2. *Bounded difference* $\overline{T}-$*conorm* $(\overline{S}_L) : \overline{S}(\overline{x},\overline{y}) = rmin(\overline{1},\overline{x}+\overline{y}-\overline{0})$
3. *Algebraic product* $\overline{T}-$*conorm* $(\overline{S}_P) : \overline{S}(\overline{x},\overline{y}) = \overline{x}\ \overline{y}$
4. *Drastic intersection:*

$$\overline{S}_D : \overline{S}(\overline{x},\overline{y}) = \begin{cases} \overline{x} & when\ \overline{y} = \overline{0} \\ \overline{y} & when\ \overline{x} = \overline{0} \\ \overline{1} & otherwise \end{cases}$$

The maximum $\overline{S}_M(\overline{x};\overline{y}) = rmax(\overline{x},\overline{y})$, *the product* $\overline{S}_P(\overline{x};\overline{y}) = \overline{x}.\overline{y}$ *and the Lukasiewicz* $\overline{T}-$*conorm* $\overline{S}_L(\overline{x};\overline{y}) = rmin(\overline{x}+\overline{y};\overline{1})\ \forall\ \overline{x},\overline{y} \in D[0,1]$ *are some of the* $\overline{T}-$*conorms.*

Definition 2.9. *[8] Let* A *be an Intuitionistic fuzzy subset of* X, *and* $s,t \in [0,1]$. *Then* $A_{s,t} = \{x, \zeta_A(x) \geq s, \eta_A(x) \leq t / x \in X\}$ *where* $0 \leq \zeta_A(x) + \eta_A(x) \leq 1$ *is called an intuitionistic level set of* X.

Definition 2.10. *[8] Let* A *be an interval valued intuitionistic (i_v_i_) fuzzy subset of* X,*and* $(\overline{s},\overline{t}) \in D[0,1]$. *Then* $A_{\overline{s},\overline{t}} = \{x, \overline{\zeta}(x) \geq \overline{s}, \overline{\eta}(x) \leq \overline{t} : x \in X\}$ *where* $\overline{0} \leq \overline{\zeta}_A(x) + \overline{\eta}_A(x) \leq \overline{1}$ *is called an interval valued intuitionistic level set of* X. *Since* $\overline{0} = [0,0]$ & $\overline{1} = [1,1]$.

Definition 2.11. *[15],[7] A* $\beta-$ *algebra is a non-empty set* X *with a constant* 0 *and two binary operations* $+$ *and* $-$ *satisfying the following axioms:*
(i) $x - 0 = x$
$(ii)(0-x)+x = 0$
$(iii)(x-y)-z = x-(z+y)\ \ \forall\ x,y,z \in X$.

Example 2.12. The following Cayley table shows $(X = \{0,1,2,3\},+,-,0)$ is a $\beta-$algebra.

Table 1. β-algebra

$+$	0	1	2	3
0	0	1	2	3
1	1	3	0	2
2	2	0	3	1
3	3	2	1	0

$-$	0	1	2	3
0	0	2	1	3
1	1	0	3	2
2	2	3	0	1
3	3	1	2	0

Definition 2.13. *[15],[1] A non empty subset A of a $\beta-$algebra $(X,+,-,0)$ is called a β-subalgebra of X, if*
(i) $x+y \in A$ *and*
(ii) $x-y \in A \quad \forall \ x,y \in A.$

Definition 2.14. *[9],[2],[10] Let X be a non-empty set. By a cubic set in X we mean a structure $C = \{\langle x, \overline{\zeta}_C(x), \eta_C(x)\rangle : x \in X\}$ in which $\overline{\zeta}_C$ is an interval valued fuzzy set in X and η_C is a fuzzy set in X.*

Definition 2.15. *[13] Let $C = \{\langle x, \overline{\zeta}_C(x), \eta_C(x)\rangle : x \in X\}$ be a cubic fuzzy set in X. Then the set C is a cubic fuzzy $\beta-$ subalgebra if it satisfies the following conditions.*

(i) $\overline{\zeta}_C(x+y) \geq rmin\{\overline{\zeta}_C(x), \overline{\zeta}_C(y)\}$ & $\overline{\zeta}_C(x-y) \geq rmin\{\overline{\zeta}_C(x), \overline{\zeta}_C(y)\}$
(ii) $\eta_C(x+y) \leq max\{\eta_C(x), \eta_C(y)\}$ & $\eta_C(x-y) \leq max\{\eta_C(x), \eta_C(y)\}$
$\forall \ x,y \in X$

Definition 2.16. *[11],[12],[16] Let X be a non-empty set. By a Cubic intuitionistic set in X we indicate a structure $\tilde{C} = \{\langle x, (x), \rho_(x)\rangle : x \in X\}$ in which is an interval valued intuitionistic fuzzy set in X and ρ is an intuitionistic fuzzy set in X. Since $= \{\langle x, \overline{\zeta}(x), \overline{\eta}(x)\rangle : x \in X\}$ and $\rho = \{\langle x, \sigma_\rho(x), \phi_\rho(x)\rangle : x \in X\}$*

3. Cubic Intuitionistic $\beta-$ subalgebras of $\beta-$algebras

This section provides the notion of cubic intuitionistic $\beta-$ subalgebras of $\beta-$algebras and also some interesting results were examined. Also throughout the paper, X is a $\beta-$algebra and $= \{\langle x, \overline{\zeta}(x), \overline{\eta}(x)\rangle : x \in X\}$ and $\rho = \{\langle x, \sigma_\rho(x), \phi_\rho(x)\rangle : x \in X\}$ unless and otherwise specified.

Definition 3.1. *Let $\tilde{C} = \{\langle x, (x), \rho_(x)\rangle : x \in X\}$ be a cubic intuitionistic set in X, where is an interval valued intuitionistic fuzzy set in X and ρ is* *an intuitionistic fuzzy set in X.Then the set $\tilde{C}$ is called a cubic intuitionistic $\beta-$subalgebra if it satisfies the following conditions:*

(i) $\overline{\zeta}(x+y) \geq rmin\{\overline{\zeta}(x), \overline{\zeta}(y)\}$ & $\overline{\zeta}(x-y) \geq rmin\{\overline{\zeta}(x), \overline{\zeta}(y)\}$
(ii) $\overline{\eta}(x+y) \leq rmax\{\overline{\eta}(x), \overline{\eta}(y)\}$ & $\overline{\eta}(x-y) \leq rmax\{\overline{\eta}(x), \overline{\eta}(y)\}$
(iii) $\sigma_\rho(x+y) \leq max\{\sigma_\rho(x), \sigma_\rho(y)\}$ & $\sigma_\rho(x-y) \leq max\{\sigma_\rho(x), \sigma_\rho(y)\}$
(iv) $\phi_\rho(x+y) \geq min\{\phi_\rho(x), \phi_\rho(y)\}$ & $\phi_\rho(x-y) \geq min\{\phi_\rho(x), \phi_\rho(y)\}$
$\forall\ x, y \in X$

Example 3.2. Let $X = \{0, 1, 2, 3\}$ be a $\beta-$algebra with constant 0 and binary operations $+$ and $-$ are defined on X as in the following cayley's table.

$+$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$-$	0	1	2	3
0	0	3	2	1
1	1	0	3	2
2	2	1	0	3
3	3	2	1	0

Define a Cubic intuitionistic set $\tilde{C} = \{\langle x, (x), \rho_(x)\rangle : x \in X\}$ in X as follows:

X	$= \langle\overline{\zeta}, \overline{\eta}\rangle$	$\rho = (\sigma_\rho, \phi_\rho)$
height0	$\langle[0.4, 0.6], [0.1, 0.4]\rangle$	$(0.4, 0.7)$
1	$\langle[0.2, 0.4], [0.3, 0.6]\rangle$	$(0.4, 0.7)$
2	$\langle[0.3, 0.5], [0.2, 0.5]\rangle$	$(0.4, 0.7)$
3	$\langle[0.2, 0.4], [0.3, 0.6]\rangle$	$(0.6, 0.5)$

Then $\tilde{C}$ is a Cubic intuitionistic $\beta-$subalgebra of X. If it is considered as below

X	$= \langle\overline{\zeta}, \overline{\eta}\rangle$	$\rho = (\sigma_\rho, \phi_\rho)$
height0	$\langle[0.4, 0.6], [0.1, 0.4]\rangle$	$(0.6, 0.5)$
1	$\langle[0.4, 0.6], [0.2, 0.5]\rangle$	$(0.4, 0.7)$
2	$\langle[0.2, 0.4], [0.2, 0.5]\rangle$	$(0.4, 0.7)$
3	$\langle[0.3, 0.5], [0.3, 0.6]\rangle$	$(0.6, 0.5)$

Then $\tilde{C}$ is not a Cubic intuitionistic $\beta-$subalgebra of X .

Proposition 3.3. *Let $\tilde{C} = \{\langle x, (x), \rho_(x)\rangle : x \in X\}$ cubic intuitionistic $\beta-$subalgebra of X. Then*
$(1)\overline{\zeta}(0) \geq \overline{\zeta}(x)$, $\overline{\eta}(0) \leq \overline{\eta}(x), \sigma_\rho(0) \leq \sigma_\rho(x)$ *and* $\phi_\rho(0) \geq \phi_\rho(x), \quad \forall\, x \in X$
$(2)\overline{\zeta}(x) \leq \overline{\zeta}(x^*) \leq \overline{\zeta}(0)$ & $\overline{\eta}(x) \geq \overline{\eta}(x^*) \geq \overline{\eta}(0)$,
$\sigma_\rho(x) \geq \sigma_\rho(x^*) \geq \sigma_\rho(0)$ & $\phi_\rho(x) \leq \phi_\rho(x^*) \leq \phi_\rho(0) \quad \forall\, x \in X$ *where* $x^* = 0 - x$
The proof is straight forward.

Proposition 3.4. *Let $\tilde{C} = \{\langle x, (x), \rho_(x)\rangle : x \in X\}$ be a cubic intuitionistic $\beta-$subalgebra of X. Then*
$(1)\overline{\zeta}(0+x) \geq \overline{\zeta}(x) \& \overline{\zeta}(0-x) \geq \overline{\zeta}(x)$
$(2)\overline{\eta}(0+x) \leq \overline{\eta}(x)$ & $\overline{\eta}(0-x) \leq \overline{\eta}(x)$
$(3)\sigma_\rho(0+x) \leq \sigma_\rho(x)$ & $\sigma_\rho(0-x) \leq \sigma_\rho(x)$
$(4)\phi_\rho(0+x) \geq \phi_\rho(x) \& \phi_\rho(0-x) \geq \phi_\rho(x) \quad \forall\, x \in X$
The proof is straight forward.

Remark 3.5. The sets $\{x \in X : \overline{\zeta}(x) = \overline{\zeta}(0)\}$, $\{x \in X : \overline{\eta}(x) = \overline{\eta}(0)\}$, $\{x \in X : \sigma_\rho(x) = \sigma_\rho(0)\}$ and $\{x \in X : \phi_\rho(x) = \phi_\rho(0)\}$ are denoted by $T_{\overline{\zeta}}$,$T_{\overline{\eta}}$, T_{σ_ρ} and T_{ϕ_ρ} respectively.

Theorem 3.6. *Let $\tilde{C} = \{\langle x, (x), \rho_(x)\rangle : x \in X\}$ be a cubic intuitionistic $\beta-$subalgebra of X. Then the sets $T_{\overline{\zeta}}$,$T_{\overline{\eta}}$, T_{σ_ρ} and T_{ϕ_ρ} are $\beta-$subalgebras of X.*

Proof: *Let $x, y \in T_{\overline{\zeta}}$ and $x, y \in T_{\overline{\eta}}$. Then $\overline{\zeta}(x) = \overline{\zeta}(0) = \overline{\zeta}(y)$ and $\overline{\eta}(x) = \overline{\eta}(0) = \overline{\eta}(y)$. Thus $\overline{\zeta}(x+y) \geq rmin\{\overline{\zeta}(x), \overline{\zeta}(y)\} = rmin\{\overline{\zeta}(0), \overline{\zeta}(0)\} = \overline{\zeta}(0)$. Therefore $\overline{\zeta}(x+y) \geq \overline{\zeta}(0)$. Similarly, $\overline{\zeta}(x-y) \geq \overline{\zeta}(0)$. Consequently, $\overline{\eta}(x+y) \leq rmax\{\overline{\eta}(x), \overline{\eta}(y)\} = rmax\{\overline{\eta}(0), \overline{\eta}(0)\} = \overline{\eta}(0)$. Hence, $\overline{\eta}(x+y) \leq \overline{\eta}(0)$. Likewise, we can obtain $\overline{\eta}(x-y) \leq \overline{\eta}(0)$. By using Proposition 3.3, it can be conclude that $\overline{\zeta}(x+y) \leq \overline{\zeta}(0)$ & $\overline{\zeta}(x-y) \leq \overline{\zeta}(0)$ and $\overline{\eta}(x+y) \geq \overline{\eta}(0)$ & $\overline{\eta}(x-y) \geq \overline{\eta}(0)$. Hence $\overline{\zeta}(x+y) = \overline{\zeta}(0)$ & $\overline{\zeta}(x-y) = \overline{\zeta}(0)$ and $\overline{\eta}(x+y) = \overline{\eta}(0)$ & $\overline{\eta}(x-y) = \overline{\eta}(0)$ or equivalently, $x+y, x-y \in T_{\overline{\zeta}}$ & $T_{\overline{\eta}}$. Let $x, y \in T_{\sigma_\rho}$ and $x, y \in T_{\phi_\rho}$. Then $\sigma_\rho(x) = \sigma_\rho(0) = \sigma_\rho(y)$ and $\phi_\rho(x) = \phi_\rho(0) = \phi_\rho(y)$. Thus $\sigma_\rho(x+y) \leq max\{\sigma_\rho(x), \sigma_\rho(y)\} = max\{\sigma_\rho(0), \sigma_\rho(0)\} = \sigma_\rho(0)$. Hence $\sigma_\rho(x+y) \leq \sigma_\rho(0)$. In the similar way, $\sigma_\rho(x-y) \leq \sigma_\rho(0)$. $\phi_\rho(x+y) \geq min\{\phi_\rho(x), \phi_\rho(y)\} = min\{\phi_\rho(0), \phi_\rho(0)\} = \phi_\rho(0)$. Therefore, $\phi_\rho(x+y) \geq \phi_\rho(0)$. Similarly, $\phi_\rho(x-y) \geq \phi_\rho(0)$. By Using Proposition 3.3, it can be conclude that $\sigma_\rho(x+y) \geq \sigma_\rho(0)$ & $\sigma_\rho(x-y) \geq \sigma_\rho(0)$ and $\phi_\rho(x+y) \leq \phi_\rho(0)$ & $\phi_\rho(x-y) \leq \phi_\rho(0)$. Hence, $\sigma_\rho(x+y) = \sigma_\rho(0)$ &*

$\sigma_\rho(x-y) = \sigma_\rho(0)$ and $\phi_\rho(x+y) = \phi_\rho(0)$ & $\phi_\rho(x-y) = \phi_\rho(0)$ or equivalently, $x+y, x-y \in T_{\sigma_\rho}$ & T_{ϕ_ρ}. Therefore the sets $T_{\overline{\zeta}}$,$T_{\overline{\eta}}$, T_{σ_ρ} and T_{ϕ_ρ} are $\beta-$subalgebras of X.

Definition 3.7. *Let* $A = \{\langle x,_A(x), \rho_A(x)\rangle : x \in X\}$ *and* $B = \{\langle x,_B(x), \rho_B(x)\rangle : x \in X\}$ *be two cubic intuitionistic sets on* X*, then the intersection of* A *and* B *is defined by* $A \cap B = \{\langle x,_{A\cap B}(x), \rho_{A\cap B}(x)\rangle\} = \{\langle x, rmin\{\overline{\zeta}_{\psi_A}(x), \overline{\zeta}_{\psi_B}(x)\}, rmax\{\overline{\eta}_{\psi_A}(x), \overline{\eta}_{\psi_B}(x)\}, max(\sigma_{\rho_A}(x), \sigma_{\rho_B}(x)), min(\phi_{\rho_A}(x), \phi_{\rho_B}(x))\rangle : x \in X\}$.

Proposition 3.8. *Let* $A = \{\langle x,_A(x), \rho_A(x)\rangle : x \in X\}$ *and* $B = \{\langle x,_B(x), \rho_B(x)\rangle : x \in X\}$ *be two cubic intuitionistic fuzzy* $\beta-$*subalgebras. Then the intersection of* A *and* B *is also a cubic intuitionistic fuzzy* $\beta-$*subalgebra.*

Proof: *Let* $x, y \in A \cap B$. *Then*

$$\begin{aligned}
\overline{\zeta}_{\psi_{A\cap B}}(x+y) &= rmin\{\overline{\zeta}_{\psi_A}(x+y), \overline{\zeta}_{\psi_B}(x+y)\} \\
&\geq rmin\{rmin\{\overline{\zeta}_{\psi_A}(x), \overline{\zeta}_{\psi_A}(y)\}, rmin\{\overline{\zeta}_{\psi_B}(x), \overline{\zeta}_{\psi_B}(y)\}\} \\
&\geq rmin\{rmin\{\overline{\zeta}_{\psi_A}(x), \overline{\zeta}_{\psi_B}(x)\}, rmin\{\overline{\zeta}_{\psi_A}(y), \overline{\zeta}_{\psi_B}(y)\} \\
&\geq rmin\{\overline{\zeta}_{\psi_{A\cap B}}(x), \overline{\zeta}_{\psi_{A\cap B}}(y)\}.
\end{aligned}$$

Similarly, $\overline{\zeta}_{\psi_{A\cap B}}(x-y) \geq rmin\{\overline{\zeta}_{\psi_{A\cap B}}(x), \overline{\zeta}_{\psi_{A\cap B}}(y)\}$. *By applying the same process, then we get* $\overline{\eta}_{\psi_{A\cap B}}(x+y) \leq rmax\{\overline{\eta}_{\psi_{A\cap B}}(x), \overline{\eta}_{\psi_{A\cap B}}(y)\}$ *In the similar way, we obtain* $\overline{\eta}_{\psi_{A\cap B}}(x-y) \leq rmax\{\overline{\eta}_{\psi_{A\cap B}}(x), \overline{\eta}_{\psi_{A\cap B}}(y)\}$.

Further,

$$\begin{aligned}
\sigma_{\rho_{A\cap B}}(x+y) &= max\{\sigma_{\rho_A}(x+y), \sigma_{\rho_B}(x+y)\} \\
&\leq max\{max\{\sigma_{\rho_A}(x), \sigma_{\rho_A}(y)\}, max\{\sigma_{\rho_B}(x), \sigma_{\rho_B}(y)\}\} \\
&\leq max\{\sigma_{\rho_A}(x), \sigma_{\rho_B}(x)\}, max\{\sigma_{\rho_A}(y), \sigma_{\rho_B}(y)\}\} \\
&\leq max\{\sigma_{\rho_{A\cap B}}(x), \sigma_{\rho_{A\cap B}}(y)\}.
\end{aligned}$$

Likewise, we have $\sigma_{\rho_{A\cap B}}(x-y) \leq max\{\sigma_{\rho_{A\cap B}}(x), \sigma_{\rho_{A\cap B}}(y)\}$. *By using the same process, we obtain* $\phi_{\rho_{A\cap B}}(x+y) \geq min\{\phi_{\rho_{A\cap B}}(x), \phi_{\rho_{A\cap B}}(y)\}$. *In the same manner, we can get* $\phi_{\rho_{A\cap B}}(x-y) \geq min\{\phi_{\rho_{A\cap B}}(x), \phi_{\rho_{A\cap B}}(y)\}$. *Therefore, the intersection of* A *and* B *are cubic intuitionistic* $\beta-$*subalgebras.*

Theorem 3.9. *If* $\tilde{C} = \{\langle x, (x), \rho(x)\rangle : x \in X\}$ *be a cubic intuitionistic* $\beta-$*subalgebra of* X. *Let* $\chi_{\tilde{C}} = \{x \in X/\overline{\zeta}(x) = \overline{\zeta}(0), \overline{\eta}(x) = \overline{\eta}(0), \sigma_\rho(x) = \sigma_\rho(0)\ \phi_\rho(x) = \phi_\rho(0)\}$. *Then* $\chi_{\tilde{C}}$ *is a* $\beta-$*subalgebra of* X.

Proof: *For any* $x, y \in \chi_{\tilde{C}}$.
$\overline{\zeta}(x) = \overline{\zeta}(0)$, $\overline{\zeta}(y) = \overline{\zeta}(0)$ *and*
$\overline{\eta}_(x) = \overline{\eta}_(0), \overline{\eta}_(y) = \overline{\eta}_(0)$
$\sigma_\rho(x) = \sigma_\rho(0)$, $\sigma_\rho(y) = \sigma_\rho(0)$ *and* $\phi_\rho(x) = \phi_\rho(0), \phi_\rho(y) = \phi_\rho(0)$

It is known that,

$$\begin{aligned}
\overline{\zeta}(x+y) &= [\zeta^L(x+y), \zeta^U(x+y)] \\
&\geq [min\{\zeta^L(x), \zeta^L(y)\}, min\{\zeta^U(x), \zeta^U(y)\}] \\
&= rmin\{[\zeta^L(x), \zeta^U(x)], [\zeta^L(y), \zeta^U(y)]\} \\
&\geq rmin\{\overline{\zeta}(x), \overline{\zeta}(y)\} \\
&= rmin\{\overline{\zeta}(0), \overline{\zeta}(0)\} \\
&= \overline{\zeta}(0) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)
\end{aligned}$$

$$\begin{aligned}
\overline{\zeta}(0) &= \overline{\zeta}(0-0) \\
&= [\zeta^L(0-0), \zeta^U(0-0)] \\
&\geq [min\{\zeta^L(0), \zeta^L(0)\}, min\{\zeta^U(0), \zeta^U(0)\}] \\
&= rmin\{[\zeta^L(0), \zeta^U(0)], [\zeta^L(0), \zeta^U(0)]\} \\
&\geq rmin\{\overline{\zeta}(0), \overline{\zeta}(0)\} \\
&= rmin\{\overline{\zeta}(x), \overline{\zeta}(y)\} \\
&= \overline{\zeta}(x+y) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)
\end{aligned}$$

From (1) and (2) we get $\overline{\zeta}(x+y) = \overline{\zeta}(0)$. *Similarly,* $\overline{\zeta}(x-y) = \overline{\zeta}(0)$. *By using the same process, we get* $\overline{\eta}(x+y) \leq \overline{\eta}(0)$ *and* $\overline{\eta}(0) \leq \overline{\eta}(x+y)$ *which yields that* $\overline{\eta}(x+y) = \overline{\eta}(0)$. *Similarly,* $\overline{\eta}(x-y) = \overline{\eta}(0)$. *Now,*

$$\begin{aligned}
\sigma_\rho(x+y) &\leq max\{\sigma_\rho(x), \sigma_\rho(y)\} \\
&= max\{\sigma_\rho(0), \sigma_\rho(0)\} \\
&= \sigma_\rho(0) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)
\end{aligned}$$

$$\begin{aligned}
\sigma_\rho(0) &= \sigma_\rho(0-0) \\
&\leq max\{\sigma_\rho(0), \sigma_\rho(0)\} \\
&= max\{\sigma_\rho(x), \sigma_\rho(y)\} \\
&= \sigma_\rho(x+y) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)
\end{aligned}$$

From (3) and (4) we obtain $\sigma_\rho(x+y) = \sigma_\rho(0)$. *In a similar way,* $\sigma_\rho(x-y) = \sigma_\rho(0)$. *By applying the same process, we can have* $\phi_\rho(x+y) \geq \phi_\rho(0)$ *and* $\phi_\rho(0) \geq \phi_\rho(x+y)$ *which gives* $\phi_\rho(x+y) = \phi_\rho(0)$. *Likewise* $\phi_\rho(x-y) = \phi_\rho(0)$
Thus $x+y, x-y \in \chi_{\tilde{C}}$. *Hence* $\chi_{\tilde{C}}$ *is a* $\beta-$*subalgebra of* X.

Theorem 3.10. *If* $\tilde{C} = \{\langle x, (x), \rho(x)\rangle : x \in X\}$ *be a cubic intuitionistic* $\beta-$*subalgebra of* X, *then*
$\overline{\zeta}(x) \leq \overline{\zeta}(x-0)$, $\overline{\eta}(x) \geq \overline{\eta}(x-0)$, $\sigma_\rho(x) \geq \sigma_\rho(x-0)$ *and* $\phi_\rho(x) \leq \phi_\rho(x-0)$

Proof: *Let $\tilde{C}$ be a cubic intuitionistic $\beta-$subalgebra of X.*

$$\begin{aligned}\overline{\zeta}(x-0) &= [\zeta^L(x-0), \zeta^U(x-0)]\\ &\geq [min\{\zeta^L(x), \zeta^L(0)\}, min\{\zeta^U(x), \zeta^U(0)\}]\\ &= rmin\{[\zeta^L(x), \zeta^U(x)], [\zeta^L(0), \zeta^U_C(0)]\}\\ &= rmin\{\overline{\zeta}(x), \overline{\zeta}(0)\}\\ &= rmin\{\overline{\zeta}(x), \overline{\zeta}(x-x)\}\\ &= rmin\{\overline{\zeta}(x), rmin\{\overline{\zeta}(x), \overline{\zeta}(x)\}\}\\ &= rmin\{\overline{\zeta}(x), \overline{\zeta}(x)\}\\ &= \overline{\zeta}(x)\end{aligned}$$

Thus, $\overline{\zeta}(x) \leq \overline{\zeta}(x-0)$. In the same way, for another component η, we can obtain $\overline{\eta}(x) \geq \overline{\zeta}(x-0)$. Further, we consider

$$\begin{aligned}\sigma_\rho(x-0) &\leq max\{\sigma_\rho(x), \sigma_\rho(0)\}\\ &= max\{\sigma_\rho(x), \sigma_\rho(x-x)\}\\ &= max\{\sigma_\rho(x), max\{\sigma_\rho(x), \sigma_\rho(x)\}\}\\ &= max\{\sigma_\rho(x), \sigma_\rho(x)\}\\ &= \sigma_\rho(x)\end{aligned}$$

Hence, $\sigma_\rho(x) \geq \sigma_\rho(x-0)$. By applying the same process for another component ϕ_ρ , we will have $\phi_\rho(x) \leq \phi_\rho(x-0)$.

Remark 3.11. Let $\tilde{C} = \{\langle x, (x), \rho(x)\rangle\}$ be a cubic intuitionistic set in a non-empty set X. Given $([u_1, v_1], [u_2, v_2]) \in D[0,1] \times D[0,1]$ and $(\theta_1, \theta_2) \in [0,1] \times [0,1]$. We consider the sets

$\overline{\zeta}[u_1, v_1] = \{x \in X/\overline{\zeta}(x) \geq [u_1, v_1]\}$; $\overline{\eta}[u_2, v_2] = \{x \in X/\overline{\eta}(x) \leq [u_2, v_2]\}$
$\sigma_\rho(\theta_1) = \{x \in X/\sigma_\rho(x) \leq (\theta_1)\}$; $\phi_\rho(\theta_2) = \{x \in X/\phi_\rho(x) \geq (\theta_2)\}$

By using the above remark, the following theorem will be proved.

Theorem 3.12. *If $\tilde{C} = \{\langle x, (x), \rho(x)\rangle\}$ be a cubic intuitionistic $\beta-$subalgebra of X then the sets $\overline{\zeta}[u, v], \overline{\eta}[u, v], \sigma_\rho(\theta)$ and $\phi_\rho(\theta)$are $\beta-$subalgebra of X for every $[u, v] \in D[0,1]$ and $\theta \in [0,1]$.*

Proof: *For every $[u, v] \in D[0,1]$ and $\theta \in [0,1]$. Let $x, y \in X$ be such that $x, y \in \overline{\zeta}[u, v] \cap \overline{\eta}[u, v] \cap \sigma_\rho(\theta) \cap \phi_\rho(\theta)$. Then $\overline{\zeta}(x) \geq [u, v]$, $\overline{\eta}(x) \leq [u, v]$, $\sigma_\rho(x) \leq \theta$, $\phi_\rho(x) \geq \theta$ and $\overline{\zeta}(y) \geq [u, v]$, $\overline{\eta}(y) \leq [u, v]$, $\sigma_\rho(y) \leq \theta$, $\phi_\rho(y) \geq \theta$. It follows that $\overline{\zeta}(x+y) \geq rmin\{\overline{\zeta}(x), \overline{\zeta}(y)\} = rmin\{[u, v], [u, v]\} = [u, v]$. Similarly, $\overline{\zeta}(x-y) \geq [u, v]$ and $\overline{\eta}(x+y) \leq rmax\{\overline{\eta}(x), \overline{\eta}(y)\} = rmax\{[u, v], [u, v]\} = [u, v]$. In the similar way, $\overline{\eta}(x-y) \leq [u, v]$. Also $\sigma_\rho(x+y) \leq max\{\sigma_\rho(x), \sigma_\rho(y)\} = max\{\theta, \theta\} = \theta$ implies $\sigma_\rho(x+y) \leq \theta$.*

Likewise, $\sigma_\rho(x-y) \leq \theta$ *and* $\phi_\rho(x+y) \geq min\{\phi_\rho(x), \phi_\rho(y)\} = min\{\theta, \theta\} = \theta$ *which gives* $\phi_\rho(x+y) \geq \theta$. *Similarly,* $\phi_\rho(x-y) \geq \theta$. *That is* $x+y, x-y \in \overline{\zeta}[u,v] \cap \overline{\eta}[u,v] \cap \sigma_\rho(\theta) \cap \phi_\rho(\theta)$. *Therefore,* $\overline{\zeta}[u,v], \overline{\eta}[u,v], \sigma_\rho(\theta), \phi_\rho(\theta)$ *are* $\beta-$*subalgebras of* X, *for all* $[u,v] \in D[0,1]$ *and* $\theta \in [0,1]$.

4. Product on Cubic Intuitionistic $\beta-$subalgebra

This section, introduces the notion of product on Cubic intuitionistic $\beta-$ subalgebras of $\beta-$ algebras and provides some fascinating results.

Definition 4.1. *Let* $A = \{\langle x,_A(x), \rho_A(x)\rangle : x \in X\}$ *and* $B = \{\langle y,_B(y), \rho_B(y)\rangle : y \in Y\}$ *be cubic intuitionistic sets in* X *and* Y *respectively. The Cartesian product of* A *and* B *denoted by* $A \times B$ *is defined to be the set* $A \times B = \{\langle (x,y),_{A\times B}(x,y), \rho_{A\times B}(x,y)\rangle : (x,y) \in X \times Y\}$ *where* $_{A\times B} = [\overline{\zeta}_{A\times B}, \overline{\eta}_{A\times B}]$ & $\rho_{A\times B} = (\sigma_{A\times B}, \phi_{A\times B})$ *and*
$\overline{\zeta}_{A\times B} : X \times Y \to D[0,1]$ *is given by* $\overline{\zeta}_{A\times B}(x,y) = rmin\{\overline{\zeta}_A(x), \overline{\zeta}_B(y)\}$,
$\overline{\eta}_{A\times B} : X \times Y \to D[0,1]$ *is given by* $\overline{\eta}_{A\times B}(x,y) = rmax\{\overline{\eta}_A(x), \overline{\eta}_B(y)\}$,
$\sigma_{A\times B} : X \times Y \to [0,1]$ *is given by* $\sigma_{A\times B}(x,y) = max\{\sigma_A(x), \sigma_B(y)\}$ *and*
$\phi_{A\times B} : X \times Y \to [0,1]$ *is given by* $\phi_{A\times B}(x,y) = min\{\phi_A(x), \phi_B(y)\}$

Theorem 4.2. *Let*$A = \{\langle x,_A(x), \rho_A(x)\rangle : x \in X\}$ *and* $B = \{\langle y,_B(y), \rho_B(y)\rangle : y \in Y\}$ *be any two cubic intuitionistic* $\beta-$*subalgebras of* X *and* Y *respectively. Then* $A \times B$ *is also an cubic intuitionistic* $\beta-$*subalgebra of* $X \times Y$.

Proof: *Let* $A = \{\langle x,_A(x), \rho_A(x)\rangle : x \in X\}$ *and* $B = \{\langle y,_B(y), \rho_B(y)\rangle : y \in Y\}$ *be cubic intuitionistic* $\beta-$*subalgebras in* X *and* Y. *Take* $(a,b) \in X \times Y$, *where* $a = (x_1, x_2)$ *and* $b = (y_1, y_2)$. *It follows that*

$$\begin{aligned} \overline{\zeta}_{A\times B}(a+b) &= \overline{\zeta}_{A\times B}((x_1,x_2)+(y_1,y_2)) \\ &= (\overline{\zeta}_A \times \overline{\zeta}_B)((x_1+y_1),(x_2+y_2)) \\ &= rmin\{\overline{\zeta}_A(x_1+y_1), \overline{\zeta}_B(x_2+y_2)\} \\ &\geq rmin\{rmin\{\overline{\zeta}_A(x_1), \overline{\zeta}_A(y_1)\}, rmin\{\overline{\zeta}_B(x_2), \overline{\zeta}_B(y_2)\} \\ &\geq rmin\{rmin\{\overline{\zeta}_A(x_1), \overline{\zeta}_B(x_2)\}, rmin\{\overline{\zeta}_A(y_1), \overline{\zeta}_B(y_2)\} \\ &= rmin\{(\overline{\zeta}_A \times \overline{\zeta}_B)((x_1,x_2), \overline{\zeta}_A \times \overline{\zeta}_B)(y_1,y_2))\} \\ &= rmin\{\overline{\zeta}_{A\times B}(a), \overline{\zeta}_{A\times B}(b)\} \end{aligned}$$

Similarly, we can get $\overline{\zeta}_{A\times B}(a-b) \geq rmin\{\overline{\zeta}_{A\times B}(a), \overline{\zeta}_{A\times B}(b)\}$. *By applying the same process we will obtain* $\overline{\eta}_{A\times B}(a+b) \leq rmax\{\overline{\eta}_{A\times B}(a), \overline{\eta}_{A\times B}(b)\}$ *and* $\overline{\eta}_{A\times B}(a-b) \leq rmax\{\overline{\eta}_{A\times B}(a), \overline{\eta}_{A\times B}(b)\}$.

Further,

$$\begin{aligned}
\sigma_{A\times B}(a+b) &= \sigma_{A\times B}((x_1,y_1)+(x_2,y_2)) \\
&= (\sigma_A \times \sigma_B)\{(x_1+y_1),(x_2+y_2)\} \\
&= max\{\sigma_A(x_1+y_1),\sigma_B(x_2+y_2)\} \\
&\le max\{max\{\sigma_A(x_1),\sigma_A(y_1)\},max\{\sigma_B(x_2),\sigma_B(y_2)\}\} \\
&\le max\{max\{\sigma_A(x_1),\sigma_B(x_2)\},max\{\sigma_A(y_1),\sigma_B(y_2)\}\} \\
&= max\{(\sigma_A\times\sigma_B)(x_1,x_2),(\sigma_A\times\sigma_B)(y_1,y_2)\} \\
&= max\{\sigma_{A\times B}(a),\sigma_{A\times B}(b)\}
\end{aligned}$$

In the similar way, one can have, $\sigma_{A\times B}(a-b) \le max\{\sigma_{A\times B}(a),\sigma_{A\times B}(b)\}$. *By applying the similar process, we can have* $\phi_{A\times B}(a+b) \ge min\{\phi_{A\times B}(a), \phi_{A\times B}(b)\}$ *and* $\phi_{A\times B}(a-b) \ge min\{\phi_{A\times B}(a),\phi_{A\times B}(b)\}$.

Theorem 4.3. *If* $A\times B$ *is an cubic intuitionistic* $\beta-$*subalgebra of* $X\times Y$, *then either* A *is a cubic intuitionistic* $\beta-$*subalgebra of* X *or* B *is a cubic intuitionistic* $\beta-$*subalgebra of* Y.

Proof: *Let* $A\times B$ *is a cubic intuitionistic fuzzy* $\beta-$*subalgebra of* $X\times Y$. *Take* (x_1,y_1) *and* $(x_2,y_2) \in X\times Y$. *Then,* $\overline{\zeta}_{A\times B}\{(x_1,y_1)+(x_2,y_2)\} \ge rmin\{\overline{\zeta}_{A\times B}(x_1,y_1),\overline{\zeta}_{A\times B}(x_2,y_2)\}$. *Put* $x_1=x_2=0$ *which implies that* $\overline{\zeta}_{A\times B}\{(0,y_1),(0,y_2)\} \ge rmin\{\overline{\zeta}_{A\times B}(0,y_1),\overline{\zeta}_{A\times B}(0,y_2)\}$. *Now consider,* $\overline{\zeta}_{A\times B}\{(0+0),(y_1+y_2)\} \ge rmin\{\overline{\zeta}_{A\times B}(0,y_1),\overline{\zeta}_{A\times B}(0,y_2)\}$. *So,* $\overline{\zeta}_B(y_1+y_2) \ge rmin\{\overline{\zeta}_B(y_1),\overline{\zeta}_B(y_2)\}$. *Similarly,* $\overline{\zeta}_B(y_1-y_2) \ge rmin\{\overline{\zeta}_B(y_1),\overline{\zeta}_B(y_2)\}$ *and also* $\overline{\eta}_{A\times B}\{(x_1,y_1)+(x_2,y_2)\} \le rmax\{\overline{\eta}_{A\times B}(x_1,y_1),\overline{\eta}_{A\times B}(x_2,y_2)\}$. *Put* $x_1=x_2=0$ *which gives* $\overline{\eta}_{A\times B}\{(0,y_1),(0,y_2)\} \le rmax\{\overline{\eta}_{A\times B}(0,y_1), \overline{\eta}_{A\times B}(0,y_2)\}$. *Now* $\overline{\eta}_{A\times B}\{(0+0),(y_1+y_2)\} \le rmax\{\overline{\eta}_{A\times B}(0,y_1),\overline{\eta}_{A\times B}(0,y_2)\}$. *Moreover,* $\overline{\eta}_B(y_1+y_2) \le rmax\{\overline{\eta}_B(y_1),\overline{\eta}_B(y_2)\}$. *In the similar way, we have* $\overline{\eta}_B(y_1-y_2) \le rmax\{\overline{\eta}_B(y_1),\overline{\eta}_B(y_2)\}$. *Further,* $\sigma_{A\times B}\{(x_1,y_1)+(x_2,y_2)\} \le max\{\sigma_{A\times B}(x_1,y_1),\sigma_{A\times B}(x_2,y_2)\}$. *Put* $x_1=x_2=0$ *gives* $\sigma_{A\times B}\{(0,y_1),(0,y_2)\} \le max\{\sigma_{A\times B}(0,y_1),\sigma_{A\times B}(0,y_2)\}$. *Then we have* $\sigma_{A\times B}\{(0+0),(y_1+y_2)\} \le max\{\sigma_{A\times B}(0,y_1),\sigma_{A\times B}(0,y_2)\}$. *It follows that* $\sigma_B(y_1+y_2) \le max\{\sigma_B(y_1),\sigma_B(y_2)\}$. *In the same manner,* $\sigma_B(y_1-y_2) \le max\{\sigma_B(y_1),\sigma_B(y_2)\}$ *and* $\phi_{A\times B}\{(x_1,y_1)+(x_2,y_2)\} \ge min\{\phi_{A\times B}(x_1,y_1), \phi_{A\times B}(x_2,y_2)\}$. *Put* $x_1=x_2=0$ *which gives* $\phi_{A\times B}\{(0,y_1),(0,y_2)\} \ge min\{\phi_{A\times B}(0,y_1),\phi_{A\times B}(0,y_2)\}$. *Then we can have* $\phi_{A\times B}\{(0+0),(y_1+y_2)\} \ge min\{\phi_{A\times B}(0,y_1),\phi_{A\times B}(0,y_2)\}$ *which yields that* $\phi_B(y_1+y_2) \ge min\{\phi_B(y_1),\phi_B(y_2)\}$. *Likewise,* $\phi_B(y_1-y_2) \ge min\{\phi_B(y_1),\phi_B(y_2)\}$. *Hence* B *is a Cubic intuitionistic* $\beta-$*subalgebra of* Y.

5. Level set of Cubic Intuitionistic $\beta-$Subalgebras

Definition 5.1. *Let $\tilde{C} = \{\langle x, (x), \rho(x)\rangle : x \in X\}$ be a cubic intuitionistic set of X. Define $\tilde{C}_{\overline{\alpha},\overline{\gamma},\lambda,\omega} = \{x \in X : \overline{\zeta} \geq \overline{\alpha}, \overline{\eta} \leq \overline{\gamma}, \sigma_\rho \leq \lambda, \phi_\rho \geq \omega\}$, where $\overline{\alpha}, \overline{\gamma} \in D[0,1]$ and $\lambda, \omega \in [0,1]$ is called a cubic intuitionistic level set of $\tilde{C}$.*

Example 5.2. Consider a subset $\tilde{C}$ of the $\beta-$algebra X, given in example 3.2. If we define $\overline{\alpha} = [0.1, 0.5]$, $\overline{\gamma} = [0.4, 0.5]$, $\lambda = 0.5$ and $\omega = 0.6$ then $\tilde{C}_{[0.1,0.5],[0.4,0.5],0.5,0.6} = \{0, 2\}$ is a cubic intuitionistic level set of $\tilde{C}$.

Theorem 5.3. *If $\tilde{C} = \{\langle x, (x), \rho(x)\rangle : x \in X\}$ be a cubic intuitionistic $\beta-$sub*
algebra in X if and only if $\tilde{C}_{\overline{\alpha},\overline{\gamma},\lambda,\omega}$ is a $\beta-$subalgebra of X, for every $\overline{\alpha}, \overline{\gamma} \in D[0,1]$ and $\lambda, \omega \in [0,1]$.

Proof. *For $x, y \in \tilde{C}_{\overline{\alpha},\overline{\gamma},\lambda,\omega}$ and $\overline{\zeta}(x) \geq \overline{\alpha}$ and $\overline{\zeta}(y) \geq \overline{\alpha}$, we can write $\overline{\zeta}(x+y) \geq rmin\{\overline{\zeta}(x), \overline{\zeta}(y)\} \geq rmin\{\overline{\alpha}, \overline{\alpha}\} = \overline{\alpha}$.*

Similarly, $\overline{\zeta}(x-y) \geq \overline{\alpha}$. For $x, y \in \tilde{C}_{\overline{\alpha},\overline{\gamma},\lambda,\omega}$ and $\overline{\eta}(x) \leq \overline{\gamma}$ and $\overline{\eta}(y) \leq \overline{\gamma}$, we can write $\overline{\eta}(x+y) \leq rmax\{\overline{\eta}(x), \overline{\eta}(y)\} \leq rmax\{\overline{\gamma}, \overline{\gamma}\} = \overline{\gamma}$. In the similar way, $\overline{\eta}(x-y) \leq \overline{\gamma}$. For $\tilde{C}_{\overline{\alpha},\overline{\gamma},\lambda,\omega}$ and $\sigma_\rho(x) \leq \lambda$ and $\sigma_\rho(y) \leq \lambda$, we have $\sigma_\rho(x+y) \leq max\{\sigma_\rho(x), \sigma_\rho(y)\} = \lambda$. Likewise, $\sigma_\rho(x-y) \leq \lambda$. For $\tilde{C}_{\overline{\alpha},\overline{\gamma},\lambda,\omega}$ and $\phi_\rho(x) \geq \omega$ and $\phi_\rho(y) \geq \omega$, we have $\phi_\rho(x+y) \geq min\{\phi_\rho(x), \phi_\rho(y)\} = \omega$. Similarly, $\phi_\rho(x-y) \geq \omega$. So, we conclude that $x+y, x-y \in \tilde{C}_{\overline{\alpha},\overline{\gamma},\lambda,\omega}$. Hence, $\tilde{C}_{\overline{\alpha},\overline{\gamma},\lambda,\omega}$ is a $\beta-$subalgebra of X.

Conversely, assume that $\tilde{C} = \{\langle x, (x), \rho(x)\rangle : x \in X\}$ is a cubic intuitionistic set in X. Since $\tilde{C}_{\overline{\alpha},\overline{\gamma},\lambda,\omega}$ is a $\beta-$subalgebra of X for $\overline{\alpha}, \overline{\gamma} \in D[0,1]$ and $\lambda, \omega \in [0,1]$, it follows that $x+y$ and $x-y \in \tilde{C}_{\overline{\alpha},\overline{\gamma},\lambda,\omega}$. Now, take $\overline{\alpha} = rmin\{\overline{\zeta}_(x), \overline{\zeta}_(y)\}$, $\overline{\gamma} = rmax\{\overline{\eta}_(x), \overline{\eta}_(y)\}$ and $\lambda = max\{\sigma_\rho(x), \sigma_\rho(y)\}$, $\omega = min\{\phi_\rho(x), \phi_\rho(y)\}$ then we obtain $x+y \in C_{\overline{\alpha},\overline{\gamma},\lambda,\omega}$ this implies that $\overline{\zeta}(x+y) \geq \overline{\alpha}$ and $\overline{\eta}(x-y) \leq \overline{\gamma}$ and $\sigma_\rho(x-y) \geq \lambda$, $\phi_\rho(x-y) \leq \omega$.

Also, $x-y \in \tilde{C}_{\overline{\alpha},\overline{\gamma},\lambda,\omega}$ which yields that $\overline{\zeta}(x-y) \geq \overline{\alpha}$, $\overline{\eta}(x-y) \leq \overline{\gamma}$ and $\sigma_\rho(x-y) \geq \lambda$, $\phi_\rho(x-y) \leq \omega$. Therefore, we conclude that $\overline{\zeta}_C(x+y) \geq rmin\{\overline{\zeta}_C(x), \overline{\zeta}_C(y)\}$, $\overline{\eta}_C(x+y) \leq rmax\{\overline{\eta}_C(x), \overline{\eta}_C(y)\}$. Similarly, we have $\overline{\zeta}(x-y) \geq rmin\{\overline{\zeta}(x), \overline{\zeta}(y)\}$, $\overline{\eta}(x-y) \leq rmax\{\overline{\eta}(x), \overline{\eta}_C(y)\}$. Also, we know that $\sigma_\rho(x+y) \leq max\{\sigma_\rho(x), \sigma_\rho(y)\}$, $\phi_\rho(x+y) \geq min\{\phi_\rho(x), \phi_\rho(y)\}$. Similarly, $\sigma_\rho(x-y) \leq max\{\sigma_\rho(x), \sigma_\rho(y)\}$, $\phi_\rho(x-y) \geq min\{\phi_\rho(x), \phi_\rho(y)\}$. Hence $\tilde{C}$ is a cubic intuitionistic $\beta-$subalgebra of X. $\square$

6. $(\overline{T}, \overline{S}, S, T)$-Normed Cubic Intuitionistic $\beta-$subalgebras

This section introduces $(\overline{T}, \overline{S}, S, T)$-normed cubic intuitionistic $\beta-$subalgebra of a $\beta-$algebra and discusses few of its associated outcomes.

Definition 6.1. *Let $(X, +, -, 0)$ be a $\beta-$algebra. A cubic intuitionistic set $\tilde{C} = \{\langle x, (x), \rho_(x)\rangle : x \in X\}$ is called $(\overline{T}, \overline{S}, S, T)$ normed cubic intuitionistic $\beta-$subalgebra of X, if it satisfies the following conditions*

(i) $\overline{\zeta}(x+y) \geq \overline{T}\{\overline{\zeta}(x), \overline{\zeta}(y)\}$ & $\overline{\zeta}(x-y) \geq \overline{T}\{\overline{\zeta}(x), \overline{\zeta}(y)\}$
(ii) $\overline{\eta}(x+y) \leq \overline{S}\{\overline{\eta}(x), \overline{\eta}(y)\}$ & $\overline{\eta}(x-y) \leq \overline{S}\{\overline{\eta}(x), \overline{\eta}(y)\}$
(iii) $\sigma_\rho(x+y) \leq S\{\sigma_\rho(x), \sigma_\rho(y)\}$ & $\sigma_\rho(x-y) \leq S\{\sigma_\rho(x), \sigma_\rho(y)\}$
(iv) $\phi_\rho(x+y) \geq T\{\phi_\rho(x), \phi_\rho(y)\}$ & $\phi_\rho(x-y) \geq T\{\phi_\rho(x), \phi_\rho(y)\}$
$\forall\ x, y \in X$

Example 6.2. Let $X = \{0, 1, 2, 3\}$ be a set with constant 0 and binary operations $+$ and $-$ are defined on X by the following cayley's tables.

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

−	0	1	2	3
0	0	3	2	1
1	1	0	3	2
2	2	1	0	3
3	3	2	1	0

Let $\overline{T}_L, \overline{S}_L : D[0,1] \times D[0,1] \to D[0,1]$ and $S_L, T_L : [0,1] \times [0,1] \to [0,1]$ be functions defined by $\overline{T}_L(\overline{x}, \overline{y}) = rmax(\overline{x} + \overline{y} - \overline{1}, \overline{0})$, $\overline{S}_L(\overline{x}, \overline{y}) = rmin(\overline{x} + \overline{y}, \overline{1})$, $S_L(x, y) = min(x+y, 1)$ and $T_L(x, y) = max(x+y-1, 0)$ $\forall x, y \in [0, 1]$. Here $\overline{T}_L$ is a $\overline{T}-$norm , $\overline{S}_L-$ is a $\overline{T}-$conorm and S_L is a T-conorm, T_L is a T-norm. In all the T-norm and T-conorm Lukasiewicz property has been used. Define a Cubic intuitionistic set $\tilde{C} = \{\langle x, (x), \rho_(x)\rangle : x \in X\}$ in X as follows:

X	$= \langle\overline{\zeta}, \overline{\eta}_\rangle$ & $\rho = (\sigma_\rho, \phi_\rho)$	
0	$\langle[0.3, 0.6], [0.2, 0.4]\rangle$	$(0.6, 0.4)$
1	$\langle[0.1, 0.3], [0.4, 0.6]\rangle$	$(0.5, 0.7)$
2	$\langle[0.2, 0.5], [0.3, 0.5]\rangle$	$(0.5, 0.7)$
3	$\langle[0.1, 0.3], [0.4, 0.6]\rangle$	$(0.5, 0.7)$

Then $\tilde{C}$ is $(\overline{T}, \overline{S}, S, T)$-normed cubic intuitionistic $\beta-$subalgebra.

Definition 6.3. *Let $f : X \to Y$ be a function. Let A and B be two $(\overline{T},\overline{S},S,T)$-normed cubic intuitionistic sets in X and Y respectively. Then inverse image of B under f is defined by $f^{-1}(B) = \{f^{-1}(\overline{\zeta}_B(x)),$ $f^{-1}(\overline{\eta}_B(x)), f^{-1}(\sigma_B(x)), f^{-1}(\phi_B(x)) : x \in X\}$ such that $f^{-1}(\overline{\zeta}_B(x)) = (\overline{\zeta}_B(f(x)), f^{-1}(\overline{\eta}_B(x)) = (\overline{\eta}_B(f(x)), f^{-1}(\sigma_B(x)) = (\sigma_B(f(x))$ and $f^{-1}(\phi_B(x)) = (\phi_B(f(x))$*

Theorem 6.4. *Let $f : X \to Y$ be a $\beta-$ homomorphism . If $\tilde{C}$ is a $(\overline{T},\overline{S},S,T)$-normed cubic intuitionistic $\beta-$subalgebra of Y, then $f^{-1}(\tilde{C})$ is a $(\overline{T},\overline{S},S,T)$-normed cubic intuitionistic $\beta-$subalgebra of X.*

Proof. *Let $\tilde{C}$ be a$(\overline{T},\overline{S},S,T)$-normed cubic intuitionistic $\beta-$subalgebra of Y,*
For $x, y \in Y$,

$$\begin{aligned} f^{-1}(\overline{\zeta}(x+y)) &= \overline{\zeta}(f(x+y)) \\ &= \overline{\zeta}(f(x)+f(y)) \\ &\geq \overline{T}\{\overline{\zeta}(f(x)), \overline{\zeta}(f(y))\} \\ &\geq \overline{T}\{f^{-1}(\overline{\zeta}(x)), f^{-1}(\overline{\zeta}(y))\} \end{aligned}$$

Similarly, $f^{-1}(\overline{\zeta}(x-y)) \geq \overline{T}\{f^{-1}(\overline{\zeta}(x)), f^{-1}(\overline{\zeta}(y))\}$. On the other hand,

$$\begin{aligned} f^{-1}(\overline{\eta}(x+y)) &= \overline{\eta}(f(x+y)) \\ &= \overline{\eta}(f(x)+f(y)) \\ &\leq \overline{S}\{\overline{\eta}(f(x)), \overline{\eta}(f(y))\} \\ &\leq \overline{S}\{f^{-1}(\overline{\eta}(x)), f^{-1}(\overline{\eta}(y))\} \end{aligned}$$

In the similar manner, $f^{-1}(\overline{\eta}(x-y)) \leq \overline{S}\{f^{-1}(\overline{\eta}(x)), f^{-1}(\overline{\eta}(y))\}$. Moreover,

$$\begin{aligned} f^{-1}(\sigma_\rho(x+y)) &= \sigma_\rho(f(x+y)) \\ &= \sigma_\rho(f(x)+f(y)) \\ &\leq S\{\sigma_\rho(f(x)), \sigma_\rho(f(y))\} \\ &\leq S\{f^{-1}(\sigma_\rho(x)), f^{-1}(\sigma_\rho(y))\} \end{aligned}$$

Similarly, one can have $f^{-1}(\sigma_\rho(x-y)) \leq S\{f^{-1}(\sigma_\rho(x)), f^{-1}(\sigma_\rho(y))\}$. Also,

$$\begin{aligned} f^{-1}(\phi_\rho(x+y)) &= \phi_\rho(f(x+y)) \\ &= \phi_\rho(f(x)+f(y)) \\ &\geq T\{\phi_\rho(f(x)), \phi_\rho(f(y))\} \\ &\geq T\{f^{-1}(\phi_\rho(x)), f^{-1}(\phi_\rho(y))\} \end{aligned}$$

In the same way, $f^{-1}(\phi_\rho(x-y)) \geq T\{f^{-1}(\phi_\rho(x)), f^{-1}(\phi_\rho(y))\}$. Hence $f^{-1}(\tilde{C})$ is a $(\overline{T},\overline{S},S,T)$-normed cubic intuitionistic $\beta-$subalgebra of X. □

Definition 6.5. *Let f be a mapping from a set X into a set Y. Let $\tilde{C}$ be a $(\overline{T},\overline{S},S,T)$-normed cubic intuitionistic set in X. Then the image of $\tilde{C}$, denoted by $f[\tilde{C}]$, is the $(\overline{T},\overline{S},S,T)$-normed cubic intuitionistic in Y with the membership function defined by*

$$f(\tilde{C}) = \{\langle x, f_{rsup}(\overline{\zeta}), f_{rinf}(\overline{\eta}), f_{sup}(\sigma_\rho), f_{inf}(\phi_\rho)\rangle :\ x \in\ Y\}, \text{ where}$$

$$f_{rsup}(\overline{\zeta})(y) = \begin{cases} rsup_{x\in\ f^{-1}(y)}\ \overline{\zeta}(x), & \text{if } f^{-1}(y) \neq \emptyset \\ \overline{0}, & \text{otherwise} \end{cases}$$

$$f_{rinf}(\overline{\eta})(y) = \begin{cases} rinf_{x\in\ f^{-1}(y)}\ \overline{\eta}(x), & \text{if } f^{-1}(y) \neq \emptyset \\ \overline{1}, & \text{otherwise} \end{cases}$$

$$f_{inf}(\sigma_\rho)(y) = \begin{cases} inf_{x\in\ f^{-1}(y)}\ \sigma_\rho(x), & \text{if } f^{-1}(y) \neq \emptyset \\ 1, & \text{otherwise} \end{cases}$$

$$f_{sup}(\phi_\rho)(y) = \begin{cases} sup_{x\in\ f^{-1}(y)}\ \phi_\rho(x), & \text{if } f^{-1}(y) \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

Theorem 6.6. *Let $f : X \to X$ be an endomorphism of $\beta-$ algebra. If $\tilde{C}$ is normed cubic intuitionistic $\beta-$subalgebra of X, then $f(\tilde{C})$ is a $(\overline{T},\overline{S},S,T)$-normed cubic intuitionistic $\beta-$subalgebra of X.*

Proof. Let $\tilde{C}$ be a $(\overline{T},\overline{S},S,T)$-normed cubic intuitionistic $\beta-$subalgebra of Y, $x, y \in X$.

$$\begin{aligned} \overline{\zeta}_f(x+y) &= \overline{\zeta}(f(x+y)) \\ &= \overline{\zeta}(f(x)+f(y)) \\ &= \overline{\zeta}(f(x)) + \overline{\zeta}(f(y)) \\ &\geq \overline{T}\{\overline{\zeta}(f(x)), \overline{\zeta}(f(y))\} \\ &= \overline{T}\{\overline{\zeta}_f(x), \overline{\zeta}_f(y)\} \end{aligned}$$

Similarly, $\overline{\zeta}_f(x-y) \geq \overline{T}\{\overline{\zeta}_f(x), \overline{\zeta}_f(y)\}$

$$\begin{aligned}
\overline{\eta}_f(x+y) &= \overline{\eta}(f(x+y)) \\
&= \overline{\eta}(f(x)+f(y)) \\
&= \overline{\eta}(f(x))+\overline{\eta}(f(y)) \\
&\leq \overline{S}\{\overline{\eta}(f(x)),\overline{\eta}(f(y))\} \\
&= \overline{S}\{\overline{\eta}_f(x),\overline{\eta}_f(y)\}
\end{aligned}$$

Similarly, $\overline{\eta}_f(x-y) \leq \overline{S}\{\overline{\eta}_f(x),\overline{\eta}_f(y)\}$

$$\begin{aligned}
\sigma_f(x+y) &= \sigma(f(x+y)) \\
&= \sigma(f(x)+f(y)) \\
&= \sigma(f(x))+\sigma(f(y)) \\
&\leq S\{\sigma(f(x)),\sigma(f(y))\} \\
&= S\{\sigma_f(x),\sigma_f(y)\}
\end{aligned}$$

Similarly, $\sigma_f(x-y) \leq S\{\sigma_f(x),\sigma_f(y)\}$

$$\begin{aligned}
\phi_f(x+y) &= \phi(f(x+y)) \\
&= \phi(f(x)+f(y)) \\
&= \phi(f(x))+\phi(f(y)) \\
&\geq T\{\phi(f(x)),\phi(f(y))\} \\
&= T\{\phi_f(x),\phi_f(y)\}
\end{aligned}$$

Similarly, $\sigma_f(x-y) \geq T\{\sigma_f(x),\sigma_f(y)\}$. *Hence* $f(\tilde{C})$ *is a normed cubic fuzzy* $\beta-$*subalgebras of* Y. □

7. Conclusion

The theory of cubic sets initiated in[9], influenced many researchers. This theory have been utilized in numerous algebraic structures like $BCK/BCI-$ algebras and so on. The concept of intuitionistic fuzzy introduced in[3], applied in various algebraic systems. In this study, we have introduced the concept of cubic intuitionistic $\beta-$subalgebras. In addition, we extended the idea into cubic intuitionistic level set and product of cubic intuitionistic $\beta-$subalgebras. Consequently, the thought of $(\overline{T},\overline{S},S,T)$-normed cubic intuitionistic fuzzy $\beta-$subalgebra has been initiated using $\overline{T}$-norm, $\overline{T}$-conorm, T-norm and T-conorm. In future, this can be extended in other substructures of different algebraic systems.

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