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On domination in the total torsion element graph of a module

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Abstract

Let R be a commutative ring with non-zero unity and M be a unitary R-module. Let $T(M)$ be the set of torsion elements of M. Atani and Habibi [6] introduced the total torsion element graph of M over R as an undirected graph $T(\Gamma(M))$ with vertex set as M and any two distinct vertices x and y are adjacent if and only if $x + y \in T(M)$. The main objective of this paper is to study the domination properties of the graph $T(\Gamma(M))$. The domination number of $T(\Gamma(M))$ and its induced subgraphs $Tor(\Gamma(M))$ and $Tof(\Gamma(M))$ has been determined. Some domination parameters of $T(\Gamma(M))$ are also studied. In particular, the bondage number of $T(\Gamma(M))$ has been determined. Finally, it has been proved that $T(\Gamma(M))$ is excellent, domatically full and well covered under certain conditions.

Keywords: *Bondage number; Domination number; Total torsion element graph; Torsion elements; Non-torsion elements.*

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1. Introduction

The study of graphs associated to algebraic structures has become an exciting research topic in the last two decades, leading to many fascinating results and questions. Many fundamental papers assigning graphs to rings and modules have appeared recently, for instance see,[1,2,4,7,19]. In 2008, Anderson and Badawi [3] have introduced the total graph of a commutative ring and later on this notion has been generalised to many algebraic structures, in particular to module over a commutative ring (see [10,11]).

The concepts of dominating sets and domination numbers play a vital role in graph theory. Dominating sets are the focus of many books of graph theory, for example see [14] and [15]. But not much research has been done on the domination parameters of graphs associated to algebraic structures such as groups, rings, modules in terms of algebraic properties. However, some works on domination of graphs associated to rings and modules have appeared recently, for instance see,[9,12,17,18,20,21].

The study of the torsion elements is one of the important aspects of module theory. Atani and Habibi [6] have generalised the notion of total graph introduced by Anderson and Badawi [3] by introducing the total torsion element graph of a module M over a commutative R, denoted by $T(\Gamma(M))$, to be an undirected graph with all elements of M as vertices, and for distinct $x, y \in M$, the vertices x and y are adjacent if and only if $x+y \in T(M)$. Let $Tor(\Gamma(M))$ be the (induced) subgraph of $T(\Gamma(M))$, with vertices $T(M)$, and let $Tof(\Gamma(M))$ be the (induced) subgraph of $T(\Gamma(M))$ with vertices $Tof(M)$. They have studied the characteristics of $T(\Gamma(M))$ and its two induced subgraphs $Tor(\Gamma(M))$ and $Tof(\Gamma(M))$ by considering two cases, $T(M)$ is a submodule of M or is not a submodule of M.

In this paper an attempt has been made to study the domination properties of the graph $T(\Gamma(M))$. The domination number of $T(\Gamma(M))$ and its induced subgraphs $Tor(\Gamma(M))$ and $Tof(\Gamma(M))$ has been determined. Some domination parameters of $T(\Gamma(M))$ has been studied. The bondage number of $T(\Gamma(M))$ has also been determined. Finally, it has been proved that $T(\Gamma(M))$ is excellent, domatically full and well covered under certain conditions.

2. Preliminaries

In this section, we recall the definitions,concepts and results which is needed in the later sections.

Throughout this paper, R is a commutative ring with non-zero unity and M is an unitary R-module, unless otherwise specified. An element a of a commutative ring R is called a zero-divisor of R if $ab = 0$ for some non-zero element b of R. Let $Z(R)$ be the set of zero-divisors of R. Let $T(M) = \{m \in M | rm = 0 \text{ for some } 0 \neq r \in R\}$ be the set of torsion elements of M. Let $T(M)^*$ be the set of non-zero torsion elements of M. So, if R is an integral domain, then $T(M)$ is a submodule of M. Let $Tof(M) = M - T(M)$ be the set of non-torsion elements of M. A module M is called torsion module if $T(M) = M$. On the other hand, a module M is called torsion-free if $T(M) = \{0\}$. For a submodule N of M, we denote by $(N :_R M)$ the set of all r in R such that $rM \subseteq N$. The annihilator of M denoted by $Ann_R(M)$ is $(0 :_R M)$. For any undefined terminology in rings and modules we refer to [5,16].

By a graph G, we mean a simple undirected graph without loops. For a graph G, we denote by $V(G)$ and $E(G)$ the set of all vertices and edges respectively. We recall that a graph is finite if both $V(G)$ and $E(G)$ are finite sets, and we use the symbol $|G|$ to denote the number of vertices in the graph G. We say that G is a null graph if $E(G) = \phi$. A subgraph of G is a graph having all of its vertices and edges in G. A spanning subgraph of G contains all vertices of it. For any set S of vertices of G, the induced subgraph $< S >$ is the maximal subgraph of G with vertex set S. Two vertices x and y of a graph G are connected if there is a path in G connecting them. Also, a graph G is connected if there is a path between any two distinct vertices. A graph G is disconnected if it is not connected. A graph G is complete if any two distinct vertices are adjacent. We denote the complete graph on n vertices by K_n. A complete subgraph of G is called a clique. A maximum clique of G is a clique with largest number of vertices. The number of vertices in a maximum clique of G is called the clique number of G and it is denoted by $\omega(G)$. If the vertex set $V(G)$ of the graph G are partitioned into two non-empty disjoint sets X and Y

of cardinality $|X| = m$ and $|Y| = n$, and two vertices are adjacent if and only if they are not in the same partite set, then G is called a bipartite graph. A graph G is called a complete bipartite graph if every vertex in X is connected to every vertex in Y. We denote the complete bipartite graph on m and n vertices by $K_{m,n}$. For vertices $x, y \in G$ one defines the distance $d(x, y)$, as the length of the shortest path between x and y, if the vertices $x, y \in G$ are connected and $d(x, y) = \infty$, if they are not. Then, the diameter of the graph G is

$$diam(G) = sup\{d(x, y)|x, y \in G\}.$$

The cycle is a closed path which begins and ends in the same vertex. The cycle of n vertices is denoted by C_n. The girth of the graph G,denoted by $gr(G)$ is the length of the shortest cycle in G and $gr(G) = \infty$ if G has no cycles.

For a subset $S \subseteq V$, $< S >$ denotes the subgraph of G induced by S. For a vertex $v \in V$, $deg(v)$ is the degree of the vertex v, $N(v) = \{u \in V \mid u$ is adjacent to $v\}$ and $N[v] = N(v) \cup \{v\}$. A subset S of V is called a dominating set if every vertex in $V - S$ is adjacent to at least one vertex in S. A dominating set S is called a strong(or weak) dominating set if for every vertex $u \in V - S$ there is a vertex $v \in S$ with $deg(v) \geq deg(u)$ (or $deg(v) \leq deg(u)$) and u is adjacent to v. The domination number $\gamma(G)$ of G is defined to be minimum cardinality of a dominating set in G and such a dominating set is called γ-set of G. If G is a trivial graph, then $\gamma(G) = 0$. In a similar way, we define the strong domination number γ_s and the weak domination number γ_w. A graph G is called excellent if for every vertex $v \in V$, there exists a γ-set S containing v. A domatic partition of G is a partition of V into dominating sets in G. The maximum number of classes of a domatic partition of G is called the domatic number of G and is denoted by $d(G)$. A graph G is called domatically full if $d(G) = \delta(G) + 1$, which is the maximum possible order of a domatic partition of $V(G)$ and $\delta(G)$ is the minimum degree of a vertex of G. The disjoint domination number $\gamma\gamma(G)$ defined by $\gamma\gamma(G)$ =min$\{|S_1| + |S_2| : S_1, S_2$ are disjoint dominating sets of $G\}$. Similarly, we can define $ii(G)$ and $\gamma i(G)$. The double domination parameters are referred to [13]. The bondage number $b(G)$ is the minimum number of edges whose removal increases the domination number. A set of vertices $S \subseteq V$ is said to be independent if no two vertices in S are adjacent in G. The independence number $\beta_0(G)$, is the maximum cardinality of an independent set in G. The minimum cardinality $i(G)$ of a maximal independent set of a graph G is called the independent domination number of G. A graph G is called well-covered if $\beta_0(G) = i(G)$. For basic definitions and results in domination we refer to [14] and for any undefined graph-theoretic terminology we refer to [8].

Now we summarize some results on domination number and bondage number of a graph which will be useful for the later sections.

Lemma 2.1:[8]

(i) If G is a graph of order n, then $1 \leq \gamma(G) \leq n$. A graph G of order n has domination number 1 if and only if G contains a vertex v of degree $n-1$; while $\gamma(G) = n$ if and only if $G \cong \overline{K_n}$.

(ii) $\gamma(K_n) = 1$ for a complete graph K_n, but the converse is not true, in general and $\gamma(\overline{K_n}) = n$ for a null graph $\overline{K_n}$.

(iii) Let G be a complete r-partite graph $(r \geq 2)$ with partite sets $V_1, V_2, ..., V_r$. If $|V_i| \geq 2$ for $1 \leq i \leq r$, then $\gamma(G) = 2$; because one vertex of V_1 and one vertex of V_2 dominate G. If $|V_i| = 1$ for some i, then $\gamma(G) = 1$.

(iv) $\gamma(K_{1,n}) = 1$ for a star graph $K_{1,n}$.

(v) If G is a partition of disjoint subgraphs $G_1, G_2, ..., G_k$, then $\gamma(G) = \gamma(G_1) + \gamma(G_2) + ... + \gamma(G_k)$.

(vi) Domination number of a bistar graph is 2; because the set consisting of two centres of the graph is a minimal dominating set.

(vii) Let C_n and P_n be a n-cycle and a path with n vertices, respectively. Then $\gamma(C_n) = \lceil \frac{n}{3} \rceil = \gamma(P_n)$.

Lemma 2.2:[15]

(i) If G is a simple graph of order n, then $1 \leq b(G) \leq n-1$.

(ii) $b(K_n) = n-1$ for a complete graph K_n, but the converse is not true, in general and $b(\overline{K_n}) = 0$ for a null graph $\overline{K_n}$.

(iii) Let G be a complete r-partite graph with partite sets $V_1, V_2, ..., V_r$. Then $b(G) = min\{|V_1|, |V_2|, ..., |V_r|\}$. In particular, $b(K_{m,n}) = min\{m, n\}$.

(iv) If G is a partition of disjoint subgraphs $G_1, G_2, ..., G_k$, then $b(G) = min\{b(G_1), b(G_2), ..., b(G_k)\}$.

(v) Let C_n and P_n be a n-cycle and a path with n vertices, respectively. Then $b(P_n) = 1$ and $b(C_n) = 2$.

3. Domination number of $T(\Gamma(M))$ and induced subgraphs

In this section, an attempt has been made to study the domination properties of the graph $T(\Gamma(M))$. In particular, the domination number of $T(\Gamma(M))$ and its induced subgraphs $Tor(\Gamma(M))$ and $Tof(\Gamma(M))$ have been determined.

We begin with the following examples.

Example 3.1:
Let $R = \mathbf{Z}_4$ and $M = \mathbf{Z}_8$. Then M is an R-module with the usual operations, and $T(M) = \{\overline{0}, \overline{4}\}$ is a submodule of M. Therefore $Tof(M) = \{\overline{1}, \overline{2}, \overline{3}, \overline{5}, \overline{6}, \overline{7}\}$.
Now we can easily observe that the induced subgraphs $Tor(\Gamma(M))$ is a complete graph K_2 and $Tof(\Gamma(M))$ is a disjoint union of a 4-cycle and K_2. Thus, we have $\gamma(Tor(\Gamma(M))) = \gamma(K_2) = 1$ and $\gamma(Tof(\Gamma(M))) = \gamma(C_4 \cup K_2) = \gamma(C_4) + \gamma(K_2) = \lceil \frac{4}{3} \rceil + 1 = 2 + 1 = 3$.

Also, we can see that the total graph $T(\Gamma(M))$ is the disjoint union of a 4-cycle and two disjoint K_2's. Hence, we have $\gamma(T(\Gamma(M))) = \gamma(C_4 \cup K_2 \cup K_2) = \gamma(C_4) + \gamma(K_2) + \gamma(K_2) = \lceil \frac{4}{3} \rceil + 1 + 1 = 2 + 1 + 1 = 4$.

Example 3.2:
Let $R = \mathbf{Z}_8$ and $M = \mathbf{Z}_4$. Then M is an R-module with the usual operations, and $T(M) = \{\overline{0}, \overline{2}, \overline{3}\}$. Thus $Tof(M) = \{\overline{1}\}$.
Now we can see that the induced subgraphs $Tor(\Gamma(M))$ is a star graph $K_{1,2}$ and $Tof(\Gamma(M))$ contains only an isolated vertex. Thus, we have $\gamma(Tor(\Gamma(M))) = \gamma(K_{1,2}) = 1$ and $\gamma(Tof(\Gamma(M))) = \gamma(\overline{K_1}) = 1$.
Also, we observe that the total graph $T(\Gamma(M))$ is a complete bipartite graph $K_{2,2}$. Hence, we have $\gamma(T(\Gamma(M))) = \gamma(K_{2,2}) = 2$.

Theorem 3.3:

Let M be a module over a commutative ring R such that $T(M)$ is a submodule of M. Then the following hold:

(1) The graph $T(\Gamma(M))$ is complete if and only if $T(M) = M$.

(2) The graph $T(\Gamma(M))$ is null if and only if $T(M) = \{0\}$ and $|M| \geq 2$.

Proof.

(1) If $T(M) = M$, then for any two vertices $x, y \in M$, one has $x + y \in T(M)$. Therefore, they are adjacent in $T(\Gamma(M))$. On the other hand, if $T(\Gamma(M))$ is complete, then every vertex is adjacent to 0. Thus, $x = x + 0 \in T(M)$ for each $m \in M$, from which the claim follows.

(2) If $T(\Gamma(M))$ is null, then for every $0 \neq x \in M$, $x = x + 0 \in Tof(M)$. So $T(M) = \{0\}$.
On the other hand, if $T(M) = \{0\}$, then $x + y \notin T(M)$ for any pair of distinct elements $x, y \in M$. Thus $T(\Gamma(M))$ is null.

□

The condition of the theorem 3.3(1) is necessarily fulfilled if $Ann_R(M) \neq \{0\}$. let us observe the following examples.

Example 3.4:
Let $R = \mathbf{Z}_n \times \mathbf{Z}_m$ and $M = \mathbf{Z}_n$ an R-module defined by $(a, b).m = am$. Then $Ann_R(M) \neq \{(\overline{0}, \overline{0})\}$ since $(\overline{0}, b) \in Ann_R(M)$ for every $b \in \mathbf{Z}_m$. Thus, $T(\Gamma(M))$ is complete and consequently $\gamma(T(\Gamma(M))) = 1$.

Example 3.5:
Every finite Abelian group M is a torsion $\mathbf{Z}$-module. In particular, if $R = \mathbf{Z}$ and $M = \mathbf{Z}_n$, an R-module with usual multiplication, then $T(\Gamma(M)) \cong K_n$ and consequently $\gamma(T(\Gamma(M))) = 1$.

Theorem 3.6:[6]
Let M be a module over a commutative ring R such that $T(M)$ is a submodule of M. Then the following hold:

(1) $Tor(\Gamma(M))$ is a complete (induced) subgraph of $T(\Gamma(M))$ and $Tor(\Gamma(M))$ is disjoint from $Tof(\Gamma(M))$.

(2) If N is a submodule of M, then $T(\Gamma(N))$ is the (induced) subgraph of $T(\Gamma(M))$.

(3) If $Ann_R(M) \neq \{0\}$, then $T(\Gamma(M))$ is a complete graph.

The following proposition is an immediate observation of the above theorems.

Proposition 3.7:
Let M be a module over a commutative ring R such that $T(M)$ is a submodule of M. Then the following hold:

(1) $\gamma(T(\Gamma(M))) = 1$ if $T(M) = M$.

(2) $\gamma(T(\Gamma(M))) = |M|$ if and only if $T(M) = \{0\}$ and $|M| \geq 2$.

(3) $\gamma(Tor(\Gamma(M))) = 1$.

(4) If $Ann_R(M) \neq \{0\}$, then $\gamma(T(\Gamma(M))) = 1$.

Corollary 3.8:
Let M be a torsion R-module, then $\gamma(T(\Gamma(M))) = 1$.

Proof. This is obvious as M being torsion, we have $T(M) = M$ and the result follows from proposition 3.7(1). □

The next theorem gives a complete description of $T(\Gamma(M))$. We allow α, β to be infinite, then of course $\beta - 1 = \frac{(\beta-1)}{2} = \beta$.

Theorem 3.9:[6]
Let M be a module over a commutative ring R such that $T(M)$ is a submodule of M, $|T(M)| = \alpha$ and $|\frac{M}{T(M)}| = \beta$.

(1) If $2 = 1_R + 1_R \in Z(R)$ then $Tof(\Gamma(M))$ is the union of $\beta - 1$ disjoint K_α's.

(2) If $2 = 1_R + 1_R \notin Z(R)$ then $Tof(\Gamma(M))$ is the union of $\frac{\beta-1}{2}$ disjoint $K_{\alpha,\alpha}$'s.

Example 3.10:
Let R be a ring and $M = R \oplus R$ a module over R.

(1) If $R = \mathbf{Z}_4$, then $T(\Gamma(M))$ is a union of 4 disjoint complete graphs K_4.

(2) If $R = \mathbf{Z}_9$, then $T(\Gamma(M))$ is a disjoint union of one complete graph K_9 and 4 bipartite graphs $K_{9,9}$.

Proposition 3.11:
Let M be a module over a commutative ring R such that $T(M)$ is a submodule of M, $|T(M)| = \alpha$ and $|\dfrac{M}{T(M)}| = \beta$, then $\gamma(T(\Gamma(M))) = \beta$.

Proof. Let us consider the following two cases for $Z(R)$.

Case 1: Suppose that $2 = 1_R + 1_R \in Z(R)$. Then we have from theorem 3.9(1) that the graph $Tof(\Gamma(M))$ is the union $\beta - 1$ disjoint K_α's and we know that $\gamma(K_\alpha) = 1$. Thus $\gamma(Tof(\Gamma(M))) = \beta - 1$. But $Tor(\Gamma(M))$ is complete by theorem 3.6(1), so $\gamma(Tor(\Gamma(M))) = 1$. Also by theorem 3.6(1), $Tor(\Gamma(M))$ is disjoint from $Tof(\Gamma(M))$. consequently, $\gamma(T(\Gamma(M))) = \gamma(Tor(\Gamma(M))) + \gamma(Tof(\Gamma(M))) = 1 + \beta - 1 = \beta$.

Case 2: Suppose that $2 = 1_R + 1_R \notin Z(R)$. Then again we have from theorem 3.9(2) that the graph $Tof(\Gamma(M))$ is the union of $\dfrac{\beta - 1}{2}$ disjoint $K_{\alpha,\alpha}$'s and we know that $\gamma(K_{\alpha,\alpha}) = 2$. Thus $\gamma(Tof(\Gamma(M))) = \dfrac{\beta - 1}{2} \times 2 = \beta - 1$. But $Tor(\Gamma(M))$ is complete by theorem 3.6(1), so $\gamma(Tor(\Gamma(M))) = 1$. Also by theorem 3.6(1), $Tor(\Gamma(M))$ is disjoint from $Tof(\Gamma(M))$. Therefore, $\gamma(T(\Gamma(M))) = \gamma(Tor(\Gamma(M))) + \gamma(Tof(\Gamma(M))) = 1 + \beta - 1 = \beta$. □

Proposition 3.12:
Let M be a non-zero torsion-free module over a commutative ring R such that $|\dfrac{M}{T(M)}| = \beta$, then $\gamma(T(\Gamma(M))) = \dfrac{\beta + 1}{2}$.

Proof. Since M is torsion-free, so we have $T(M) = \{0\}$. Therefore, $|\dfrac{M}{T(M)}| = |M| = \beta$. Now, we show that $Z(R) = \{0\}$. Let $0 \neq x \in Z(R)$, then there exist $0 \neq y \in R$ such that $xy = 0$. Let us consider an element $0 \neq m \in M$, and we have $(xy)m = 0$ which yields $x(ym) = 0$. Then $ym = 0$ as $x \neq 0$ which yields either $y = 0$ or $m = 0$, a contradiction. Therefore, $Z(R) = 0$. So, $2 = 1_R + 1_R \notin Z(R)$ and from theorem 3.9(2) we have the graph $Tof(\Gamma(M))$ is the union of $\dfrac{\beta - 1}{2}$ disjoint $K_{1,1}$'s. Thus $\gamma(Tof(\Gamma(M))) = \dfrac{\beta - 1}{2} \times 1 = \dfrac{\beta - 1}{2}$. But $Tor(\Gamma(M))$

is complete by theorem 3.6(1), so $\gamma(Tor(\Gamma(M))) = 1$. Also by theorem 3.6(1), $Tor(\Gamma(M))$ is disjoint from $Tof(\Gamma(M))$. Therefore we will have $\gamma(T(\Gamma(M))) = \gamma(Tor(\Gamma(M))) + \gamma(Tof(\Gamma(M))) = 1 + \frac{\beta - 1}{2} = \frac{\beta + 1}{2}$. □

Theorem 3.13:[6]
Let M be a module over a commutative ring R such that $T(M)$ is a submodule of M with $M - T(M) \neq \emptyset$. Then

(1) $Tof(\Gamma(M))$ is complete if and only if either $|\frac{M}{T(M)}| = 2$ or $|\frac{M}{T(M)}| = |M| = 3$.

(2) $Tof(\Gamma(M))$ is connected if and only if either $|\frac{M}{T(M)}| = 2$ or $|\frac{M}{T(M)}| = 3$.

(3) $Tof(\Gamma(M))$ (and hence $Tor(\Gamma(M))$) and $T(\Gamma(M))$ are totally disconnected if and only if $T(M) = \{0\}$ and $2 \in Z(R)$.

Theorem 3.14:[6]
Let M be a module over a commutative ring R such that $T(M)$ is a submodule of M. Then

(1) $diam(Tof(\Gamma(M))) = 0$ if and only if $T(M) = \{0\}$ or $|M| = 2$.

(2) $diam(Tof(\Gamma(M))) = 1$ if and only if either $T(M) \neq \{0\}$ and $|\frac{M}{T(M)}| = 2$ or $T(M) = \{0\}$ and $|M| = 3$.

(3) $diam(Tof(\Gamma(M))) = 2$ if and only if $T(M) \neq \{0\}$ and $|\frac{M}{T(M)}| = 3$.

(4) Otherwise, $diam(Tof(\Gamma(M))) = \infty$.

Note that $m + 0 \in T(M)$ for each $m \in T(M) - \{0\}$. So 0 is adjacent to any vertex of $T(M) - \{0\}$ in $Tor(\Gamma(M))$. Thus, $S = \{0\}$ is a containing set for $Tor(\Gamma(M))$ and hence $\gamma(Tor(\Gamma(M))) = 1$.

We now establish a relationship between the domination number of $T(\Gamma(M))$ and the same of $Tof(\Gamma(M))$.

Proposition 3.15:
Let M be a module over a commutative ring R such that $T(M)$ is a submodule of M. Then the following are equivalent:

(1) $\gamma(T(\Gamma(M))) = 2$.

(2) $\gamma(Tof(\Gamma(M))) = 1$.

(3) $|\dfrac{M}{T(M)}| = 2$ or $|\dfrac{M}{T(M)}| = |M| = 3$.

Proof. (1) $\Leftrightarrow$ (2):
Since $T(M)$ is a submodule of M, so by theorem 3.6(1), $Tor(\Gamma(M))$ and $Tof(\Gamma(M))$ are disjoint and $Tor(\Gamma(M))$ is complete. Therefore, $\gamma(Tor(\Gamma(M))) = 1$ and hence $\gamma(T(\Gamma(M))) = \gamma(Tor(\Gamma(M))) + \gamma(Tof(\Gamma(M)))$ which yields $\gamma(T(\Gamma(M))) = 1 + \gamma(Tof(\Gamma(M)))$.

(2) $\Rightarrow$ (3):
Suppose $\gamma(Tof(\Gamma(M))) = 1$. Then clearly $Tof(\Gamma(M))$ is connected. If $2 \in Z(R)$, then $\beta - 1 = 1$ and hence $\beta = 2$, where $\beta = |\dfrac{M}{T(M)}|$, by theorem 3.9(1). Thus $|\dfrac{M}{T(M)}| = 2$.

If $2 \notin Z(R)$, then $\dfrac{\beta - 1}{2} = 1$ and so $\beta = |\dfrac{M}{T(M)}| = 3$, by theorem 3.9(2). Also, by assumption, $\alpha = |T(M)| = 1$ and hence $T(M) = \{0\}$. Thus $|\dfrac{M}{T(M)}| = |M| = 3$.

(3) $\Rightarrow$ (2):
Assume $|\dfrac{M}{T(M)}| = 2$ or $|\dfrac{M}{T(M)}| = |M| = 3$. Then by theorem 3.13(1), $Tof(\Gamma(M))$ is complete and hence $\gamma(Tof(\Gamma(M))) = 1$. □

In the following a relationship between diameter and domination number of $Tof(\Gamma(M))$ has been established.

Corollary 3.16:
Let M be a module over a commutative ring R such that $T(M)$ is a submodule of M. Then

(1) $diam(Tof(\Gamma(M))) = 1$ if and only if $\gamma(Tof(\Gamma(M))) = 1$.

(2) $diam(Tof(\Gamma(M))) = 2$ if and only if $\gamma(Tof(\Gamma(M))) = 2$.

Proof. (1) It is clear by theorem 3.14(2) and proposition 3.15.
(2) If $diam(Tof(\Gamma(M))) = 2$, then $T(M) \neq \{0\}$ and $|\frac{M}{T(M)}| = 3$, by theorem 3.14(3). Hence $Tof(\Gamma(M))$ is connected, by theorem 3.13(2). Therefore $Tof(\Gamma(M))$ is a complete bipartite graph $K_{\alpha,\alpha}$ with $\alpha \geq 2$. So $\gamma(Tof(\Gamma(M))) = 2$.
Conversely, if $\gamma(Tof(\Gamma(M))) = 2$, then $Tof(\Gamma(M))$ is the union of two K_α's or is a complete bipartite graph $K_{\alpha,\alpha}$ with $\alpha \geq 2$, by theorem 3.9(1) and theorem 3.9(2). So $\beta - 1 = 2$ or $\frac{\beta - 1}{2} = 1$. In either case, $|\frac{M}{T(M)}| = 3$ and $|T(M)| \geq 2$. Thus $|T(M)| \neq \{0\}$ and $diam(Tof(\Gamma(M))) = 2$, by theorem 3.14(3). □

4. Bondage number of $T(\Gamma(M))$

In this section, the bondage number of the graph $T(\Gamma(M))$ has been studied. We begin with the following proposition.

Proposition 4.1:
Let M be a module over a commutative ring R such that $T(M)$ is a submodule of M, $|T(M)| = \alpha$ and $|\frac{M}{T(M)}| = \beta$. Then $b(T(\Gamma(M))) = \alpha - 1$.

Proof. Suppose that $2 = 1_R + 1_R \in Z(R)$. Then, by theorem 3.9(1), the graph $Tof(\Gamma(M))$ is the union of $\beta - 1$ disjoint K_α's and we know that $b(K_\alpha) = \alpha - 1$. Hence $b(Tof(\Gamma(M))) = \alpha - 1$. Also $T(M)$ is a submodule of M, so $Tor(\Gamma(M))$ is complete, by theorem 3.6(1). Thus, $b(Tor(\Gamma(M))) = \alpha - 1$. On the other hand, $Tor(\Gamma(M))$ and $Tof(\Gamma(M))$ are disjoint, by theorem 3.6(1). Therefore, $b(T(\Gamma(M))) = \alpha - 1$.

Now, suppose that $2 = 1_R + 1_R \notin Z(R)$. Then, by theorem 3.9(2), $Tof(\Gamma(M))$ is the union of $\frac{\beta - 1}{2}$ disjoint $K_{\alpha,\alpha}$'s and we know that $b(K_{\alpha,\alpha}) = \alpha$. Thus $b(Tof(\Gamma(M))) = \alpha$. But $Tor(\Gamma(M))$ is complete and disjoint from $Tof(\Gamma(M))$, by theorem 3.6(1). So, $b(Tor(\Gamma(M)))$ and hence $b(T(\Gamma(M)))$ is equal to $\alpha - 1$. □

Example 4.2:

(i) If $T(\Gamma(M))$ is complete, then $b(T(\Gamma(M))) = n - 1$. But $T(M) = M$, by theorem 3.3(1). So, $b(T(\Gamma(M))) = |T(M)| - 1$.

(ii) If $\gamma(G) = |V(G)|$, then $b(G) = 0$. So, by proposition 3.3(2), if $T(M) = 0$ and $|M| \geq 2$, then $b(T(\Gamma(M))) = 0$.

(iii) If M be a R-module such that $T(M)$ is a submodule of M, then $b(Tor(\Gamma(M))) = |T(M)| - 1$.

Theorem 4.3:[6]
Let M be a module over a commutative ring R such that $T(M)$ is a submodule of M. Then the following hold:

(1)(a) $gr(Tof(\Gamma(M))) = 3$ if and only if $2 \in Z(R)$ and $|T(M)| \geq 3$.

(1)(b) $gr(Tof(\Gamma(M))) = 4$ if and only if $2 \notin Z(R)$ and $|T(M)| \geq 2$.

(1)(c) Otherwise, $gr(Tof(\Gamma(M))) = \infty$.

(2)(a) $gr(T(\Gamma(M))) = 3$ if and only if $|T(M)| \geq 3$.

(2)(b) $gr(T(\Gamma(M))) = 4$ if and only if $2 \notin Z(R)$ and $|T(M)| = 2$.

(2)(c) Otherwise, $gr(T(\Gamma(M))) = \infty$.

In the following a relationship between girth and bondage number of $Tof(\Gamma(M))$ has been established.

Proposition 4.4:
Let M be a module over a commutative ring R such that $T(M)$ is a submodule of M, $|T(M)| = \alpha$ and $|\frac{M}{T(M)}| = \beta$. Then

(1) $gr(Tof(\Gamma(M))) = 3$ if and only if $b(Tof(\Gamma(M))) = \alpha - 1$ and $|T(M)| \geq 3$.

(2) $gr(Tof(\Gamma(M))) = 4$ if and only if $b(Tof(\Gamma(M))) = \alpha$ and $|T(M)| \geq 2$.

Proof.

(1) If $gr(Tof(\Gamma(M))) = 3$, then $2 \in Z(R)$ and $|T(M)| \geq 3$, by theorem 4.3(1)(a). Since $2 \in Z(R)$ so $Tof(\Gamma(M))$ is the union of $\beta - 1$ disjoint K_α's, by theorem 3.9(1). Therefore, $b(Tof(\Gamma(M))) = \alpha - 1$.

Now assume that $b(Tof(\Gamma(M))) = \alpha - 1$ and $|T(M)| \geq 3$. If $2 \notin Z(R)$, then $Tof(\Gamma(M))$ is the union of $\dfrac{\beta - 1}{2}$ disjoint $K_{\alpha,\alpha}$'s, by theorem 3.9(2) and hence $b(Tof(\Gamma(M))) = \alpha$, a contradiction by assumption. Therefore $2 \in Z(R)$, and then $gr(Tof(\Gamma(M))) = 3$, by theorem 4.3(1)(a).

(2) If $gr(Tof(\Gamma(M))) = 4$, then $2 \notin Z(R)$ and $|T(M)| \geq 2$, by theorem 4.3(1)(b). So $b(Tof(\Gamma(M))) = \alpha$, by the same argument to above. Now, let $b(Tof(\Gamma(M))) = \alpha$ and $|T(M)| \geq 2$. If $2 \in Z(R)$, then $b(Tof(\Gamma(M))) = \alpha - 1$, by theorem 3.9(1), a contradiction. So $2 \notin Z(R)$. Therefore, $Tof(\Gamma(M))$ is the union of $K_{\alpha,\alpha}$'s, where $\alpha \geq 2$. Thus $gr(K_{\alpha,\alpha})$ and hence $gr(Tof(\Gamma(M)))$ is equal to 4.

□

5. When is $T(\Gamma(M))$ is excellent, domatically full and well covered ?

In this section, some domination parameters of $T(\Gamma(M))$ has been studied. It has been proved that $T(\Gamma(M))$ is excellent, domatically full and well covered under some conditions.

We begin with the following proposition.

Proposition 5.1:
Let M be a module over a commutative ring R such that $T(M)$ is a submodule of M, $|T(M)| = \alpha \neq 0$ and $|\dfrac{M}{T(M)}| = \beta$. A set $S = \{x_1, x_2, ..., x_\beta\} \subset V(T(\Gamma(M)))$ is a γ-set of $T(\Gamma(M))$ if and only if $x_j \notin x_i + T(M)$ for all $1 \leq i, j \leq \beta$ and $i \neq j$.

Proof. If part follows directly from proposition 3.11 as $\gamma(T(\Gamma(M))) = \beta$. Conversely, let S be a γ-set of $T(\Gamma(M))$. Let us assume that there exist $j, k \in \{1, 2, ..., \beta\}$ such that $x_j \in x_k + T(M)$. Since $|S| = \beta$, so there exist a coset $x + T(M)$ such that $x_i \notin x + T(M)$ for all $x_i \in S$. Now, the vertices in $-x + T(M)$ cannot be dominated by S, a contradiction. $\square$

Theorem 5.2:[11]
Let x be a vertex of the graph $T(\Gamma(M))$. Then

$$deg(x) = \begin{cases} |T(M)| - 1, & \text{if } 2 \in Z(R) \text{ or } x \in T(M) \\ |T(M)|, & \text{otherwise.} \end{cases}$$

Proposition 5.3:
Let M be a module over a commutative ring R such that $T(M)$ is a submodule of M, $|T(M)| = \alpha \neq 0$ and $|\frac{M}{T(M)}| = \beta$, then

(1) $T(\Gamma(M))$ is excellent.

(2) the domatic number $d(T(\Gamma(M))) = \alpha$.

(3) $T(\Gamma(M))$ is domatically full.

Proof. The proof for (1) and (2) are trivial.
(3) By (2) we have $d(T(\Gamma(M))) = \alpha = |T(M)|$. Also, we have by theorem 5.2 that $\delta(T(\Gamma(M))) = |T(M)| - 1 = \alpha - 1$. Therefore, we have $d(T(\Gamma(M))) = \delta(T(\Gamma(M))) + 1$. Hence, $T(\Gamma(M))$ is domatically full. $\square$

Theorem 5.4:[6]
Let M be a module over a commutative ring R such that $T(M)$ is not a submodule of M. Then $T(\Gamma(M))$ is connected if and only if $M =< T(M) >$ (that is, $M =< a_1, a_2, ..., a_k >$ for some $a_1, a_2, ..., a_k \in T(M)$).

Lemma 5.5:
Let M be a module over a commutative ring R and N be a maximum annihilator submodule in M such that $|N| = \alpha \neq 0$ and $|\frac{M}{N}| = \beta$. If $\gamma(T(\Gamma(M))) = \mu$, then the set $S = \{x_1, x_2, ..., x_\mu\} \subset V(T(\Gamma(M)))$ is a γ-set of $T(\Gamma(M))$ where $x_j \notin x_i + N$ for all $1 \leq i, j \leq \beta$ and $i \neq j$.

Proposition 5.6:
Let M be a module over a commutative ring R. If $T(M)$ is not a submodule of M, $M =< T(M) >$ (that is, M is generated by $T(M)$) and

$\gamma(T(\Gamma(M))) = \mu$,
then $\gamma_t(T(\Gamma(M))) = \gamma_c(T(\Gamma(M))) = \mu$.

Proof. If $T(M)$ is not a submodule of M and $M =< T(M) >$, then by theorem 5.4, $T(\Gamma(M))$ is connected. Let N be a maximum annihilator submodule in M and $x_1 \in N$. Since $T(\Gamma(M))$ is connected, there exists a vertex $x_2 \in a_1 + N$ for some $a_1 \in M - N$ such that x_2 is adjacent to x_1. Again by connectedness of $T(\Gamma(M))$, there exists a coset $a_2 + N$ for some $a_2 \notin N$ as well as $a_2 \notin a_1 + N$ such that at least one element of $a_2 + N$ is adjacent to either a vertex in N or in $a_1 + N$, say N.

If there exists an element $a \in a_i+N$ which is adjacent to some $b \in a_j+N$ with $a \notin a_j+N$, then each vertex in a_i+N is adjacent to at least one vertex in $a_j + N$. For, if $a + b = c$ for some $c \in T(M)$, then $c \in a_i + a_j + N$. Let $d_1 \in a_i + N$ and take $d_2 \in M$ such that $d_1 + d_2 = c$. From this $d_2 \in a_j + N$ and d_1 is adjacent to d_2. Therefore, each vertex in $a_i + N$ is adjacent to at least one vertex in $a_j + N$.

Thus x_1 is adjacent to some vertex $x_3 \in a_2 + N$. Similarly, we can choose coset representatives x_i, for $4 \leq i \leq \mu$, in distinct cosets of N in M other than N, a_1+N and a_2+N such that $< x_1, x_2, ..., x_\mu >\subseteq T(\Gamma(M))$ is connected. Then by lemma 5.5, $\{x_1, x_2, ..., x_\mu\}$ is a γ_c-set of $T(\Gamma(M))$ and so $\gamma_c(T(\Gamma(M))) = \mu$. Since, for any graph G, we have $\gamma(G) \leq \gamma_t(G) \leq \gamma_c(G)$, so $\gamma_t(T(\Gamma(M))) = \mu$. □

Lemma 5.7:
Let M be a finite module over a commutative ring R such that $T(M)$ is a submodule of M, $|T(M)| = \alpha$ and $|\dfrac{M}{T(M)}| = \beta$. Then

$$T(\Gamma(M)) = \begin{cases} K_\alpha \cup \underbrace{K_\alpha \cup K_\alpha \cup ... \cup K_\alpha}_{(\beta-1) copies}, \text{if } 2 \in Z(R) \\ K_\alpha \cup \underbrace{K_{\alpha,\alpha} \cup K_{\alpha,\alpha} \cup ... \cup K_{\alpha,\alpha}}_{(\frac{\beta-1}{2}) copies}, \qquad \text{if } 2 \notin Z(R). \end{cases}$$

Proof. It follows from theorem 3.9 directly. □

Proposition 5.8:
Let M be a finite module over a commutative ring R such that $T(M)$ is a submodule of M, $|T(M)| = \alpha$ and $|\frac{M}{T(M)}| = \beta$. Then $T(\Gamma(M))$ is well covered.

Proof. If $2 \in Z(R)$, then by lemma 5.7 we have $i(T(\Gamma(M))) = \beta$.
If $2 \notin Z(R)$, then all the vertices in one partition of $K_{\alpha,\alpha}$ together with a vertex of $T(M)$, form an i-set of $T(\Gamma(M))$ and so $i(T(\Gamma(M))) = (\frac{\beta-1}{2})\alpha + 1$. Similarly $\beta_0(T(\Gamma(M)))$ is same as $i(T(\Gamma(M)))$. Thus

$$i(T(\Gamma(M))) = \beta_0(T(\Gamma(M))) = \begin{cases} \beta, & \text{if } 2 \in Z(R) \\ (\frac{\beta-1}{2})\alpha + 1, & \text{otherwise.} \end{cases}$$

Hence, $T(\Gamma(M))$ is well covered. □

Corollary 5.9:
Let M be a finite module over a commutative ring R such that $T(M)$ is a submodule of M and $|T(M)| = \alpha$, then $\omega(T(\Gamma(M))) = \alpha$.

As proved above, we can prove the following.

Proposition 5.10:
Let M be a finite module over a commutative ring R such that $T(M)$ is a submodule of M, $|T(M)| = \alpha$ and $|\frac{M}{T(M)}| = \beta$. Then

(1)
$$\gamma_t(T(\Gamma(M))) = \begin{cases} 2\beta, & \text{if } 2 \in Z(R) \\ \beta + 1, & \text{otherwise.} \end{cases}$$
.

(2) $\gamma_s(T(\Gamma(M))) = \gamma_w(T(\Gamma(M))) = \beta$.

(3) $\gamma_p(T(\Gamma(M))) = \beta$.

Proposition 5.11:
Let M be a finite module over a commutative ring R such that $T(M)$ is a submodule of M, $|T(M)| = \alpha$ and $|\frac{M}{T(M)}| = \beta$. Then

(1) $\gamma\gamma(T(\Gamma(M))) = 2\beta$.

(2)
$$\gamma i(T(\Gamma(M))) = \begin{cases} 2\beta, & \text{if } 2 \in Z(R) \\ \beta + (\frac{\beta-1}{2})\alpha + 1, & \text{otherwise.} \end{cases}$$
.

(3)
$$ii(T(\Gamma(M))) = \begin{cases} 2\beta, & \text{if } 2 \in Z(R) \\ (\beta-1)\alpha + 2, & \text{otherwise.} \end{cases}$$
.

(4)
$$tt(T(\Gamma(M))) = \begin{cases} 4\beta, & \text{if } 2 \in Z(R) \text{ and } \alpha \geq 4 \\ 2(\beta+1), & \text{if } 2 \notin Z(R) \\ \text{does not exist}, & \text{otherwise.} \end{cases}$$
.

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